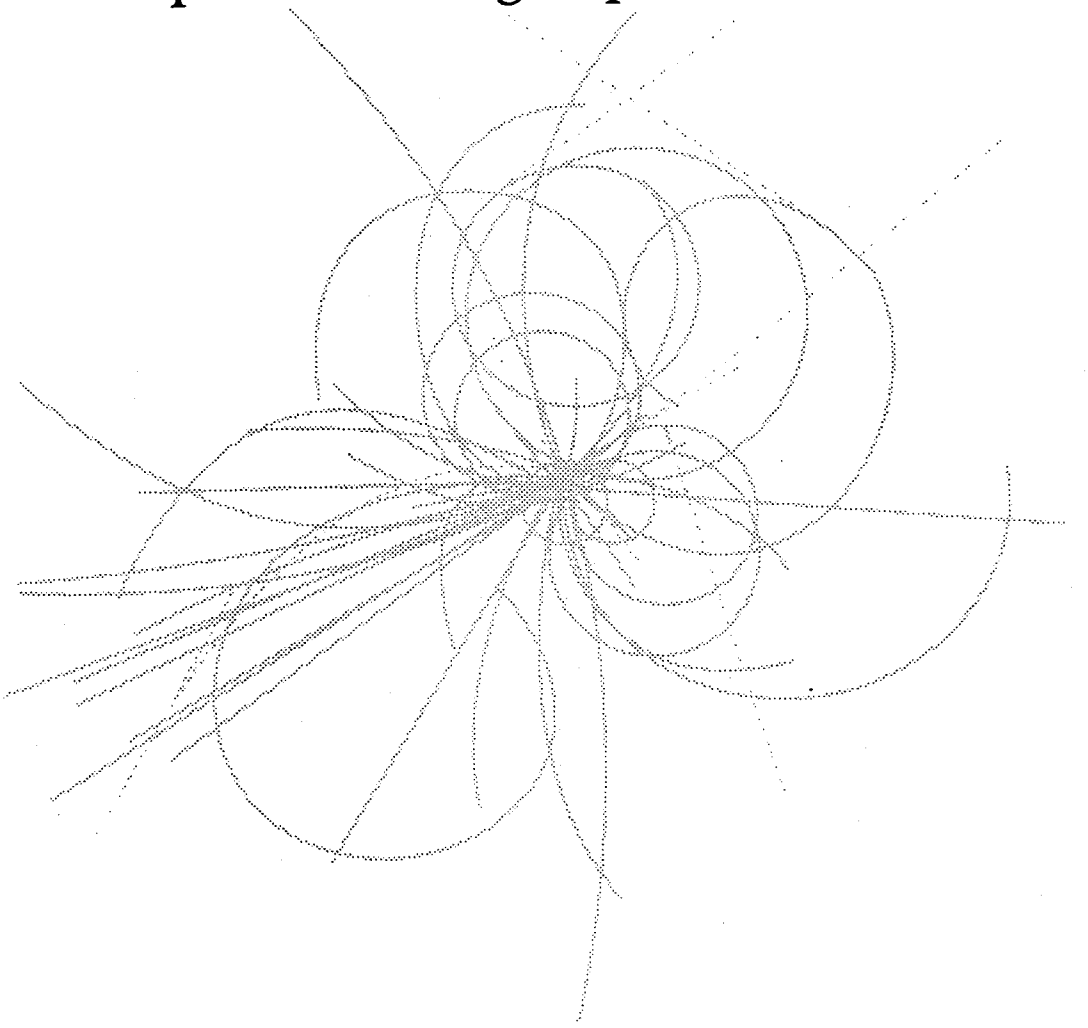
SSCL-499

SSCL-499

Superconducting Super Collider Laboratory



Uncertainties in the Foldy-Wouthuysen Transformation

Dan-di Wu

August 1991

SSCL-499
UM-P-91/61

Uncertainties in the Foldy-Wouthuysen Transformation*

Dan-di Wu

School of Physics
University of Melbourne
Parkville, Victoria, 3052
Australia

Current Address:

Superconducting Super Collider Laboratory†
2550 Beckleymeade Avenue
Dallas, TX 75237
USA
E-mail address: Danwu@SSCVX1.SSC.GOV

August 1991

* This work is supported in part by the Australia Research Council.
† Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

SSCL-499
UM-P-91/61

Uncertainties in the Foldy-Wouthuysen Transformation

Dan-di Wu

Abstract

The Foldy-Wouthuysen transformation is not uniquely defined in general, which causes some ambiguities when interaction is considered. Neither is the reduction of the Pauli equation from the Dirac equation.

