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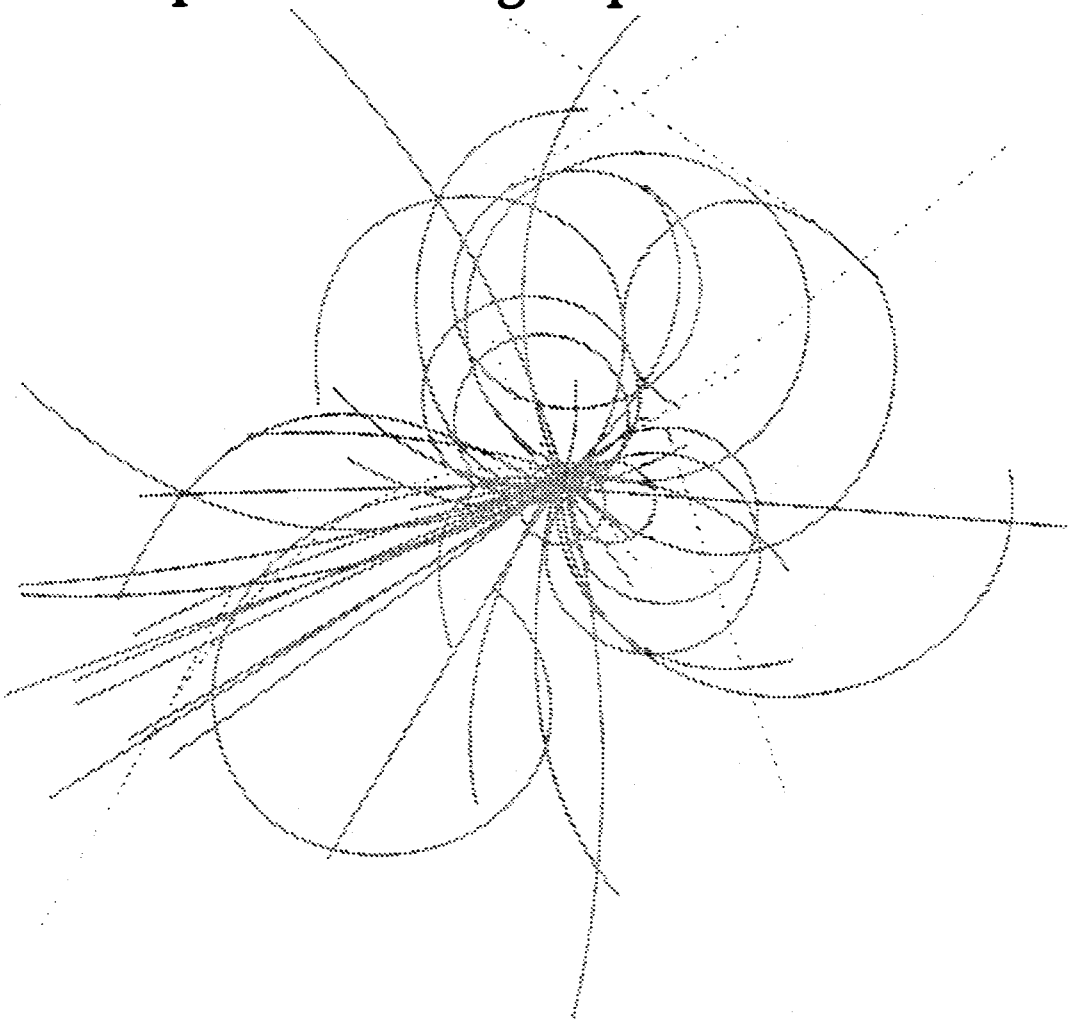

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May 1991

SSCL-469

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* Submitted to *Physical Review D*.

† Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC02-89ER40486.

Geometric Phases in Self–Induced Transparency

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Abstract

We consider the geometric phases arising in the lossless propagation of light pulses through a medium composed of near resonant two–level atoms. A reformulation of the coupled Maxwell–Schrödinger equations allows us to construct conservation laws in a general context. There exist periodic solutions of these equations which lead to the possibility of cyclical evolution of the state vector and the appearance of a geometric phase. We first show that if the ground state is the initial state of the system, then it acquires a geometric phase after the passage of the soliton pulses of McCall and Hahn. More generally if the initial state is a superposition of the two levels, continuous pulse trains can propagate without appreciable loss. We also find in this case that the state vector develops a geometric phase provided the parameters take on the particular values required for cyclical evolution. In both cases we exhibit the geometric character of the calculated phases by showing that they equal half the solid angle subtended by a closed curve traced by the Bloch vector on the Bloch sphere. We verify a recent assertion of Anandan and Aharonov that the energy uncertainty in the state is directly related to the speed at which the tip of the Bloch vector moves along the curve on the Bloch sphere (or in more general terms the energy uncertainty is related to the speed in the projective Hilbert space).

1 Introduction

The paradigm for Berry's phase [1] has been the system of a spin half particle in a magnetic field. If a time varying magnetic field vector traces out a closed loop and its evolution is adiabatic (i.e. the time period for the magnetic field is much larger than the period of precessional motion of the particle) then the state vector initially chosen to be an energy eigenstate remains an eigenstate at all times and furthermore picks up an additional phase besides the usual dynamical phase at the end of the cyclical evolution. The additional phase is geometric in that this phase equals one half the solid angle subtended by the closed loop (traced by the magnetic field) at the origin. It has also been shown by Aharonov and Anandan [2] that whenever the state vector undergoes a cyclic evolution (irrespective of whether the Hamiltonian varies adiabatically or not) , the state vector picks up a geometric phase which depends on the closed loop traced in the projective Hilbert space or equivalently the density matrix space . In particular, the state vector for a spin half particle in a constant magnetic field undergoes a cyclic evolution and if initially the state vector is not an energy eigenstate , the geometric phase equals one half the solid angle subtended by the closed loop traced out by the spin vector. Both the Berry phase with an adiabatically varying magnetic field and the Aharonov–Anandan phase for the state vector have been measured in nuclear magnetic resonance experiments [3, 4].

There are close analogies between magnetic resonance and optical resonance. By the result of Feynman, Vernon and Hellwarth [5], the interaction of two–level atoms with resonant electromagnetic radiation can be mapped onto the problem of a spin half particle in a magnetic field. This has been used recently to show that the energy eigenstate of a two–level atom develops a Berry's phase when placed in an external, nearly resonant electric field and that the phase affects the Rabi frequency of oscillation [6]. It has also been generalised to study the Berry's phase arising in a two-level atomic system described by a density matrix (either because it is prepared in a mixed state or to include the effects of dissipation) and interacting with an external classical laser field [7]. However there is one significant aspect in which optical resonance differs from magnetic resonance and that is the emission of radiation

by the atomic dipoles at wavelengths close to that of the applied field. Under the influence of the resonant field the atoms may emit radiation which then propagates through the medium composed of two level atoms.

In this work we will consider the phenomenon of self–induced transparency (SIT) where optical radiation propagates without loss in an absorbing medium and show that the state vector for the atom may develop a geometric phase as a consequence of the interaction. Note that in contrast to most other work on the geometric phase, the phase in this system develops as a consequence of the self consistent interaction between the medium of two–level atoms and the electric field and not due to some external parameters. As a consequence of this, the state and the electric field vary on the same time scale and the assumptions for the appearance of Berry's phase are not met. However, under certain conditions as we will see in the main text, the appearance of the Aharonov-Anandan (A-A) phase is possible. Therefore in this paper we will concentrate on the study of this A-A phase for SIT. By this study we aim to show explicitly the intimate connection between the geometry of the curves traced on the projective Hilbert space (here the Bloch sphere) and the dynamics of the system.

The phenomenon of self–induced transparency has been studied and observed for nearly two decades [8, 9, 10], but attention has been focussed on the components of the macroscopic polarization and the inversion. To study the phase we need to solve the Schrödinger equation for the state vector itself. In section 2 we set up the coupled system of Schrödinger and Maxwell equations describing the interaction of the two level atoms with the resonant electric field. With the slowly varying envelope approximation this can be reduced to a system of six first order differential equations. These equations enable us to view this composite system of quantum and classical sub–systems as a classical Hamiltonian system. We show that there exist three conservation laws, generalizing the forms of the laws usually stated. We solve this system in section 3 for two choices of initial conditions, the first corresponding to the usual case of self–induced transparency viz. all atoms initially in the ground state which results in a single *sech* pulse propagating through the medium. The second choice corresponds to taking the initial state vector to be a coherent superposition of the ground state and the

excited state. Here, instead of a single pulse, a continuous pulse train propagates through the medium taking the state vector through a cyclical evolution for a certain choice of initial parameters. We follow the dynamical evolution of the system by tracing the path of the Bloch vector on the Bloch sphere and explore the geometry of this path. At the end of a cyclical evolution of the state vector this path closes on itself. To emphasize the point made earlier, this loop is determined by the dynamics of our self consistent system and the initial conditions. This is in contrast to previous studies of the geometrical phase, where the closed loop was determined by the choice of the external magnetic field. In section 4 we propose a method by which the phase of the state vector may be measured following the set up used to observe the A-A phase in magnetic resonance [4]. We end with a summary in section 5.

2 Coupled Maxwell–Schrödinger Equations

The phenomenon of self–induced transparency was first observed nearly two decades ago by McCall and Hahn [8] and later followed up by Slusher and Gibbs [11], Matulic and Eberly [12] and others. When light at low intensities is incident upon a resonant medium it gets absorbed within a characteristic length (known as Beer's length) of the medium. However as the intensity is increased beyond a threshold value, the medium behaves as though it were transparent to the light pulse which passes through practically unabsorbed. In addition, the velocity of the pulse is much slower than the phase velocity in the medium. This behaviour is non–linear in nature and cannot be understood as perturbations around the behaviour at low intensities. Instead this may be understood by modelling the system as a collection of two–level atoms whose dynamics is governed by Schrödinger's equation in the presence of a classical electric field which itself evolves according to Maxwell's equation. Our point of departure from previous work is that we focus on both the amplitude and the *phase* developed by the state vector and not just the bilinear combinations of the amplitudes and the relative phase occurring in the components of the Bloch vector (although we certainly obtain these as a byproduct). Consequently, the system of equations we study are somewhat

different from that considered in the earlier literature.

