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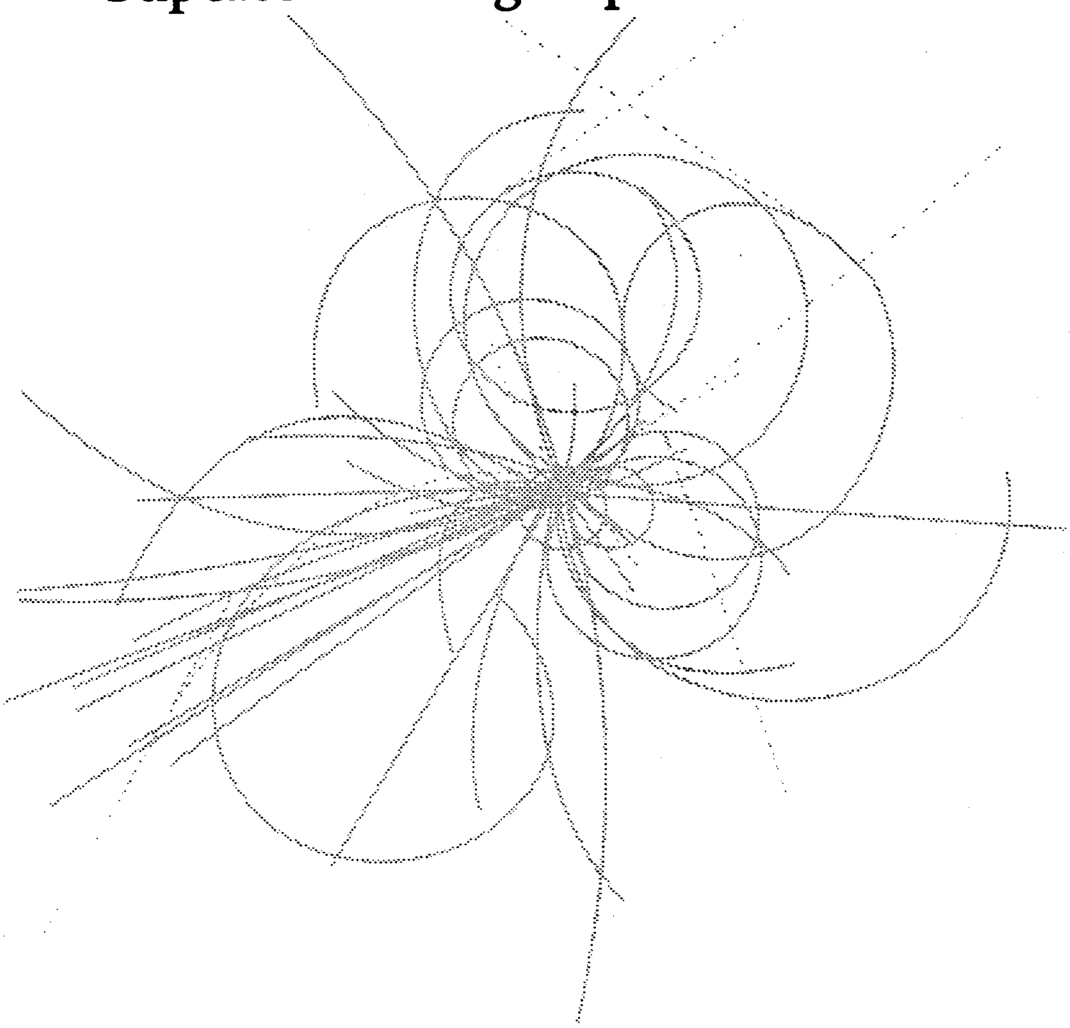

Hamiltonians and Lagrangians for One-Dimensional Autonomous Systems and Dissipative Systems

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Abstract

The Lagrangian and the Generalized Linear Momentum are given in terms of a constant of motion for a time-independent system. The possibility of having an explicit Hamiltonian expression is also analyzed. To illustrate the method, the approach is applied to some dissipative systems.