The Foldy-Wouthuysen transformation[1] is designed to transform the Hamiltonian of a fermion to a block diagonalized one in order to make the physical contents explicit. At the limit of zero interaction, this diagonalization can be done completely. With interactions, the diagonalization is incomplete and can be done only in the nonrelativistic limit to the extent of neglecting higher orders of $1/m$ where m is the mass of the fermion. Although the Foldy-Wouthuysen transformations are not uniquely defined in either case, the ambiguity in the final results appears only in the case of incomplete diagonalization. For some very special interactions, this ambiguity has been studied by a few authors;[2] however, this article will show that the ambiguity exists in general. Let us first study the Hamiltonian of a free fermion (in the Bjorken-Drell representation):

$$H_o = \vec{\alpha} \cdot \vec{p} + \gamma_o m. \tag{1}$$

The most general unitary transformation to do the job is $U_o = \exp(S_o)$, where

$$S_o = (\gamma_o + ia\gamma_5)\vec{\alpha} \cdot \vec{p}\theta_o, \tag{2}$$

where θ_o is defined to meet $\tan(2|\vec{p}|\theta_o) = |\vec{p}|/m$ and a is an arbitrary parameter. It is easy to find

$$H_o' = UH_oU^{-1} = \gamma_o\sqrt{\vec{p}^2 + m^2}. \tag{3}$$

When there is an interaction (e.g., the electromagnetic interaction),

$$H = \theta + e\Phi + \gamma_o m, \tag{4}$$

where $\theta = \vec{\alpha} \cdot (\vec{p} - e\vec{A})$. With $U = \exp(S)$ and $S = (\gamma_o + ia\gamma_5 + ib\gamma_5\gamma_o)\theta/2m$, we find that

$$H' = UHU^{-1} = \gamma_o m + e\Phi + \frac{e\gamma_o}{2m}[\theta, \Phi] + \frac{\gamma_o}{2m}\theta^2 + a\frac{ie\gamma_5}{2m}[\theta, \Phi] + 0\left(\frac{1}{m}\right), \tag{5}$$

where skew block diagonal terms (the odd terms) are already suppressed to the order of $1/m$. Let the sum of these odd terms be θ', and let $S' = (\gamma_o + ia'\gamma_5 + ib'\gamma_5\gamma_o)\theta'/2m$; then

$$H'' = U'H'U'^{-1} \tag{6}$$

will have odd terms only at the order of $1/m^2$ where $U' = \exp(+S')$. One can push the odd terms to any higher orders of $1/m$, with the arbitrary parameters a, b, a', b' ... in the resulting Hamiltonian. When $a = b = a' = b' \ldots = 0$, we obtain the standard result.

Looking at the fifth term of Eq. (5), which is a diagonalized one, we can understand what the most important new physics is bringing in by the new parameters:

$$a\frac{ie\gamma_5}{2m}[\theta,\Phi] = -\frac{ea}{2m}\vec{\Sigma}\cdot\vec{E}, \tag{7}$$

where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & o \\ o & \vec{\sigma} \end{pmatrix}$. This is an electric dipole moment (EDM) interaction with an EDM for the electron $\frac{ea}{2m}$. If we write down the complete result of the transformation, we find that all terms are new-parameter-sensitive except the EM potential and the magnetic moment. Actually all operators which commute with γ_o can be added to S or S'. Therefore we can also consider terms like $i(p_o - m - e\Phi)(c + d\gamma_o)/2m$ in S, where p_o is the total energy of the electron in a stationary state and similar terms in S'.

Similarly, if we believe that the Dirac equation is fundamental, we can reduce the Pauli equation for two-component spinor from it;[3] we find different results if we do not do it most "naively." The Dirac equation for the electron can be easily turn into the corresponding Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi = [\vec{\alpha}\cdot(\vec{p} - e\vec{A}) + \gamma_o m + e\Phi]\Psi. \tag{8}$$

If we let $\Psi = e^{iEt}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ with $E^2 = \vec{p}^2 + m^2$ and $E > 0$, then for the eigenstate of energy we have

$$(E+\varepsilon)\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \vec{\alpha}\cdot(\vec{p} - e\vec{A})\begin{pmatrix} \chi \\ \varphi \end{pmatrix} + m\begin{pmatrix} \varphi \\ -\chi \end{pmatrix} + e\Phi\begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{9}$$

where ε is the electromagnetic energy of the electron in the external EM field. Imaginably, the leading order of ε is proportional to e. To the first order in e we find

$$\varepsilon\varphi = \left\{ e\Phi - \frac{e\vec{A}\cdot\vec{p}}{E} - \frac{e\vec{\sigma}\cdot\vec{B}}{2E} - \frac{e\vec{\sigma}\cdot\vec{E}\times\vec{p}}{2E(E+m)} + i\frac{e\vec{E}\cdot\vec{p}}{2E(E+m)} \right\}\varphi. \tag{10}$$