We consider the interaction of a medium of two–level atoms with an electric field whose frequency is nearly resonant with the frequency of transition between the two levels. Let $|g\rangle$ and $|e\rangle$ be the lower and upper energy eigenstates differing in energy by $\hbar\omega_0$, in the absence of the electric field. The Hamiltonian in the presence of the electric field is

$$H = \frac{\hbar\omega_0}{2}\sigma_3 - \vec{p}.\vec{E} \tag{2.1}$$

for a single atom whose dipole moment $\vec{p}$ couples to the electric field. We neglect any dipole–dipole interaction between the atoms so that the atoms interact with one another only via the electric field. We assume that all other atomic levels are very distant in energy and hence do not couple resonantly to the electric field. We consider the electric field to be a circularly polarized plane wave, propagating in the positive z direction, the field vector being

$$\vec{E} = \mathcal{E}(z,t)[\hat{x}\cos\Phi(z,t) + \hat{y}\sin\Phi(z,t)] \tag{2.2}$$

where $\mathcal{E}(z,t)$ is the real envelope of the field,

$$\Phi(z,t) = \omega t - Kz + \phi(z,t)$$

is the total field phase and ω is the frequency of the light pulse in the medium. Circularly polarized light couples to a $|\Delta m| = 1$ transition between the two levels and the dipole moment vector for an individual atom can be written as

$$\vec{p} = p(\hat{x}\sigma_1 + \hat{y}\sigma_2).$$

where $p = p_x = p_y$. In matrix form the Hamiltonian may be written as

$$H = \begin{pmatrix} \frac{\hbar\omega_0}{2} & -p\mathcal{E}e^{-i\Phi} \\ -p\mathcal{E}e^{i\Phi} & -\frac{\hbar\omega_0}{2} \end{pmatrix} \tag{2.3}$$

We assume that the electric field amplitude $\mathcal{E}$ is sufficiently strong that the field may be regarded as classical, yet small enough so that

$$|p\mathcal{E}| << \hbar\omega_0.$$

This last condition is necessary so that the atom does not get ionized in the presence of the electric field and is well satisfied in practice. The wave number of the field is

$$K = (\omega\eta)/c + \delta k$$

where η is the refractive index of the medium and δk reflects the possible existence of dispersion due to the interaction between light and the atoms. The refractive index is assumed to be unaffected by the light pulse intensity so that no Kerr like effects are present. Note that since we wish to identify ω as the carrier frequency, any constant term arising in $\partial_t\phi$ must be set to zero.

It is assumed that the optical resonance line in the medium is inhomogeneously broadened, caused by Doppler broadening in a gas and by a distribution of static crystalline electric and magnetic fields in solids. We neglect the effect of other linewidth contributions due to homogeneous broadening e.g. caused by radiative decay, collisions with other atoms etc. This is a valid approximation as long as we consider pulse widths of a few nanoseconds, that being somewhat less than the time scale over which homogeneous broadening occurs. The distribution of transition frequencies ω_0 of the atoms in the medium is described by the spectral density function $g(\gamma)$ where $\gamma = \omega_0 - \bar{\omega}$. To a good approximation, $g(\gamma)$ may be taken to be symmetrical about a central frequency $\bar{\omega}$. It is normalized as

$$\int_{-\infty}^{\infty} g(\gamma)d\gamma = 1 \ .$$

The applied frequency ω of the electric field is tuned to $\bar{\omega}$, i.e the center of $g(\gamma)$ as closely as possible. The macroscopic polarization density of the medium, obtained by summing over the microscopic dipole moments of all atoms is

$$\vec{P} = Np\int_{-\infty}^{\infty} g(\gamma)[\hat{x}\langle\sigma_1\rangle + \hat{y}\langle\sigma_2\rangle]d\gamma. \tag{2.4}$$

N is the number of atoms per unit volume, $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ are the expectation values of the Pauli matrices in the atomic states.

We can write the state vector $|\Psi\rangle$ for an individual atom at time t as a superposition of the unperturbed eigenstates,

$$|\Psi(t)\rangle = a(t)|e\rangle + b(t)|g\rangle \tag{2.5}$$

The state vector obeys Schrödinger's equation

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle \tag{2.6}$$

which with the given form of the Hamiltonian (2.3) leads to the coupled equations

$$i\hbar\frac{\partial a}{\partial t} = \frac{\hbar\omega_0}{2}a - p\mathcal{E}e^{-i\Phi}b \tag{2.7}$$

$$i\hbar\frac{\partial b}{\partial t} = -p\mathcal{E}e^{i\Phi}a - \frac{\hbar\omega_0}{2}b. \tag{2.8}$$

The state vector is normalized to unity so that we have the condition

$$|a|^2 + |b|^2 = 1\ . \tag{2.9}$$

Now we perform a transformation to go from the laboratory coordinate system $(\hat{x}, \hat{y}, \hat{z})$ to a rotating coordinate system whose unit vectors $\hat{e}_1, \hat{e}_2$ rotate about $\hat{e}_3 \equiv \hat{z}$ with angular speed $\partial_t\Phi$. This transformation is effected by the unitary matrix

$$U(z,t) = e^{(i\Phi\sigma_3)/2}.$$

The transformed state vector and the Hamiltonian in the rotating frame are

$$\begin{aligned} |\Psi_r\rangle &= U|\Psi\rangle \\ H_r &= UHU^{-1} + i\hbar\frac{\partial U}{\partial t}U^{-1} \end{aligned}$$

We find that the transformed Hamiltonian has the explicit form

$$H_r = \frac{\hbar}{2}(\omega_0 - \partial_t\Phi)\sigma_3 - p\mathcal{E}\sigma_1 \tag{2.10}$$

while the state vector is

$$|\Psi_r\rangle = A|e\rangle + B|g\rangle \tag{2.11}$$

where we have defined

$$\begin{aligned} A &= ae^{(i\Phi)/2} \\ B &= be^{(-i\Phi)/2}. \end{aligned} \tag{2.12}$$

The Schrödinger equation in the rotating coordinate system is

$$\frac{\partial A}{\partial t} = -\frac{i}{2}(\omega_0 - \partial_t \Phi)A + \frac{ip}{\hbar}\mathcal{E}B \tag{2.13}$$

$$\frac{\partial B}{\partial t} = \frac{ip}{\hbar}\mathcal{E}A + \frac{i}{2}(\omega_0 - \partial_t \Phi)B \tag{2.14}$$

The unit vectors in the lab frame $(\hat{x}, \hat{y}, \hat{z})$ and those in the rotating frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are related as

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} .$$

In terms of the unit vectors in the rotating frame, the electric field simplifies to

$$\vec{E} = \mathcal{E}\hat{e}_1 \tag{2.15}$$

In this frame the electric field vector has a constant direction and changes slowly in time compared to the variation in the lab frame. The polarization vector expressed in the new basis is

$$\vec{P} = Np\int_{-\infty}^{\infty} g(\gamma)[\hat{e}_1\langle\sigma_1\rangle_r + \hat{e}_2\langle\sigma_2\rangle_r]d\gamma \tag{2.16}$$

where $\langle\sigma_i\rangle_r$ are the expectation values calculated in the rotating frame. We may define a pseudo–spin vector or Bloch vector $\vec{s}$ whose components are

$$u \equiv \langle\sigma_1\rangle_r = AB^* + A^*B \tag{2.17}$$

$$v \equiv \langle\sigma_2\rangle_r = i(AB^* - A^*B) \tag{2.18}$$

$$w \equiv \langle\sigma_3\rangle_r = |A|^2 - |B|^2 \tag{2.19}$$

As the name suggests, the pseudo–spin vector has a constant magnitude equal to unity, i.e.

$$u^2 + v^2 + w^2 = 1 .$$

u and v are the components of the polarization which are in phase and out of phase (by $\pi/2$) respectively with the electric field. w is known as the *inversion* and measures the difference

in population levels. We remark in passing that equations (2.13) and (2.14) lead to the following equation of motion for the Bloch vector

$$\frac{\partial}{\partial t}\vec{s} = \vec{\Omega} \times \vec{s} \tag{2.20}$$

where

$$\vec{\Omega} = \begin{pmatrix} -\frac{2p}{\hbar}\mathcal{E} \\ 0 \\ \omega_0 - \partial_t \Phi \end{pmatrix}.$$

Equation (2.20) has the same form as the equation describing the spinning of a classical top driven by a torque vector $\vec{\Omega}$. In magnetic resonance this equation is known as the Bloch equation (often with damping included).

The propagation of the electric field in the medium is governed by one of Maxwell's equations. The electric field induces a polarization in the medium of two–level atoms which in turn acts as a source for the propagation of the field. Assuming that the field is propagating in the z direction, Maxwell's equation for the electric field reduces to

$$[\frac{\partial^2}{\partial z^2} - \frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t} - \frac{\eta^2}{c^2}\frac{\partial^2}{\partial t^2}]\vec{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P} \tag{2.21}$$

where σ is the conductivity of the medium and η is the refractive index. Here we will assume that there are no Ohmic losses of energy so that we can set $\sigma = 0$.

We will now make the slowly varying approximation on the amplitude and phase of the complex envelope, viz.

$$\frac{\partial \mathcal{E}}{\partial t} << \omega\mathcal{E},\ \frac{\partial \mathcal{E}}{\partial z} << K\mathcal{E};\ \frac{\partial \phi}{\partial t} << \omega\phi,\ \frac{\partial \phi}{\partial z} << K\phi$$

i.e. $\mathcal{E}, \phi$ vary slowly on the time and length scale set by the carrier oscillations. We will drop all second derivatives and products of first derivatives of $\mathcal{E}$ and ϕ. With these approximations and also using the condition $\omega_0 >> \frac{2p\mathcal{E}}{\hbar}$ we obtain the following equations on equating the components of $\hat{e}_1$ and $\hat{e}_2$,

$$\frac{\partial \mathcal{E}}{\partial z} + \frac{\eta}{c}\frac{\partial \mathcal{E}}{\partial t} = \frac{2\pi N p}{\omega\eta c}\int_{-\infty}^{\infty} \omega_0^2 v(z,t;\gamma)g(\gamma)d\gamma \tag{2.22}$$

$$(\frac{\partial \phi}{\partial z} + \frac{\eta}{c}\frac{\partial \phi}{\partial t} - \delta k)\mathcal{E} = -\frac{2\pi N p}{\omega\eta c}\int_{-\infty}^{\infty} \omega_0^2 u(z,t;\gamma)g(\gamma)d\gamma \tag{2.23}$$

These are two real equations for the two real quantities $\mathcal{E}$ and ϕ. We write down the real versions of the complex equations (2.13) and (2.14) for the complex amplitudes A and B by introducing the real variables Q_a, Q_b, ϕ_a, ϕ_b, defined as

$$A = Q_a(z,t;\gamma)e^{i\phi_a(z,t;\gamma)} \tag{2.24}$$

$$B = Q_b(z,t;\gamma)e^{i\phi_b(z,t;\gamma)}. \tag{2.25}$$

Then, the complex equations (2.13) and (2.14) split into the four real equations

$$\frac{\partial Q_a}{\partial t} = -\frac{p}{\hbar}\mathcal{E}Q_b\sin(\phi_b - \phi_a) \tag{2.26}$$

$$\frac{\partial Q_b}{\partial t} = \frac{p}{\hbar}\mathcal{E}Q_a\sin(\phi_b - \phi_a) \tag{2.27}$$

$$\frac{\partial \phi_a}{\partial t} + \frac{1}{2}(\Delta - \frac{\partial \phi}{\partial t}) = \frac{p}{\hbar}\frac{\mathcal{E}Q_b}{Q_a}\cos(\phi_b - \phi_a) \tag{2.28}$$

$$\frac{\partial \phi_b}{\partial t} - \frac{1}{2}(\Delta - \frac{\partial \phi}{\partial t}) = \frac{p}{\hbar}\frac{\mathcal{E}Q_a}{Q_b}\cos(\phi_b - \phi_a) \tag{2.29}$$

where $\Delta = \omega_0 - \omega$, and we have omitted the dependences of Q_a, Q_b, ϕ_a and ϕ_b on $(z,t;\gamma)$ for clarity. Recall that the linear variation in time of Φ is contained in the definition of ω.