1.0 INTRODUCTION

The new generation of charged particle accelerators must deal with energy dissipation of the particles, due mainly to the synchrotron radiation light emission during the bending process. This phenomenon breaks the nice phase-space picture made for the particle behavior with the Hamiltonian formalism without dissipation. In fact, almost any dissipative system does it. The reason is simple: there is not yet a consistent Lagrangian and Hamiltonian formulation for the dissipative systems. This problem is a particular case of the well-known "inverse problem of the mechanics."

"The Inverse Problem of the Mechanics" or "the Inverse Problem of the Calculus of Variations" is seen when obtaining the derivation of the Lagrangian from the equations of motion of the mechanical system. This topic has been studied by many mathematicians and theoretical physicists since the end of the last century.[1–4] The interest of the physicists in this problem has grown recently because of the quantization of dissipative system.[5,6] A mechanical system can be quantized once its Hamiltonian[7] is known, and this Hamiltonian is usually obtained from a "equivalent" Lagrangian.[8] Therefore, it is important to know the correct expression for the Lagrangian in order to be sure of what it is quantizing. On the other hand, the Hamiltonian formulation of a dissipative system would allow the use of the powerful tool of canonical transformation and the study of the motion of the particles in the phase-space. This paper deals with this problem, restricting itself to the study of one-dimensional autonomous systems.

A mechanical system is said to be "autonomous" if the forces acting on the system do not depend explicitly on time. Otherwise, the system is said to be "nonautonomous." For one-dimensional autonomous systems, Kobussen[9] has given the Lagrangian in terms of a constant of motion of the system. This same expression was found by Leubner[10] using a similar approach, and it was used by Okubo[6] to study some quantization examples. For the same type of systems and using a different approach, Yan[11] has found the generalized linear momentum in terms of the same constant of motion. For these expressions and for the approach to follow, the constant of motion plays the most important part of the analysis. For this reason, it is shown how to obtain this constant through the characteristics method which, under certain restrictions, is defined as the "Generalized Energy."

Using a new approach and this constant of motion, the Lagrangian and the generalized linear momentum are given in terms of it. These expressions coincide with those of Kobussen-Leubner and of Yan. The transformation from one expression to the other is explained, and finally, the method is illustrated with some dissipative system examples.

2.0 THE CONSTANT OF MOTION AND GENERALIZED ENERGY

Newton's equation for a one-dimensional autonomous system can be written as the following dynamical system:

$$dv/dt = F(x, v) \tag{1a}$$

and

$$dx/dt = v, \tag{1b}$$

where v is the velocity, x is the distance, and $F(x, v)$ is an arbitrary force (the mass is included in this function).

The constant of motion, denoted by $K = K(x, v)$, of this system is the first integral of Eqs. (1). That is, K satisfies the partial differential equation

$$F(x, v)\,\partial K/\partial v + v\,\partial K/\partial x = 0. \tag{2}$$

The solutions (the integral surfaces) of Eq. (2) can be deduced from the equation for the characteristics:[12]

$$dv/F(x, v) = dx/dv = dK/0. \tag{3}$$

From the first two terms of Eq. (3), the characteristic curve, $C(x, v)$, can be obtained, and the general solution of Eq. (2) can be given by

$$K(x, v) = G(C(x, v)), \tag{4}$$

where G is an arbitrary function of the characteristic curve. For those cases in which the function $F(x, v)$ depends on one parameter, α, and satisfies the following limit:

$$\lim_{\alpha \to 0} F(x, v) = f(x), \tag{5}$$

where $f(x)$ is an arbitrary position-depending function, the function G in Eq. (4) might be chosen such that the constant of motion, $K(x, v)$, represents the usual energy of the system in this limit. This particular constant of motion is defined as the "Generalized Energy" (see appendix for conditions).

3.0 THE LAGRANGIAN AND GENERALIZED LINEAR MOMENTUM

In this section, the relation between the Lagrangian and the Generalized Linear Momentum (GLM) with the constant of motion is studied.