This is the naive Pauli equation reduced from Eq. (9). However, if we simply change φ to φ':

$$\varphi' = \sqrt{\frac{E+m}{E+m_1}}\left(\cos\lambda - i\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\sin\lambda\right)\varphi, \tag{11}$$

where $m_1 = m\cos 2\lambda$, $m_2 = m\sin 2\lambda$ (which will appear later), we find that

$$\varepsilon\varphi' = \left\{ e\Phi - \frac{e\vec{A}\cdot\vec{p}}{E} - \frac{e\vec{\sigma}\cdot\vec{B}}{2E} - \frac{e\vec{\sigma}\cdot\vec{E}\times\vec{p}}{2E(E+m_1)} + i\,\frac{\vec{E}\cdot\vec{p}}{2E(E+m_1)} - \frac{em_2\vec{\sigma}\cdot\vec{E}}{2E(E+m_1)} + \dots \right\}\varphi'. \tag{12}$$

Note the appearance of the EDM term and the fact there is not a standard to determine whether it is φ or φ' to describe the electron. Again we notice that the EM potential and magnetic moment terms are not changeable.

Part of these ambiguities can probably be traced back to the degeneracy of the even- and odd-parity solutions of the free Dirac equation. Both u_+ and u_- are eigenstates with the same energy E of Hamiltonian Eq. (1), where u_+ and u_- are

$$u_+ = \sqrt{\frac{E+m}{2E}}\begin{pmatrix} \varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi \end{pmatrix} \tag{13}$$

$$u_- = \sqrt{\frac{E-m}{2E}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\varphi \\ \varphi \end{pmatrix} \tag{14}$$

$$\gamma_o u_\pm(-\vec{p}) = \pm u_\pm(\vec{p}). \tag{15}$$

Of course, the general solutions can be

$$u' = au_+ + bu_-, \tag{16}$$

with arbitrary complex numbers a and b. A special series of the mixed parity solutions is

$$\tilde{u}(\lambda) = \frac{1}{\sqrt{2E(E+m_1)}}\left[\begin{pmatrix} (E+m)\varphi \\ \vec{\sigma}\cdot\vec{p}\varphi \end{pmatrix}\cos\lambda + \begin{pmatrix} \vec{\sigma}\cdot\vec{p}\varphi \\ (E-m)\varphi \end{pmatrix} i\sin\lambda\right]. \tag{17}$$

By identifying this as the wave function of a free electron, one obtains the first order EM energy in the external fields as

$$E'(\lambda) = \langle e|e\bar{\Psi}\,\not{A}\Psi|e\rangle = e\tilde{u}^\dagger(\lambda)\gamma_o\,\not{A}\tilde{u}(\lambda), \tag{18}$$

which is exactly the written terms in the bracket of Eq. (12) sandwiched between $\varphi^\dagger$ and φ.* A detailed study of this degeneracy can be found in a collaborative work of the author

* Eq. (12) can also be obtained by the use of $[(\not{p} - e\,\not{A})^2 - m^2]\Psi = 0$, if Ψ is replaced by u in Eq. (17) as the lowest order solution.

and others.[4] It is argued that instead of simply choosing the naive solution u_+ to start perturbation calculations, some procedure similar to perturbation with degenerate wave functions[5] in quantum mechanics should be invented, although it does not seem to be practical. It might turn out that the choice of u_+ is made more upon intuition than on rigorous theoretical considerations. As emphasized in that text, the solutions of the Dirac equation with Coulomb potential may provide a source of the needed hint.

ACKNOWLEDGEMENT

Enlightening discussions with Z. Huang and B.J. McKellar are appreciated.

REFERENCES

1. L.L. Foldy and S.A. Wouthuysen, *Phys. Rev.* **78** (1950) 29.
2. D.L. Pursey, *Nucl. Phys.* **13** (1958) 595; E. Eriksen, *Phys. Rev.* **111** (1958) 1011; M.V. Barnhill III, *Nucl. Phys.* **A131** (1969) 106.
3. See J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, 1964, p. 12.
4. V. Gupta, B.J. McKellar, and D.D. Wu, Melbourne University preprint.
5. See L. Schiff, *Quantum Mechanics*, McGraw-Hill, 1949, p. 157.