Now we look for travelling wave solutions of the system. We are interested in pulses which maintain a constant shape as they propagate through the medium. Thus we assume that all the quantities to be determined depend on the space–time coordinates through the single variable ξ, defined as

$$\xi = t - \frac{z}{v_0}$$

where v_0 is the steady velocity of the pulse. With this ansatz, our system of coupled Schrödinger–Maxwell equations reduces to a system of six ordinary differential equations,

$$\frac{d}{d\xi}Q_a = -\frac{p}{\hbar}\mathcal{E}Q_b\sin(\phi_b - \phi_a) \tag{2.30}$$

$$\frac{d}{d\xi}Q_b = \frac{p}{\hbar}\mathcal{E}Q_a\sin(\phi_b - \phi_a) \tag{2.31}$$

$$\frac{d}{d\xi}\mathcal{E} = -\alpha\int_{-\infty}^{\infty}\omega_0^2 Q_a Q_b\sin(\phi_b - \phi_a)g(\gamma)d\gamma \tag{2.32}$$

$$\frac{d}{d\xi}\phi_a + \frac{1}{2}(\Delta - \frac{d}{d\xi}\phi) = \frac{p}{\hbar}\frac{\mathcal{E}Q_b}{Q_a}\cos(\phi_b - \phi_a) \tag{2.33}$$

$$\frac{d}{d\xi}\phi_b - \frac{1}{2}(\Delta - \frac{d}{d\xi}\phi) = \frac{p}{\hbar}\frac{\mathcal{E}Q_a}{Q_b}\cos(\phi_b - \phi_a) \tag{2.34}$$

$$\frac{d}{d\xi}\phi + \Gamma = \alpha\int_{-\infty}^{\infty}\omega_0^2\frac{Q_aQ_b}{\mathcal{E}}\cos(\phi_b - \phi_a)g(\gamma)d\gamma \tag{2.35}$$

where we have defined the parameters

$$\alpha = \frac{4\pi Np}{\omega\eta c}(\frac{1}{v_0} - \frac{\eta}{c})^{-1}; \quad \Gamma = \delta k(\frac{1}{v_0} - \frac{\eta}{c})^{-1}.$$

Using equations (2.33) to (2.35) we can write down an equation for the variable $\phi_{ba} = \phi_b - \phi_a$ as

$$\frac{d}{d\xi}\phi_{ba} = \frac{p}{\hbar}\mathcal{E}(\frac{Q_a}{Q_b} - \frac{Q_b}{Q_a})\cos\phi_{ba} - \frac{\alpha}{\mathcal{E}}\int_{-\infty}^{\infty}\omega_0^2Q_aQ_b\cos\phi_{ba}g(\gamma)d\gamma + \Delta + \Gamma \tag{2.36}$$

This equation together with the three equations (2.30)–(2.32) form a closed system by themselves. From these equations we can derive with complete generality the following three conservation laws

$$1 = Q_a^2 + Q_b^2 \tag{2.37}$$

$$I_2 = \alpha\int_{-\infty}^{\infty}\omega_0^2Q_a^2g(\gamma)d\gamma - \frac{p}{\hbar}\mathcal{E}^2 \tag{2.38}$$

$$I_3 = \int_{-\infty}^{\infty}\omega_0^2[p\mathcal{E}Q_aQ_b\cos\phi_{ba} - \frac{\hbar(\Delta+\Gamma)}{2}Q_a^2]g(\gamma)d\gamma \tag{2.39}$$

where I_2, I_3 are constants. The first law is a statement of the normalization of the state vector. In terms of the components of the Bloch vector, the other two conservation laws may be written as

$$I_2' = \int_{-\infty}^{\infty}\omega_0^2w(\xi,\gamma)g(\gamma)d\gamma - \frac{2p}{\hbar\alpha}\mathcal{E}^2(\xi) \tag{2.40}$$

$$I_3' = \int_{-\infty}^{\infty}\omega_0^2[p\mathcal{E}u(\xi,\gamma) - \frac{\hbar(\Delta+\Gamma)}{2}w(\xi,\gamma)]g(\gamma)d\gamma \tag{2.41}$$

where we have renamed the constants as I_2' and I_3'. The second conservation law is the Manley-Rowe relation, indicating the transfer of energy from the field to the excited state

[15]. The third conservation law relates the energy of interaction between the electric field and the atoms to the energy stored in the atoms which depends on the inversion. Matulic and Eberly [12] do obtain a form of the second conservation law (2.40) but only after making the "factorization assumption", viz. that the solution for the second component of the Bloch vector $v(\xi,\gamma)$ factorizes in the form

$$v(\xi,\gamma) = F(\gamma)v(\xi,0) \tag{2.42}$$

an assumption which is not valid in general. Here, we have shown that (2.42) is not a requirement to obtain the conservation laws and (2.37)–(2.39) are valid in a more general context.

Furthermore, assuming the electric field to vanish at the initial instant, the second conservation law can be used to obtain an expression for the electric field

$$\mathcal{E}^2(\xi) = \frac{\hbar\alpha}{2p}\int_{-\infty}^{\infty}\omega_0^2[w(\xi,\gamma) - w_0(\gamma)]g(\gamma)d\gamma \tag{2.43}$$

where we have defined $w_0(\gamma)$ to be the initial value of the inversion. This expression corresponds closest to eq. (3.2) of Matulic and Eberly [12]. An alternative form of the third conservation law may be written down by using (2.40) and (2.41) to obtain

$$I_3'' = \mathcal{E}\int_{-\infty}^{\infty}\omega_0^2 u g(\gamma)d\gamma - \frac{\Delta+\Gamma}{\alpha}\mathcal{E}^2(\xi) \tag{2.44}$$

with I_3'' being a combination of I_2' and I_3'. With zero electric field initially, this conservation law reduces to

$$0 = \mathcal{E}(\xi)\int_{-\infty}^{\infty}\omega_0^2 u(\xi,\gamma)g(\gamma)d\gamma - \frac{\Delta+\Gamma}{\alpha}\mathcal{E}^2(\xi) \tag{2.45}$$

Again under the factorization assumption Matulic and Eberly obtain a somewhat different form of this conservation law (their equation (3.7)).

The complete solution of the system of equations (2.30)–(2.35) will give us the amplitudes and the phases occurring in the state vector at any arbitrary time as well as the polarization and the inversion. In the literature on self–induced transparency the only solutions obtained for an arbitrary choice of the spectral density function $g(\gamma)$ have been by use of the "factorization assumption" (2.42). This is the case because the system (2.30)-(2.35) corresponds to

an infinite set of coupled equations that decouple only for specific choices of $g(\gamma)$. For these choices, analytical solutions have been found. Recently, by assuming $g(\gamma)$ to be a sum of delta-functions located at different values of γ, it was proved that for certain values of the parameters the system exhibits the Painlevé property and consequently is likely to be integrable (for specifics see ref. [13]). Here, however, we avoid the complications of constructing a more general solution and consider only the simplest choice for $g(\gamma)$ viz. a single Dirac delta function, in what follows.

3 Sharp Line Self–Induced Transparency

Here we assume that the effects of inhomogeneous broadening on the width of the optical resonance line can also be ignored. All the atoms have the same transition frequency and consequently each atom experiences the same detuning from the applied field frequency ω. The spectral density function $g(\gamma)$ reduces to a Dirac delta function

$$g(\gamma) = \delta(\gamma - \Delta)$$

where $\Delta = \omega_0 - \omega$. In a gas inhomogeneous broadening would be absent if we are dealing with atoms either stationary or moving with the same velocity. Sharp line SIT has been observed in a beam of atomic rubidium with a few nanoseconds pulse from a ^{202}Hg II laser [14]. The Doppler width was about one-fourth the homogeneous width of the pulse. It has also been found that the difference between broad-line and sharp-line SIT is not very marked physically although the analysis of the latter case is considerably simpler.

With this assumption the equations for the amplitudes and the relative phase $\phi_{ba} = \phi_b - \phi_a$ decouple and reduce to

$$\frac{d}{d\xi}Q_a = -\frac{p}{\hbar}\mathcal{E}Q_b \sin\phi_{ba} \tag{3.1}$$

$$\frac{d}{d\xi}Q_b = \frac{p}{\hbar}\mathcal{E}Q_a \sin\phi_{ba} \tag{3.2}$$

$$\frac{d}{d\xi}\mathcal{E} = -\alpha_s Q_a Q_b \sin\phi_{ba} \tag{3.3}$$

$$\frac{d}{d\xi}\phi_{ba} = [\frac{p}{\hbar}\mathcal{E}(\frac{Q_a}{Q_b} - \frac{Q_b}{Q_a}) - \frac{\alpha_s}{\mathcal{E}}Q_a Q_b]\cos\phi_{ba} + \Delta + \Gamma \tag{3.4}$$

where

$$\alpha_s \equiv \omega_0^2\alpha = \frac{4\pi\omega_0^2 Np}{\omega\eta c}(\frac{1}{v_0} - \frac{\eta}{c})^{-1} > 0 \ .$$

The equations for the individual phases are

$$\frac{d}{d\xi}\phi_a + \frac{1}{2}(\Delta - \frac{d}{d\xi}\phi) = \frac{p}{\hbar}\frac{\mathcal{E}Q_b}{Q_a}\cos\phi_{ba} \tag{3.5}$$

$$\frac{d}{d\xi}\phi_b - \frac{1}{2}(\Delta - \frac{d}{d\xi}\phi) = \frac{p}{\hbar}\frac{\mathcal{E}Q_a}{Q_b}\cos\phi_{ba} \tag{3.6}$$

$$\frac{d}{d\xi}\phi = \alpha_s\frac{Q_a Q_b}{\mathcal{E}}\cos\phi_{ba} - \Gamma \tag{3.7}$$

Note that Q_a, Q_b, ϕ_a, ϕ_b, and ϕ_{ba} depend only on (ξ, Δ).