3.1 The Euler-Lagrange Equation and the Constant of Motion

It is known from Darboux[1] that the Lagrangian for one-dimensional systems always exists. So the relation of the Lagrangian with the constant of motion is carried out through the Legendre transformation,

$$v\,\partial L/\partial v - L = K, \tag{6}$$

where the Lagrangian, L, and the function K depend on the variables x and v. Taking the total time-derivative of Eq. (6) and knowing that this time-derivative is expressed as

$$d/dt = F(x,v)\,\partial/\partial v + v\,\partial/\partial x, \tag{7}$$

it follows that

$$vF(x,v)\,\partial^2 L/\partial v^2 + v^2\,\partial^2 L/\partial x\partial v - v\,\partial L/\partial x = dK/dt. \tag{8}$$

Factoring v is Eq. (8), rearranging terms in the resulting expression, and using Eq. (7) again, we obtain the next relation:

$$v\left[\frac{d}{dt}(\partial L/\partial v) - \partial L/\partial x\right] = dK/dt. \tag{9}$$

If v does not equal zero, the following conclusion is reached: the Euler-Lagrange equation (the homogeneous equation deduced from the expression enclosed by the square brackets) is satisfied if and only if the function K is a constant of motion. When v does equal zero, K is a constant of motion.

Thus, it can be seen that the Legendre transformation is compatible with the Euler-Lagrange equation if the function K, appearing on the right side of Eq. (6), is a constant of motion.

3.2 The Lagrangian

The Lagrangian can be deduced by solving Eq. (6) by the characteristics method, where K is the constant of motion given by Eq. (4). The equation for the characteristics of Eq. (6) are

$$dL/(L+K) = dv/v = dx/0. \tag{10}$$

Using the new variable

$$y = \log(v), \tag{11}$$

and considering the first two terms in Eq. (10), the following equation is obtained:

$$dL/dy = L + K. \tag{12}$$

Solving this equation and expressing the solution in terms of the original variable, v, the Lagrangian can be written as

$$L(x,v) = A(x)v + v\int^{v} K(x,\xi)\,d\xi/\xi^2, \tag{13}$$

where $A(x)$ is an arbitrary function. The first term on the right side of Eq. (13) corresponds to the solution of the homogeneous part of Eq. (6). This term does not affect Newton's equation of motion. When two Lagrangians differ only by a term of this type, it is said that both Lagrangians are "equivalents." Eq. (13) is the expression given by Kobussen[9] and Leubner.[10]

3.3 Generalized Linear Momentum and Hamiltonian

The GLM, $p(x,v)$, can be obtained from the known relation,[7]

$$p(x,v) = \partial L/\partial v. \tag{14}$$

Whenever possible, the inverse transformation can be written in the explicit form,

$$v = v(x,p). \tag{15}$$

The Hamiltonian, $H(x,p)$, of the system can be obtained using Eqs. (4) and (5), and it can be given by

$$H(x,p) = K(x, v(x,p)). \tag{16}$$

It is clear from Eq. (14) that it is not always possible to do this, as will be seen in the examples. Assume the Hamiltonian (Eq. (16)) has been obtained; then the Hamilton's equations,

$$\partial H/\partial x = -dp/dt \tag{17}$$

and

$$\partial H/\partial p = dx/dt, \tag{18}$$

are modified in terms of the variables x and v in the following form:

$$(\partial K/\partial v)(\partial v/\partial p)_x + (\partial K/\partial x) = -[F(x,v)(\partial p/\partial v)_x + v(\partial p/\partial x)_v] \tag{19}$$

and

$$(\partial K/\partial v)(\partial v/\partial p)_x = v, \tag{20}$$

where the subindex means that this variable is taken as constant. Substituting $F(x,v)$ obtained from Eq. (2) in Eq. (17) yields the expression

$$v(\partial p/\partial v)_x(\partial v/\partial x)_p + \partial K/\partial x = v(\partial K/\partial x)(\partial p/\partial v)_x/(\partial K/\partial v) - v(\partial p/\partial x)_v, \tag{21}$$

where $(\partial K/\partial v) \neq 0$; otherwise, the constant of motion would be a real number. On the other hand, using the derivation of implicit functions in Eq. (14), the following expression can be reduced to

$$(\partial v/\partial x)_p = -(\partial p/\partial x)_v/(\partial p/\partial v)_x \tag{22}$$

which can be used in Eq. (21) to obtain a more simplified relation:

$$1 = v(\partial p/\partial v)_x/(\partial K/\partial v). \tag{23}$$

This equation can be integrated easily to obtain the following expression for the GLM:

$$p(x,v) = B(x) + \int^{v} (\partial K(x,\xi)/\partial \xi)\, d\xi/\xi, \tag{24}$$

where $B(x)$ is an arbitrary function. This expression is the relation obtained by Yan.[11]