The solution of this system depends on the choice of initial conditions. We solve it first for the usual case of self–induced transparency, viz. all atoms are initially in the ground state and second in the more general situation where the initial state vector is a coherent superposition of the ground and excited states. We remark in passing, that precisely the same form of the equations as above occurs in the description of three–wave interactions in plasmas [15].

3.1 *Sech* pulses

With all the atoms initially in the ground state, the initial amplitudes for the states and electric field are

$$Q_a^2(-\infty) = 0 \ ; \ Q_b^2(-\infty) = 1 \ ; \ \mathcal{E}(-\infty) = 0$$

respectively. Consequently the conservation laws (2.37)–(2.39) simplify to

$$Q_a^2 + Q_b^2 = 1 \tag{3.8}$$

$$\mathcal{E}^2 = \frac{\hbar\alpha_s}{p}Q_a^2 \tag{3.9}$$

$$\mathcal{E}Q_aQ_b\cos\phi_{ba} = \frac{\hbar(\Delta+\Gamma)}{2p}Q_a^2\,. \tag{3.10}$$

Defining $r(\xi;\Delta) = Q_a^2(\xi;\Delta)$, we find that equations (3.1), (3.9) and (3.10) lead to

$$\frac{dr}{d\xi} = -2r\sqrt{\kappa(r_0 - r)} \tag{3.11}$$

where

$$r_0 = 1 - \frac{(\Delta+\Gamma)^2}{4\kappa}$$

is the maximum value of r while

$$\kappa = \frac{p\alpha_s}{\hbar} = \frac{4\pi\omega_0^2Np^2}{\hbar\omega\eta c}\left(\frac{1}{v_0} - \frac{\eta}{c}\right)^{-1}$$

is a parameter with the dimension of frequency squared. The equation (3.11) has the solution

$$r = r_0\,\mathrm{sech}^2[\sqrt{\kappa r_0}\xi] \tag{3.12}$$

It follows that the solutions for the amplitudes are

$$Q_a = \sqrt{r_0}\,\mathrm{sech}[\sqrt{\kappa r_0}\xi] \tag{3.13}$$

$$Q_b = \sqrt{1 - r_0\,\mathrm{sech}^2[\sqrt{\kappa r_0}\xi]} \tag{3.14}$$

$$\mathcal{E} = \frac{\hbar}{p}\sqrt{\kappa r_0}\,\mathrm{sech}[\sqrt{\kappa r_0}\xi]\,. \tag{3.15}$$

The secant hyperbolic form of the electric field in (3.15) is the well known "soliton" pulse of self–induced transparency, first found by McCall and Hahn. It is also known as a 2π pulse since the area under the envelope of the electric field is 2π. This is the only possible single pulse which can propagate without appreciable loss through the medium. The phases associated with the above amplitudes are

$$\phi_a = \frac{1}{4}(\Delta+\Gamma)\xi + c_a \tag{3.16}$$

$$\phi_b = \frac{1}{4}(\Delta+\Gamma)\xi + \arctan\left[\sqrt{\frac{r_0}{1-r_0}}\tanh(\sqrt{\kappa r_0}\xi)\right] + c_b \tag{3.17}$$

$$\phi = \frac{1}{2}(\Delta-\Gamma)\xi + \phi_0 \tag{3.18}$$

The constant c_a occurring in the phase ϕ_a of the excited state at the initial instant has no physical significance and for convenience we can choose $c_a = c_b = c_{-\infty}$ to be such that

$$\phi_b(-\infty) = 0 \ .$$

Earlier we had stated that in order to consistently identify ω as the carrier frequency of the pulse we must set any constant term in $\partial_t \phi$ to zero. Doing that for the solution in (3.18) yields $\Gamma = \Delta$ or equivalently the dispersion relation

$$\delta k = \Delta(\frac{1}{v_0} - \frac{\eta}{c}) \tag{3.19}$$

From the argument of the *sech* function in (3.15) we see that $\sqrt{\kappa r_0}$ can be identified with the inverse of the pulse width (τ_p), i.e.

$$\frac{1}{\tau_p^2} \equiv \frac{4\pi\omega_0^2 N p^2}{\hbar\omega\eta c}(\frac{1}{v_0} - \frac{\eta}{c})^{-1} - \Delta^2 \ . \tag{3.20}$$

and consequently (3.15) reads

$$\mathcal{E} = \frac{\hbar}{p\tau_p}\text{sech}\frac{\xi}{\tau_p} \tag{3.21}$$

Alternatively we may use (3.20) to express the steady velocity of the secant hyperbolic pulse in terms of the other parameters viz.

$$\frac{1}{v_0} = \frac{\eta}{c} + \frac{4\pi\omega_0^2 N p^2}{\hbar\omega\eta c}(\frac{1}{\tau_p^2} + \Delta^2)^{-1} \ . \tag{3.22}$$

This differs from the expression given by McCall and Hahn by the addition of the term involving Δ^2. In terms of the two independent parameters Δ and τ_p, we can express r_0 and κ as

$$r_0 = \frac{1}{1 + (\Delta\tau_p)^2} \ ; \ \kappa = \Delta^2 + \frac{1}{\tau_p^2} \ .$$

Now that we have obtained the complete solution of the Schrödinger equation, we may calculate the dynamic and geometric phases developed by the state vector at the end of the cyclic evolution. The dynamic phase is defined as [2]

$$\theta = -\frac{1}{\hbar}\int_{-\infty}^{\infty} \langle \Psi_r | H_r | \Psi_r \rangle dt \ , \tag{3.23}$$

this being the generalization of the usual dynamic phase $-\int E\ dt/\hbar$ to the case where the state vector is not an energy eigenstate. Using the expressions for the state vector and the Hamiltonian we find

$$\begin{aligned}\langle\Psi_r|H_r|\Psi_r\rangle &= \frac{\hbar}{2}(\omega_0 - \partial_t\Phi)[Q_a^2 - Q_b^2] - 2p\mathcal{E}Q_aQ_b\cos\phi_{ba} && (3.24)\\ &= -\frac{\hbar}{2}\Delta[2r+1] && (3.25)\end{aligned}$$

We note here that for atoms on resonance the dynamical phase vanishes at all times. At the end of the cyclical evolution, the dynamical phase for any group of atoms is found, by integrating the above expression, to be

$$\theta = \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2} + \frac{\Delta}{2}\lim_{T\to\infty}\xi|_{-T}^{T}\ . \tag{3.26}$$

To find the geometric phase we need to calculate the total phase change undergone by the state vector. The state vector is

$$|\Psi_r\rangle = Q_a e^{i\phi_a}|e\rangle + Q_b e^{i\phi_b}|g\rangle \tag{3.27}$$

We have $|\Psi_r(-\infty)\rangle = |g\rangle$, our choice of the initial phase being $\phi_b(-\infty) = 0$ which corresponds to setting the integration constant $c_{-\infty}$ in ϕ_b to

$$c_{-\infty} = \frac{1}{2}\lim_{\xi\to\infty}\Delta\xi + \arctan(\frac{1}{\Delta\tau_p})\ . \tag{3.28}$$

After the pulse has left, the atom returns to the ground state times a phase. We find

$$|\Psi_r(\infty)\rangle = e^{i\phi_b(\infty)}|g\rangle \tag{3.29}$$

where the total phase change in the state vector is (using equation (3.17))

$$\phi_b(\infty) = \lim_{\xi\to\infty}\Delta\xi + 2\arctan(\frac{1}{\Delta\tau_p})\ . \tag{3.30}$$

Hence the geometric phase at the end of the cyclical evolution is

$$\begin{aligned}\beta &\equiv \phi_b(\infty) - \theta && (3.31)\\ &= 2\arctan[\frac{1}{\Delta\tau_p}] - \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2} && (3.32)\end{aligned}$$

Fig. 1 shows the variation of β with the dimensionless quantity $\Delta\tau_p$. Exactly at resonance where $\Delta = 0$,

$$\beta_{res} = \pi \ , \tag{3.33}$$

and this is the entire phase change undergone by the ground state since the dynamical phase is zero for these atoms. This phase change encodes the history of the atom which makes it differ from an atom that has not been excited and de–excited by a pulse.

Now we will show that β is indeed a geometrical quantity. It is straightforward to show that the density matrix can be parametrized by the three components (u, v, w) of the Bloch vector. Consequently, a closed loop in the projective space corresponds to the loop traced by the Bloch vector on the unit sphere (recall that $u^2 + v^2 + w^2 = 1$ due to conservation of probability). The components of the Bloch vector are

$$u = 2Q_aQ_b\cos\phi_{ba} = \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2}\mathrm{sech}\frac{\xi}{\tau_p} \tag{3.34}$$

$$v = 2Q_aQ_b\sin\phi_{ba} = \frac{2}{1+(\Delta\tau_p)^2}\tanh\frac{\xi}{\tau_p}\mathrm{sech}\frac{\xi}{\tau_p} \tag{3.35}$$

$$w = Q_a^2 - Q_b^2 = \frac{2}{1+(\Delta\tau_p)^2}\mathrm{sech}^2\frac{\xi}{\tau_p} - 1 \tag{3.36}$$

If we label the polar and azimuthal angles on the sphere by μ and ν respectively, then we find

$$\mu \equiv \arccos w = \arccos\left(\frac{2}{1+(\Delta\tau_p)^2}\mathrm{sech}^2\frac{\xi}{\tau_p} - 1\right) \tag{3.37}$$

$$\nu \equiv \phi_{ba} = \arctan\left[\frac{1}{\Delta\tau_p}\tanh\frac{\xi}{\tau_p}\right] \tag{3.38}$$

The solid angle subtended by the closed loop is

$$\Omega = \int_{\nu_0}^{-\nu_0} d\nu \int_{\pi}^{\mu(\nu)} \sin\mu \, d\mu \tag{3.39}$$

$$= -\int_{\nu_0}^{-\nu_0} d\nu[\cos\mu(\nu) + 1] \tag{3.40}$$

where the equation of the loop defines the polar angle μ to be a function of the azimuthal angle ν and

$$\nu_0 = -\arctan\left[\frac{1}{\Delta\tau_p}\right]$$

is the initial azimuthal angle determined by our choice of the initial phase $\phi_b(-\infty)$. This is a mathematical value picked by continuity of the function ν defined in (3.38), the actual physical value of the initial azimuthal angle being undefined since the initial Bloch vector is along the negative $\hat{e}_3$ axis.

Integration yields the solid angle to be

$$\Omega = -4\arctan[\frac{1}{\Delta\tau_p}] + \frac{4\Delta\tau_p}{1+(\Delta\tau_p)^2} \tag{3.41}$$

Hence on comparing (3.32) and (3.41) we find that the geometric phase is related to the solid angle as

$$\beta = -\frac{1}{2}\Omega\ . \tag{3.42}$$

At resonance ($\Delta = 0$), the tip of the pseudo–spin vector traces a great circle on the Bloch sphere. As the detuning increases, the size of the loop traced by this vector (and the solid angle subtended by this loop) shrinks correspondingly. The extent of the excursion on the Bloch sphere is a measure of the maximum value of the inversion w_{max}. Since

$$w_{max} = \frac{1-(\Delta\tau_p)^2}{1+(\Delta\tau_p)^2}$$

we can relate the geometric phase β to w_{max} as

$$\beta = 2\arctan[\sqrt{\frac{1+w_{max}}{1-w_{max}}}] - \sqrt{1-w_{max}^2}\ . \tag{3.43}$$

This expression relates the geometric phase developed by the ground state at the end of the cyclical evolution to the maximum excitation, due to energy absorption from the electric field, from the ground state into the excited state during the course of the evolution. Fig. 2 shows that β is a monotonically increasing function of w_{max}. We point out however that the above relation between β and the maximum value of the inversion is true only in this particular frame rotating with the field, since the value of the geometric phase depends on the particular frame while the inversion is invariant under unitary transformations.

We explore the geometry of the curves on the Bloch sphere a little further in order to make the connection with recent work of Anandan and Aharonov [16]. They have shown

that the time integral of the uncertainty in energy for any quantum evolution is the same for all systems whose motion on the projective Hilbert space is identical. To be specific they show that the speed on the projective space is related to the energy uncertainty as

$$\frac{ds}{dt} = \frac{2}{\hbar}\Delta E \tag{3.44}$$

where

$$ds^2 = d\langle\Psi|\, d|\Psi\rangle$$

is the Fubini-Study metric on the projective Hilbert space and the energy uncertainty is defined as usual as

$$\Delta E^2 = \langle H^2\rangle - \langle H\rangle^2 \ .$$

We can provide an explicit test of this claim by calculating both sides of equation (3.44) independently. The projective space for our two-level system is the unit Bloch sphere for which the Fubini-Study metric reduces to the natural metric (see e.g [17])

$$ds^2 = d\mu^2 + \sin^2\mu\, d\nu^2 \tag{3.45}$$

Using our results above for the polar and azimuthal angles, we obtain the speed along the curve traced by the Bloch vector on the Bloch sphere to be

$$\frac{ds}{d\xi} = \frac{2}{\tau_p(1+(\Delta\tau_p)^2)}\mathrm{sech}\frac{\xi}{\tau_p}\sqrt{1+(\Delta\tau_p)^2\tanh^2\frac{\xi}{\tau_p}} \tag{3.46}$$

We can also calculate the energy uncertainty. We have already found the expectation value of the Hamiltonian to be

$$\langle H_r\rangle = \frac{\hbar\Delta}{2}w - p\mathcal{E}u \tag{3.47}$$

while the expectation value of H_r^2 turns out to be

$$\langle H_r^2\rangle = (\frac{\hbar\Delta}{2})^2 + (p\mathcal{E})^2 \ . \tag{3.48}$$

With the calculated values for u, w and $\mathcal{E}$ it is straightforward to show that $(2/\hbar)\Delta E_r$ agrees exactly with the speed $ds/d\xi$ calculated above. This system thus provides a non-trivial verification of the claim made in [16].

We can also use the above expression for the speed along the curve to calculate the circumference C of the closed loop traced by the Bloch vector at the end of a cyclical evolution. We obtain

$$C = \oint ds = \frac{4}{\sqrt{1+(\Delta\tau_p)^2}} E\left(\frac{\Delta\tau_p}{\sqrt{1+(\Delta\tau_p)^2}}\right) \tag{3.49}$$

where E is the complete elliptic integral of the second kind. At resonance with $\Delta = 0$, the circumference reduces to 2π, i.e the Bloch vector traces out a great circle on the sphere. Figure 3 shows that the circumference C decreases as the parameter $\Delta\tau_p$ increases. Figures of the closed loops traced by the Bloch vector for various values of $\Delta\tau_p$ can be found in the original paper of McCall and Hahn [8].

Thus from the calculations above, we see that the geometry of the curve on the Bloch sphere – namely the path length and the solid angle subtended by it (for a closed curve) , contains information about the dynamical evolution of the system. The study of the underlying geometry is particularly useful in cases where we are able to extract these geometric quantities from the symmetries of the system alone and without having to solve the Schrödinger equation ([18],[19]).

3.2 Continuous pulse trains

Apart from the single pulse solution found in the previous sub–section, infinite pulse trains may also propagate through the medium if we choose the initial conditions differently. Here we take the initial state vector to be a superposition of the ground state and the excited state. Our initial conditions on the amplitudes and the electric field are

$$Q_a^2(\xi_0) = \sin^2\frac{\lambda}{2}\ ;\ Q_b^2(\xi_0) = \cos^2\frac{\lambda}{2}\ ;\ \mathcal{E}(\xi_0) = 0$$

where ξ_0 denotes the initial value of the independent variable while the initial state vector is

$$|\Psi_r(\xi_0)\rangle = \sin\frac{\lambda}{2}e^{i\phi_a(\xi_0)}|e\rangle + \cos\frac{\lambda}{2}e^{i\phi_b(\xi_0)}|g\rangle \tag{3.50}$$

where $\phi_a(\xi_0)$ and $\phi_b(\xi_0)$ are the initial phases of the excited and ground states respectively. These phases may be assigned any arbitrary value but their difference may be known by

measuring the components of the transverse polarisation at ξ_0. We will see later that λ is the angle made by the Bloch vector with the negative $\hat{e}_3$ axis at the initial instant. The conservation laws (2.37)–(2.39) reduce to

$$Q_a^2 + Q_b^2 = 1 \tag{3.51}$$

$$\mathcal{E}^2 = \frac{\hbar\alpha_s}{p}(Q_a^2 - \sin^2\frac{\lambda}{2}) \tag{3.52}$$

$$\mathcal{E}Q_aQ_b\cos\phi_{ba} = \frac{\hbar(\Delta+\Gamma)}{2p}(Q_a^2 - \sin^2\frac{\lambda}{2}) \tag{3.53}$$

Defining the variable

$$r(\xi) = Q_a^2(\xi) - Q_a^2(\xi_0) = Q_a^2 - \sin^2\frac{\lambda}{2} \tag{3.54}$$

and using eq. (3.1) together with (3.52)-(3.53) we find that $r(\xi)$ obeys

$$\frac{dr}{d\xi} = -2\sqrt{\chi(r)} \tag{3.55}$$

where

$$\chi(r) = \kappa r[-r^2 + (\cos\lambda - \frac{(\Delta+\Gamma)^2}{4\kappa})r + \frac{1}{4}\sin^2\lambda] \ , \tag{3.56}$$

We may factorise $\chi(r)$ as

$$\chi(r) = \kappa r(r_+ - r)(r - r_-) \tag{3.57}$$

where

$$r_\pm = \frac{1}{2}[\zeta \pm \sqrt{\zeta^2 + \sin^2\lambda}]$$

$$\zeta = \cos\lambda - \frac{(\Delta+\Gamma)^2}{4\kappa}$$

Since $\chi(r)$ has negative slope at both $r = r_+$ and $r = r_-$ and positive slope at the origin, it follows that $\chi(r)$ has the shape shown in Figure 4. Furthermore, from (3.55) it follows that the region of physical interest is where $\chi(r) \geq 0$. In addition, from the equality

$$r = \frac{p\mathcal{E}^2}{\hbar\alpha_s}$$

we can exclude the possibility of r being negative since α_s is a positive parameter. The region of interest is thus restricted to $0 \leq r \leq r_+$. Integrating the equation for $r(\xi)$ we

obtain the solution

$$r(\xi) = r_+ \mathrm{cn}^2[\sqrt{\kappa(r_+ - r_-)}(\xi - \xi_0) + K(l),\ l] \tag{3.58}$$

where cn is one of the Jacobi elliptic functions and $K(l)$ is the complete elliptic integral of the first kind with modulus

$$l = \sqrt{\frac{r_+}{r_+ - r_-}}.$$

We may obtain the amplitudes for the ground and excited states and for the electric field from the expression for r. The solutions for the phases takes a little more work. It is straightforward to find the phase in the electric field to be

$$\phi = \frac{1}{2}(\Delta - \Gamma)\xi + \phi_0 \tag{3.59}$$

As in the previous section, setting the constant in $\partial_t \phi$ to zero yields $\Gamma = \Delta$ or equivalently the dispersion relation

$$\delta k = \Delta(\frac{1}{v_0} - \frac{\eta}{c}), \tag{3.60}$$

as it should be, since the dispersion relations do not depend on the initial conditions. The equations for the phases in the state vector are

$$\phi_a(\xi) = \Delta \int_{\xi_0}^{\xi} \frac{r}{r + \sin^2 \frac{\lambda}{2}}\, d\xi - \frac{\Delta}{2}(\xi - \xi_0) + \phi_a(\xi_0) \tag{3.61}$$

$$\phi_b(\xi) = \Delta \int_{\xi_0}^{\xi} \frac{r}{\cos^2 \frac{\lambda}{2} - r}\, d\xi + \frac{\Delta}{2}(\xi - \xi_0) + \phi_b(\xi_0) \tag{3.62}$$