3.4 Relation between Kobussen-Leubner and Yan Expressions

Substituting Eq. (13) in the definition of Eq. (14) yields

$$p(x,v) = A(x) + \int^{v} K(x,\xi)\, d\xi/\xi + K(x,v)/v. \tag{25}$$

Now, integrating by parts the second term in the right side of Eq. (25), there is a term which cancels the third one, and there is another term that helps to reproduce Eq. (24) if the equality $A(x) = B(x)$ is imposed. In this way, Yan's expression is obtained from Kobussen-Leubner's expression.

Using Eq. (24) in Eq. (14) and integrating, the following Lagrangian expression is obtained:

$$L(x,v) = B(x)v + \int^{v} d\xi \int^{\xi} (\partial K(x,\rho)/\partial\rho)\, d\rho/\rho + C(x). \tag{26}$$

Integrating twice the second term of the right side of Eq. (26), we can write

$$\int^{v} d\xi \int^{\xi} (\partial K(x,\rho)/\partial\rho)\, d\rho/\rho = v \int^{v} K(x,\xi)\, d\xi/\xi^2, \tag{27}$$

which brings about the Kobussen-Leubner's expression if the relations $A(x) = B(x)$ and $C(x) = 0$ are chosen.

3.5 Obtaining the Lagrangian for a Given Hamiltonian

A different situation arises if the Hamiltonian is known. For the trivial case where the Hamiltonian is of the form $p^2/2m + V(x)$, the GLM is given by $p = mv$. But for nontrivial cases, this relation may not be true anymore. In this event, it is better first to find the Lagrangian through the solution of the nonlinear partial differential equation which results from Eqs. (6), (16), and (14):

$$v\frac{\partial L}{\partial v} - L = H\left(x, \frac{\partial L}{\partial v}\right), \tag{28}$$

and then to calculate the GLM from Eq. (14).

4.0 EXAMPLES

In this section the above approach is used in some dissipative systems, and the limit of nondissipation is discussed.

4.1 Constant Force for Relativistic and Nonrelativistic Particles

Consider a relativistic particle of mass at rest, m, which is moving under a constant force, β, in a dissipative medium where the dissipative force is proportional to an arbitrary function of the velocity, $g(v)$. Let the constant of proportionality be α. The equation of motion can be written as the following autonomous dynamical system:

$$mdv/dt = (1 - v^2/c^2)^{3/2}[\beta - \alpha g(v)] \tag{29a}$$

and

$$dx/dt = v, \tag{29b}$$

where c represents the speed of light. The equation for the characteristic curve, $C(x,v)$, is written as

$$mdv/(1 - v^2/c^2)^{3/2}[\beta - \alpha g(v)] = dx/v, \tag{30}$$

and from the solution of this equation, the characteristic curve is given by

$$C(x,v) = m\int^v \xi d\xi/(1 - \xi^2/c^2)^{3/2}[1 - \alpha g(\xi)/\beta] - \beta x. \tag{31}$$

If the parameter α goes to zero, this expression is the usual relativistic energy in a nondissipative medium. Therefore, it represents the generalized energy. Substituting this constant of motion in Eqs. (13) and (24), the Lagrangian and the GLM are expressed as

$$L(x,v) = mv\int^v d\xi/\xi^2 \int^\xi \rho d\rho/(1 - \rho^2/c^2)^{3/2}[1 - \alpha g(\rho)/\beta] + \beta x \tag{32}$$

and

$$p(x,v) = m\int^v d\xi/(1 - \xi^2/c^2)^{3/2}[1 - \alpha g(\xi)/\beta]. \tag{33}$$

These expressions also have their usual form in a nondissipative medium if the parameter α goes to zero. For $\alpha \neq 0$ it is not possible, in general, to obtain the velocity explicitly in terms of the generalized linear momentum of the particle, so it is not possible to obtain the explicit expression for the Hamiltonian.