Using the table of integrals in [20], we obtain the following results

$$\phi_a(\xi) = \frac{\Delta}{\sqrt{\kappa(r_+ - r_-)}}(q_a^2 - 1)[\Pi(\mathrm{am}\ U, q_a^2, l)]_{K(l)}^{U} + \frac{\Delta}{2}(\xi - \xi_0) + \phi_a(\xi_0) \tag{3.63}$$

$$\phi_b(\xi) = \frac{\Delta}{\sqrt{\kappa(r_+ - r_-)}}(1 - q_b^2)[\Pi(\mathrm{am}\ U, q_b^2, l)]_{K(l)}^{U} - \frac{\Delta}{2}(\xi - \xi_0) + \phi_b(\xi_0) \tag{3.64}$$

where we have defined the variable

$$U = \sqrt{\kappa(r_+ - r_-)}(\xi - \xi_0) + K(l)$$

and the constants

$$q_a^2 = \frac{r_+}{r_+ + \sin^2\frac{\lambda}{2}} > 0 \; ; q_b^2 = -\frac{r_+}{\cos^2\frac{\lambda}{2} - r_+} < 0$$

which are related by the expression

$$\frac{\cos^2\frac{\lambda}{2}}{q_a^2} + \frac{\sin^2\frac{\lambda}{2}}{q_b^2} = 1 \, .$$

$\Pi(\mathrm{am}\, U, q^2, l)$ is the elliptic integral of the third kind. From equation (3.54) it follows that the amplitudes are

$$Q_a^2 = r_+ \mathrm{cn}^2(U,l) + \sin^2\frac{\lambda}{2} \tag{3.65}$$

$$Q_b^2 = \cos^2\frac{\lambda}{2} - r_+ \mathrm{cn}^2(U,l) \tag{3.66}$$

$$\mathcal{E}^2 = \frac{\hbar^2}{p^2}\kappa r_+ \mathrm{cn}^2(U,l) \tag{3.67}$$

We introduce the parameter τ as a measure of the pulse width. Writing the electric field as

$$\mathcal{E} = \mathcal{E}_m \mathrm{cn}(\frac{\xi - \xi_0}{\tau} + K(l), l) \, , \tag{3.68}$$

τ is related to the other parameters as

$$\tau = \frac{1}{\sqrt{\kappa(r_+ - r_-)}} \tag{3.69}$$

and $\mathcal{E}_m$ is the maximum value of the electric field given by

$$\mathcal{E}_m = \frac{\hbar}{\sqrt{2}\, p\tau}\sqrt{1 - (\Delta\tau)^2 \sin^2\lambda + \cos\lambda\sqrt{1 - (\Delta\tau)^4 \sin^2\lambda}} \, . \tag{3.70}$$

The velocity v_0 of the pulse can be expressed in terms of the initial angle λ , the detuning Δ and the pulse width τ as

$$\frac{1}{v_0} = \frac{\eta}{c} + \frac{4\pi\omega_0^2 N p^2}{\hbar\omega\eta c}[\Delta^2\cos\lambda + \frac{1}{\tau^2}\sqrt{1 - (\Delta\tau)^4\sin^2\lambda}\,]^{-1} \, . \tag{3.71}$$

We turn our considerations to the possibility of cyclical evolution for this system. Since $\mathrm{cn}^2(U,l)$ is a periodic function with a real period $2K(l)$ (and a complex period which will not concern us) the amplitudes will return to their initial values at

$$\xi_{P_n} = \xi_0 + 2nK(l)\tau \tag{3.72}$$

where n is an integer. The atomic inversion w will also be periodic (with period $2K(l)$) since it depends only on the amplitudes Q_a^2 and Q_b^2. However the transverse components u , v are not necessarily periodic at these times because they also depend on the relative phase ϕ_{ba}. For the Bloch vector to have a cyclical evolution at the end of this period the state vector must return to its initial self except for a phase. Now we must determine the state vector at $\xi = \xi_{P_n}$. Let us denote

$$C_a^{(n)} = \phi_a(\xi_{P_n}) - \phi_a(\xi_0) \;\; ; C_b^{(n)} = \phi_b(\xi_{P_n}) - \phi_b(\xi_0)$$

The state vector at $\xi = \xi_{P_n}$ may be written as

$$|\Psi_r(\xi_{P_n})\rangle = e^{iC_a^{(n)}}[e^{i\phi_a(\xi_0)} \sin\frac{\lambda}{2}|e\rangle + e^{i(C_b^{(n)} - C_a^{(n)})} e^{i\phi_b(\xi_0)} \cos\frac{\lambda}{2}|g\rangle] \tag{3.73}$$

In order to have a cyclical evolution we must have $|\Psi_r(\xi_{P_n})\rangle$ related to $|\Psi_r(\xi_0)\rangle$ by a phase. That will happen if

$$\phi_{ba}(\xi_{P_n}) - \phi_{ba}(\xi_0) = C_b^{(n)} - C_a^{(n)} = 0 \;\; mod\; 2\pi \; . \tag{3.74}$$

This condition amounts to requiring that the components of the transverse polarization return to their initial values. Using the relation ,

$$\Pi(n\pi + \frac{\pi}{2}, q^2, l) = (2n+1)\Pi(\frac{\pi}{2}, q^2, l) = (2n+1)\Pi(q^2, l)$$

(obtained from equation (110.04) in [20]) where $\Pi(q^2, l)$ is the complete elliptic integral of the third kind, the above condition for cyclical evolution may be written as

$$2n\Delta\tau[(1-q_b^2)\Pi(q_b^2, l) + (1-q_a^2)\Pi(q_a^2, l) + K(l)] = 0 \;\; mod\; 2\pi. \tag{3.75}$$

This equation expresses the fact that a chosen value of the initial angle λ allows only certain values of the quantity $\Delta\tau$ for which the state vector undergoes a cyclical evolution. We solve this equation numerically to find the dimensionless parameter $\Delta\tau$ for a given λ. Figure 5 is a plot of these solutions of the cyclical evolution condition as a function of λ. As we expect, these solutions are symmetric about $\lambda = \pi$, the initial populations and the evolution of the system both possess this same symmetry. The physical observables of the system depend

on the two dimensional parameter space variables $(\lambda, \Delta\tau)$ which must be chosen initially. Cyclical evolution in this system comes about not by taking the parameters through a closed loop (as is the case for Berry's phase) but by choosing the initial values to lie on the one-dimensional curve shown in Fig. 5.

We also examine how the relative phase $\phi_{ba}(\xi)$ evolves with time for a particular choice of the parameters $(\lambda, \Delta\tau)$ lying on this curve. From (3.63) and (3.64) it follows that

$$\phi_{ba}(\xi) - \phi_{ba}(\xi_0) = \Delta\tau\{[(1-q_b^2)\Pi(\text{am } U, q_b^2, l) + (1-q_a^2)\Pi(\text{am } U, q_a^2, l)]_{K(l)}^{U} - \frac{\xi - \xi_0}{\tau}\}. \quad (3.76)$$

Figure 6 shows the variation of $\phi_{ba}(\xi) - \phi_{ba}(\xi_0)$ with the dimensionless scaled variable $(\xi - \xi_0)/\tau$ for the arbitrarily chosen value of $\lambda = \pi/10$. Plots for other values of λ are similar. Superimposed on the linear variation (with slope $-\Delta\tau$) is the behaviour due to the elliptic integrals. This relative phase difference goes through integral multiples of 2π at various times and at each of these times the Bloch vector completes a closed loop on the Bloch sphere.

If $\lambda = 0$, i.e. we return to the case dealt with in the previous sub-section, the condition for cyclical evolution becomes irrelevant since the amplitude for the excited state vanishes at the end of the evolution. If this condition is fulfilled for $\lambda \neq 0$ then the total phase change after n periods is

$$C_a^{(n)} = n\Delta\tau[-2(1-q_a^2)\Pi(q_a^2, l) + K(l)] \quad (3.77)$$

The dynamical part of this total phase may be evaluated as before,

$$\theta(\xi_{P_n}) = -\frac{1}{\hbar}\int_{\xi_0}^{\xi_{P_n}} \langle\Psi_r|H_r|\Psi_r\rangle d\xi \ . \quad (3.78)$$

The state vector at any instant being

$$|\Psi_r(\xi)\rangle = Q_a e^{i\phi_a}|e\rangle + Q_b e^{i\phi_b}|g\rangle \ , \quad (3.79)$$

we find

$$\langle\Psi_r|H_r|\Psi_r\rangle = -\frac{\hbar}{2}\Delta(2r + \cos\lambda) \quad (3.80)$$

Integrating the above expression and using the following relation

$$E(n\pi + \frac{\pi}{2}, l) = (2n+1)E(\frac{\pi}{2}, l) = (2n+1)E(l)$$

(equation (113.02) in ref. [12]) where $E(l)$ is the complete elliptic integral of the second kind, we find the dynamical phase to be

$$\theta(\xi_{P_n}) = 2n\Delta\tau[(r_+ - r_-)E(l) + (r_- + \frac{1}{2}\cos\lambda)K(l)] \tag{3.81}$$

Hence, the geometrical phase may be obtained as before, i.e.

$$\begin{aligned} \beta_n &= C_a^{(n)} - \theta(\xi_{P_n}) \\ &= -2n\Delta\tau[(1-q_a^2)\Pi(q_a^2, l) + (r_+ - r_-)E(l) + (r_- - \sin^2\frac{\lambda}{2})K(l)] \end{aligned} \tag{3.82}$$

We list the expressions for the various symbols occurring in the above equation in Appendix A in terms of the parameters $\Delta\tau$ and λ. Figure 7 shows the variation of β with λ. It is also symmetric about $\lambda = \pi$ at which value the geometric phase vanishes. This corresponds to the state vector remaining constant at all times which is the expected behaviour if all the atoms are initially in the excited state . At $\lambda = 0 \pmod{2\pi}$ the considerations of the previous sub-section hold with β varying as a smooth function of $\Delta\tau$. Between the values of $\lambda = 0$ and $\lambda = \pi$ the absolute value of β reaches a maximum at $\lambda \cong 0.76$ which corresponds to the maximum inversion that can be achieved during the course of a cyclical evolution with $\lambda \neq 0$.

Now we will examine the behavior of the Bloch vector $\vec{s}$ during the course of the cyclical evolution and relate the geometric phase to the solid angle subtended by the closed loop traversed by $\vec{s}$. The initial values of its components are

$$u(\xi_0) = \sin\lambda\cos\phi_{ba}(\xi_0) \tag{3.83}$$

$$v(\xi_0) = \sin\lambda\sin\phi_{ba}(\xi_0) \tag{3.84}$$

$$w(\xi_0) = -\cos\lambda \tag{3.85}$$

This shows that at the initial instant the Bloch vector makes an angle of λ with the negative $\hat{e}_3$ axis. The third conservation law (3.47) may be written as

$$\frac{p}{\hbar}\mathcal{E}u = 2\Delta(Q_a^2 - \sin^2\frac{\lambda}{2}) \tag{3.86}$$

This can be used to determine the component u at all times except when the electric field $\mathcal{E}$ and $(Q_a^2 - \sin^2\lambda/2)$ both vanish as indeed they do at ξ_0 . At times after the initial instant the components of $\vec{s}$ are given by

$$u = 2\Delta\sqrt{\frac{r_+}{\kappa}}\mathrm{cn}(U,l) \equiv \sin\mu\cos\nu \tag{3.87}$$

$$v = 2\sqrt{r_+(r_+ - r_-)}\mathrm{sn}(U,l)\mathrm{dn}(U,l) \equiv \sin\mu\sin\nu \tag{3.88}$$

$$w = 2r_+\mathrm{cn}^2(U,l) - \cos\lambda \equiv \cos\mu \tag{3.89}$$

where $\mathrm{sn}(U,l)$, $\mathrm{cn}(U,l)$, $\mathrm{dn}(U,l)$ are the Jacobian elliptic functions and μ, ν are the polar and azimuthal angles respectively. The explicit expressions for the angles as functions of ξ are

$$\mu(\xi) = \arccos[2r_+\mathrm{cn}^2(U,l) - \cos\lambda] \tag{3.90}$$

$$\begin{aligned}\nu(\xi) &= \phi_{ba} \\ &= \Delta\tau[(1-q_b^2)\Pi(\mathrm{am}\ U, q_b^2, l) + (1-q_a^2)\Pi(\mathrm{am}\ U, q_a^2, l)]_{K(l)}^{U} \\ &\quad -\Delta(\xi - \xi_0) + \phi_{ba}(\xi_0).\end{aligned} \tag{3.91}$$

Provided the condition (3.74) for cyclical evolution holds, the Bloch vector returns to its initial position at the instant $\xi = \xi_P$, having traced out a closed loop on the unit sphere. The solid angle Ω subtended by the closed loop at the centre of the unit sphere is found by integrating (for $n = 1$)

$$\Omega = \int_{\phi_{ba}(\xi_0)}^{\phi_{ba}(\xi_P)} d\nu \int_{\pi-\lambda}^{\mu(\nu)} \sin\mu\ d\mu\ . \tag{3.92}$$

We obtain

$$\begin{aligned}\Omega &= 4\Delta\tau[\frac{\cos^4\frac{\lambda}{2}}{\cos^2\frac{\lambda}{2} - r_+}\Pi(q_b^2, l) + \frac{\sin^4\frac{\lambda}{2}}{\sin^2\frac{\lambda}{2} + r_+}\Pi(q_a^2, l) \\ &\quad +(r_+ - r_-)E(l) + (\cos\lambda + r_-)K(l)]\ .\end{aligned} \tag{3.93}$$

Using the explicit form of the condition for cyclical evolution (3.75), this reduces to

$$\begin{aligned}\Omega &= 4\Delta\tau[(1-q_a^2)\Pi(q_a^2, l) + (r_+ - r_-)E(l) + (r_- - \sin^2\frac{\lambda}{2})K(l)] \\ &= -2\beta.\end{aligned} \tag{3.94}$$

Hence we have shown by an explicit calculation that the geometric phase is minus one–half the solid angle subtended by the closed loop traced on the Bloch sphere. All two–level systems which develop a geometric phase must in fact exhibit this relationship.

In both this and the previous sub-section we have chosen the amplitude of the electric field to be zero initially. As a consequence the slowly varying phase ϕ of the electric field did not develop in time. We point out here that the electric field will be chirped if we choose non-zero values for the initial amplitude of the electric field. Thus there exist other possible solutions for the amplitudes and phases in this two-level system but we do not consider them in this work.

4 Observing the phases

Here we consider the question of observing the phases associated with the state experimentally. The state vector for the two–level system at any time may be written as

$$|\Psi_r(\xi)\rangle = e^{i\phi_a}[Q_a(\xi)|e\rangle + e^{i\phi_{ba}}Q_b(\xi)|g\rangle] \tag{4.1}$$

The relative phase difference ϕ_{ba} between the eigenstates can be known by measuring the components (u, v) of the polarization vector, since

$$\phi_{ba} = \arctan(\frac{v}{u}).$$

However the overall phase would appear to be unobservable since it disappears in calculating the expectation value of any physical quantity. As was pointed out by Bouchiat and Gibbons [21], the overall phase after a cyclical evolution may be observable either if the state vector above represents only some of the degrees of freedom and is made to interfere with an identical system whose corresponding degrees of freedom have evolved unchanged or it may be observable by subsuming the system as part of a larger system. Indeed the first experiment (by Suter et.al. [4]) to report the observation of the Aharonov–Anandan phase opted for the second scheme by considering a two–level system as part of a three level system. They studied the NMR transitions of a spin 1 system with three levels 1,2,3 and experimentally

observed the geometric phase associated with the cyclic evolution of the subsystem 2–3 by its effect on the magnetization of 1–2. We will consider the optical analog of that experiment, the crucial difference here being that the cyclical evolution of 2–3 will be governed by its own dynamics and not externally controlled.

When considering three–level atomic systems there are three possible configurations known as the cascade, V and lambda configurations, all of which have level 2 in common. These are shown in Fig. 8. For definiteness we will consider only the phases arising in the cascade configuration by the propagation of sech pulses in what follows. The other two configurations and the phases due to continuous pulse trains may also be studied but we do not consider them here.

The basic idea,as enunciated for example by Stoll, Wolff and Mehring [22] is as follows. Assuming that our three level system starts in the ground state $|1\rangle$, we prepare a coherent superposition of states $|1\rangle$ and $|2\rangle$ by applying a $\pi/2$ pulse resonant with the 1–2 transition. Now if a pulse resonant with the 2–3 transition (we have in mind an SIT pulse) propagates through the system, it will create a superposition of states $|2\rangle$ and $|3\rangle$ causing the population and phase of $|2\rangle$ to change relative to that of $|1\rangle$. Measuring the 1–2 transverse polarization (or equivalently the off-diagonal elements in the 1–2 block of the 3x3 density matrix) will then give us information about the phase change induced in the 2–3 subsystem by the propagating soliton pulse. Since we are considering the case of sharp line self–induced transparency, all spectral lines are assumed to be only homogeneously broadened. Consequently the decoherence of the 1–2 pulse due to inhomogeneous broadening may be neglected. With line widths around 10 MHz, the decay time for the 1–2 pulse would be around 30 nanoseconds. Typical pulse lengths for SIT pulses are $\sim$ 5 nanoseconds. Feeding the 1–2 pulse into a phase–sensitive detector would allow us to extract the phase of state $|2\rangle$ relative to $|1\rangle$. Following the passage of the SIT pulse, we inject another $\pi/2$ pulse to the 1–2 transition. This later pulse creates an echo pulse at a later time, this echo pulse is then measured with a phase sensitive detector to extract the total phase change undergone by state $|2\rangle$. The sequence of pulses is shown in Figure 9.

With the initial population all in the lowest energy state, the density matrix at time $t = 0$ is

$$\rho(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

We now inject a pulse of constant amplitude with area $\pi/2$ and resonant with the 1–2 transition. This pulse equalizes the populations of levels 1 and 2 and induces a coherence between them. After the passage of this pulse, the state vector at time t_1 is

$$|\Psi\rangle(t_1) = \frac{1}{\sqrt{2}}[e^{i\phi_1(t_1)}|1\rangle + |2\rangle] \tag{4.3}$$

to which corresponds the density matrix

$$\rho(t_1) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & e^{-i\phi_1(t_1)} \\ 0 & e^{i\phi_1(t_1)} & 1 \end{bmatrix} \tag{4.4}$$

Next this three–level system is subjected to a propagating pulse nearly resonant with the 2–3 transition. Assuming as in the previous section that this pulse is circularly polarized, the electric field with carrier frequency ω can be written as

$$\vec{E} = \mathcal{E}(z,t)[\hat{x}\cos\Phi + \hat{y}\sin\Phi] \tag{4.5}$$

where $\mathcal{E}$ and Φ are the amplitude and phase respectively of the field. The Hamiltonian in the presence of this field is

$$H = \begin{bmatrix} E_3 & -p_2\mathcal{E}e^{-i\Phi} & 0 \\ -p_2\mathcal{E}e^{i\Phi} & E_2 & 0 \\ 0 & 0 & E_1 \end{bmatrix} \tag{4.6}$$

where E_1, E_2,and E_3 are the energies of levels 1,2 and 3 respectively in the absence of the field and p_2 is the dipole moment between states $|2\rangle$ and $|3\rangle$ induced by the field. The resonance frequencies associated with the transitions 1–2 and 2–3 are

$$\omega_1 = \frac{E_2 - E_1}{\hbar} \quad ; \quad \omega_2 = \frac{E_3 - E_2}{\hbar}$$

respectively. We transform to a frame rotating with the field by using the unitary matrix

$$U(t) = \exp\{i \begin{bmatrix} (\frac{E_2 t}{\hbar} + \Phi) & 0 & 0 \\ 0 & \frac{E_2 t}{\hbar} & 0 \\ 0 & 0 & \frac{E_1 t}{\hbar} \end{bmatrix}\} \tag{4.7}$$

The transformed state vector is

$$|\Psi\rangle_{tr} = U(t)|\Psi\rangle \tag{4.8}$$

while the transformed Hamiltonian is

$$H_{tr} = \begin{bmatrix} \hbar\Delta_2 & -p_2\mathcal{E} & 0 \\ -p_2\mathcal{E} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.9}$$

where we have defined the detuning parameter

$$\Delta_2 = \omega_2 - \omega \, .$$

Our aim is to obtain the density matrix after the passage of the pulse. We are assuming that the system has been prepared in a pure state. In this case it turns out to be easier to calculate the density matrix from the state vector rather than solve the Liouville equation of motion for the density matrix. In this frame Schrödinger's equation for the state vector

$$|\Psi\rangle_{tr}(t) = C_1(t)|1\rangle + C_2(t)|2\rangle + C_3(t)|3\rangle \tag{4.10}$$

where C_1, C_2 and C_3 are complex coefficients, reduces to the system of equations

$$i\hbar\frac{\partial C_1}{\partial t} = 0 \tag{4.11}$$

$$i\hbar\frac{\partial C_2}{\partial t} = -p_2\mathcal{E}C_3 \tag{4.12}$$

$$i\hbar\frac{\partial C_3}{\partial t} = -p_2\mathcal{E}C_2 + \hbar\Delta_2 C_3 \tag{4.13}$$