It must be observed that when the parameter c goes to infinity, Eqs. (31), (32), and (33) become, respectively, the generalized energy, the Lagrangian, and the GLM of the nonrelativistic case. An interesting nonrelativistic particular case is obtained by making $g(v) = v$. Thus, the nonrelativistic dynamic quantities obtained from Eqs. (31), (32), (33), and (16) are given by

$$K(x,v) = -m(\beta/\alpha)^2 \log(1 - \alpha v/\beta) - (\beta/\alpha)mv - \beta x, \tag{34}$$

$$L(x,v) = -(m\beta/\alpha)[(v - \beta/\alpha)\log(1 - \alpha v/\beta) - v] + \beta x, \tag{35}$$

$$p(x,v) = -(m\beta/\alpha)\log(1 - \alpha v/\beta), \tag{36}$$

and

$$H(x,p) = (\beta/\alpha)p + m(\beta/\alpha)^2[\exp(-\alpha p/m\beta) - 1] - \beta x. \tag{37}$$

Note that all of these quantities have their correct usual expression for a nondissipative medium when the parameter α goes to zero.

4.2 Harmonic Oscillator in a Dissipative Medium

Consider a nonrelativistic particle of mass, m, moving in a dissipative medium, characterized by a linear dependence in the velocity, and under the action of Hooke's law of force. Newton's dynamic-equations of motion are written as

$$m dv/dt = -m\omega^2 x - 2m\omega_\alpha v \tag{38a}$$

and

$$dx/dt = v, \tag{38b}$$

where ω is the angular frequency of the oscillations in the nondissipative medium, and ω_α is defined in terms of the dissipative constant, α, as

$$\omega_\alpha = \alpha/2m. \tag{39}$$

This example is of particular interest in accelerator physics because the radiation damping phenomenon[13] can be written exactly like the above equations. Solving Eq. (2) and using the criterion given in the appendix, we obtain the following constant of motion:[14]

$$K(x,v) = (m/2)(v^2 + 2\omega_\alpha x v + \omega^2 x^2)\exp(-2\omega_\alpha G(v/x, \omega, \omega_\alpha)), \tag{40a}$$

where the function G divides the solution into three possible cases: Strong Dissipative case $(\omega^2 < {\omega_\alpha}^2)$, Critical Dissipative case $(\omega^2 = {\omega_\alpha}^2)$, and Weak Dissipative case $(\omega^2 > {\omega_\alpha}^2)$.

And it is defined as

$$G = \begin{cases} \dfrac{1}{2\sqrt{\omega_\alpha{}^2 - \omega^2}} \log\left[\dfrac{(\omega_\alpha + v/x) - \sqrt{\omega_\alpha{}^2 - \omega^2}}{(\omega_\alpha + v/x) + \sqrt{\omega_\alpha{}^2 - \omega^2}}\right], & \text{if } \omega^2 < \omega_\alpha{}^2; \\ \dfrac{1}{\omega_\alpha + v/x}, & \text{if } \omega^2 = \omega_\alpha{}^2; \text{ and} \\ \dfrac{1}{\sqrt{\omega^2 - \omega_\alpha{}^2}} \operatorname{Arctan}\left[\dfrac{\omega_\alpha + v/x}{\sqrt{\omega^2 - \omega_\alpha{}^2}}\right], & \text{if } \omega_o^2 > \omega_\alpha{}^2. \end{cases} \tag{40b}$$

For very weak dissipative medium, $\omega_\alpha \ll \omega$, the constant of motion can be written as

$$K(x,v) = (v^2 + \omega^2 x^2)m/2 + (m\omega_\alpha/\omega)\left[\omega x v - (v^2 + \omega^2 x)\operatorname{Arctan}(\omega_\alpha/\omega + v/x\omega)\right]. \tag{41}$$

This is a multi-valuated function, and it is not difficult to see that its phase is increased by the amount π anytime there is a crossing in the x-axis, which is due to the continuity of K and the *arctan* function. In this approximation, the Lagrangian is given by

$$\begin{aligned} L(x,v) &= m(v^2 - \omega^2 x^2)/2 \\ &+ -m\omega_\alpha x v \left\{ \left(\frac{\omega_\alpha}{\omega} + \frac{v}{x\omega}\right) \arctan\left(\frac{\omega_\alpha}{\omega} + \frac{v}{x\omega}\right) - \ln\sqrt{1 + \left(\frac{\omega_\alpha}{\omega} + \frac{v}{x\omega}\right)^2} \right\} \\ &- \frac{m\omega_\alpha \omega^2 x v}{\omega_\alpha^2 + \omega^2} \left\{ \ln\frac{v/x}{\sqrt{1 + (\omega_\alpha/\omega + v/x\omega)^2}} - \frac{\omega^2 + \omega_\alpha^2 + \omega_\alpha v/x}{\omega v/x} \arctan\left(\frac{\omega_\alpha}{\omega} + \frac{v}{x\omega}\right) \right\}, \end{aligned} \tag{42}$$

and the GLM may be calculated using Eq. (14), but the Hamiltonian of the system is given only in implicit form. From this example, the conclusion is that the dissipative harmonic oscillator (radiation damping in particular) has no explicit Hamiltonian formulation.

4.3 Hamiltonian for a Particular Dissipative System

The Hamiltonian for a free particle of mass, m, in a dissipative medium where the force is proportional to the speed square, is given by

$$H(x,p) = \frac{p^2}{2m}\exp(-2\alpha x/m), \tag{43}$$

where α is the dissipative constant. To find the Lagrangian associated to the system, Eq. (43) is substituted in Eq. (28) to obtain the following equation:

$$\frac{1}{2m}\exp(-2\alpha x/m)\left(\frac{\partial L}{\partial v}\right)^2 - v\left(\frac{\partial L}{\partial v}\right) + L = 0. \tag{44}$$

The solution of this equation brings about the Lagrangian and, consequently, the GLM is given by

$$L(x,v) = \frac{1}{2}mv^2 \exp(2\alpha x/m) \tag{45}$$

and

$$p(x,v) = mv \exp(2\alpha x/m). \tag{46}$$

All of these expressions have their right limit expression when the dissipation parameter, α, goes to zero.

5.0 CONCLUSION

Using a constant of motion for a one-dimensional autonomous system, the Lagrangian and the Generalized Linear Momentum are given in terms of it, but the Hamiltonian associated to the system is not always given explicitly. The approach is applied to some dissipative systems to get their Lagrangian and Hamiltonian formulation. All the quantities obtained in the examples have the right limit expression, when the parameter which characterize the dissipation goes to zero.

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APPENDIX
GENERALIZED ENERGY

Let $F(x, v, \alpha)$ be a continuous and nonsingular function (force) with respect to the parameter α which satisfies the following limit:

$$\lim_{\alpha \to 0} F(x, v, \alpha) = f(x). \tag{a1}$$

Newton's dynamic equations with this force are given by

$$dv/dt = F(x, v, \alpha) \tag{a2}$$

and

$$dx/dt = v. \tag{a3}$$

When α goes to zero, $\alpha \to 0$, these equations become

$$dv/dt = f(x) \tag{a4}$$

and

$$dx/dt = v, \tag{a5}$$

and the usual mechanical energy, $E = v^2/2 + \int^x f(\xi)d\xi$, is its associated constant of motion. Now, let $C(x, v, \alpha)$ be the characteristic curve obtained when the function $F(x, v, \alpha)$ is used in Eq. (3), so the constant of motion in Eq. (4) is given by

$$K(x, v, \alpha) = G\Big(C(x, v, \alpha)\Big). \tag{a6}$$

If it is possible to select the function G in such a way that it satisfies the following relation:

$$\lim_{\alpha \to 0} G\Big(C(x, v, \alpha)\Big) = E, \tag{a7}$$

then the constant of motion, $K(x, v, \alpha)$, is called "Generalized Energy." This is, in fact, the concept of reducibility[3] which is required as a guide for acceptable constant of motion of a mechanical system.