The evolution of the electric field is given by Maxwell's equations. The dynamics of the 2–3 transition is independent of level 1 and can be treated exactly as in the previous section. Hence the state vector at any time may be written as

$$|\Psi\rangle_{tr}(t_1 + t) = C_1(t_1)|1\rangle + Q_2 e^{i\phi_2}|2\rangle + Q_3 e^{i\phi_3}|3\rangle \tag{4.14}$$

where Q_2, ϕ_2, Q_3, ϕ_3 are obtained by the same procedure as in the previous section. Corresponding to this solution, the density matrix is

$$\rho_{tr}(t_1+t) = \begin{bmatrix} Q_3^2 & Q_2Q_3e^{i(\phi_3-\phi_2)} & C_1^*Q_3e^{i\phi_3} \\ Q_2Q_3e^{-i(\phi_3-\phi_2)} & Q_2^2 & C_1^*Q_2e^{i\phi_2} \\ C_1Q_3e^{-i\phi_3} & C_1Q_2e^{-i\phi_2} & |C_1|^2 \end{bmatrix} \tag{4.15}$$

After a time T, following the passage of the soliton pulse, the levels 2 and 3 go through a cyclical evolution and the amplitudes Q_2, Q_3 return to their initial values at time $t = t_1$, i.e at time $t = t_1 + T$

$$Q_2(t_1+T) = \frac{1}{\sqrt{2}}\ ,\ Q_3(t_1+T) = 0$$

The state vector in the transformed frame is

$$|\Psi\rangle_{tr}(t_1+T) = \frac{1}{\sqrt{2}}[e^{i\phi_1(t_1)}|1\rangle + e^{i\phi_2(t_1+T)}|2\rangle] \tag{4.16}$$

where $\phi_2(t_1+T)$ is the total phase change undergone by state $|2\rangle$. Correspondingly the density matrix is

$$\rho_{tr}(t_1+T) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & e^{i[\phi_2(t_1+T)-\phi_1(t_1)]} \\ 0 & e^{-i[\phi_2(t_1+T)-\phi_1(t_1)]} & 1 \end{bmatrix} \tag{4.17}$$

The components of the transverse polarization of the two–level subsystem 1–2 may be obtained from the elements of the density matrix as

$$\langle P_x\rangle^{1-2} = c(\rho_{23}+\rho_{32}), \qquad \langle P_y\rangle^{1-2} = ic(\rho_{32}-\rho_{23}) \tag{4.18}$$

where c is a constant. From (4.18) it follows that

$$\langle P_x\rangle^{1-2} = c\cos(\phi_2(t_1+T)-\phi_1(t_1)) \tag{4.19}$$

$$\langle P_y\rangle^{1-2} = c\sin(\phi_2(t_1+T)-\phi_1(t_1)) \tag{4.20}$$

In principle, by measuring the free induction decay of the 1–2 levels following the passage of the soliton pulse, the total phase change $\phi_2(t_1+T)-\phi_1(t_1)$ of state $|2\rangle$ with respect to the

arbitrary initial phase of $|1\rangle$ can be obtained. In practice however, homogeneous broadening effects will cause a loss of the 1–2 polarization during the propagation of the 2–3 pulse. To counteract this, another $\pi/2$ pulse may applied to the 1–2 transition to equalize their populations and thus enhance the signal. This pulse is represented by the field

$$\vec{E}(t_2 + t) = 2\mathcal{E}_0 \cos(\omega_1 t + \alpha)\hat{x} \tag{4.21}$$

where $\mathcal{E}_0$ is a constant amplitude field which lasts for an interval τ and we have added a phase shift of α to the field. $t_2 \geq t_1 + T$ is the instant at which this pulse is applied. Applying the unitary transformation effected by $U(t)$ (given by equation (4.7)) transforms the Hamiltonian to

$$H_{tr}(t_2 + t) = \begin{bmatrix} \hbar\Delta_2 & 0 & 0 \\ 0 & 0 & p_1\mathcal{E}_0 e^{-i\alpha} \\ 0 & p_1\mathcal{E}_0 e^{i\alpha} & 0 \end{bmatrix} \tag{4.22}$$

where p_1 is the dipole matrix element between levels 1 and 2. Solving Schrödinger's equation for the state vector leads to the following density matrix at the end of the pulse

$$\rho_{tr}(t_2+\tau) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\sin(\phi_2(t_1+T)-\phi_1(t_1)+\alpha) & e^{-i\alpha}\cos(\phi_2(t_1+T)-\phi_1(t_1)+\alpha) \\ 0 & e^{i\alpha}\cos(\phi_2(t_1+T)-\phi_1(t_1)+\alpha) & 1-\sin(\phi_2(t_1+T)-\phi_1(t_1)+\alpha) \end{bmatrix} \tag{4.23}$$

We find then that the echo signal following the pulse has the following polarization components

$$\langle P_x\rangle^{1-2}(t_2 + \tau) = c\cos(\phi_2(t_1 + T) - \phi_1(t_1) + \alpha)\cos\alpha \tag{4.24}$$

$$\langle P_y\rangle^{1-2}(t_2 + \tau) = c\sin(\phi_2(t_1 + T) - \phi_1(t_1) + \alpha)\sin\alpha \tag{4.25}$$

In particular with $\alpha = \pi/2$ the 1–2 polarization pulse is along the y direction. This pulse contains the information about the phase change $\phi_2(t_1 + T) - \phi_1(t_1)$ undergone by level 2 during the passage of the *sech* pulse.

5 Summary

The phenomenon of self–induced transparency is ideally suited for studying various aspects of geometric phases. In this paper we have focussed on the abelian quantum geometric phase developed in two-level atoms as a consequence of their interaction with a classical electric field which is shaped by the atoms while propagating through the optical medium. This system is somewhat distinct from other systems considered in the literature on geometric phases in that the entire dynamics is self consistently determined and not driven by the variation of external parameters. By rewriting the equations of motion in terms of the amplitudes and phases (instead of the usual Bloch form) we are able to construct three conservation laws which are generally valid under the usual assumptions of SIT. These conservation laws may be useful in obtaining solutions with inhomogeneous broadening present but here we have considered only the simplest case — sharp line SIT. The well known *sech* pulse solutions of McCall and Hahn take the ground state through a cyclical evolution. As a consequence the ground state acquires a geometric phase depending on the detuning and the pulse width. Using the relation between the geometric phase and the solid angle subtended on the Bloch sphere we are able to relate the geometric phase to the maximum value of the atomic inversion. If the initial state is partially excited out of the ground state, continuous pulse trains are obtained as the analytical solutions for the propagating electric field. Here cyclical evolution must be brought about by choosing the parameters to take values lying on a particular curve in the parameter space. In such cases too the geometric phase is equal to half the solid angle subtended by the closed loop traced by the Bloch vector on the Bloch sphere. We outlined an experimental procedure involving three levels of the atom whereby the total phase change undergone by the state vector for a two-level sub-system can be measured. In the particular case of the *sech* electromagnetic pulse exactly on resonance with the atoms the total phase change is entirely geometric.

We mention two ways of extending the present work. The first, analysis of the case when the electromagnetic field is also quantized, is in progress and will be the subject of a future publication. The second refers to the possibility of associating geometric phases with non-

cyclic paths – as done for example by Samuel and Bhandari [23]. However, in this direction, we have shown that even for open paths the time integral of the energy uncertainty in the state is a geometric quantity related to the distance along the path on the Bloch sphere, providing independent confirmation of the result proved in [16].

Acknowledgement

This work was begun while the authors were at the Dept. of Applied Physics, Columbia University and we thank Professors A. Bhattacharjee and M. Tabor for their generous hospitality. One of us (T.S) thanks Theodore Garavaglia for his comments on the manuscript. The work of T.S was supported in part by the U.S. Department of Energy under contract No. DE-AC02-89ER40486.

Appendix A

Here we list the symbols appearing in the expression for the geometric phase (3.82) in terms of $\Delta\tau$ and λ where

$$\Delta = \omega_0 - \omega$$

is the detuning, τ–a measure of the pulse width, expressed in terms of the other variables is

$$\tau = \frac{1}{\sqrt{\kappa(r_+ - r_-)}},$$

and λ is the initial angle made by the Bloch vector with the negative $\hat{e}_3$ axis. The various quantities are

$$\begin{aligned}
\kappa &= \frac{1}{\tau^2}[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda] \\
q_a^2 &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + 2(\Delta\tau)^2\sin^2\lambda/2 + 1} \\
q_b^2 &= -\frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + 2(\Delta\tau)^2\cos^2\lambda/2 - 1} \\
l &= \frac{1}{\sqrt{2}}\sqrt{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1} \\
r_+ &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{2[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda]} \\
r_- &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda - 1}{2[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda]}
\end{aligned} \tag{A.1}$$

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Figure Captions

Fig. 1. Variation of the geometric phase β with $\Delta\tau_p$ for the *sech* pulse. $\beta(0) = \pi$.

Fig. 2. Variation of the geometric phase β with w_{max}, the maximum value of the atomic inversion, for the *sech* pulse.

Fig. 3. The circumference C of the closed loop as a function of $\Delta\tau_p$ for the *sech* pulse. $C(0) = 2\pi$.

Fig. 4. Plot of the cubic polynomial $\chi(r)$, given by equation (3.57), as a function of r. $\chi(r) = 0$ at $r = r_-,\ 0,\ r_+$.

Fig. 5. Solution of the condition (3.75). This curve picks the value of $\Delta\tau$ for a given λ in order that the evolution be cyclic.

Fig. 6. Variation of the relative phase change $\phi_{ba}(\xi) - \phi_{ba}(\xi_0)$,given by equation (3.76), with the dimensionless "time" variable $\frac{\xi-\xi_0}{\tau}$ for $\lambda = \pi/10$.

Fig. 7. Variation of the geometric phase with λ, the initial angle made by the Bloch vector with the negative $\hat{e}_3$ axis. β vanishes at $\lambda = \pi$.

Fig. 8. Possible configurations of optical transitions in a three level atom.

Fig. 9. Sequence of pulses applied to the 1-2 and 2-3 transitions in a cascade configuration. Time increases along the horizontal axis to the right.

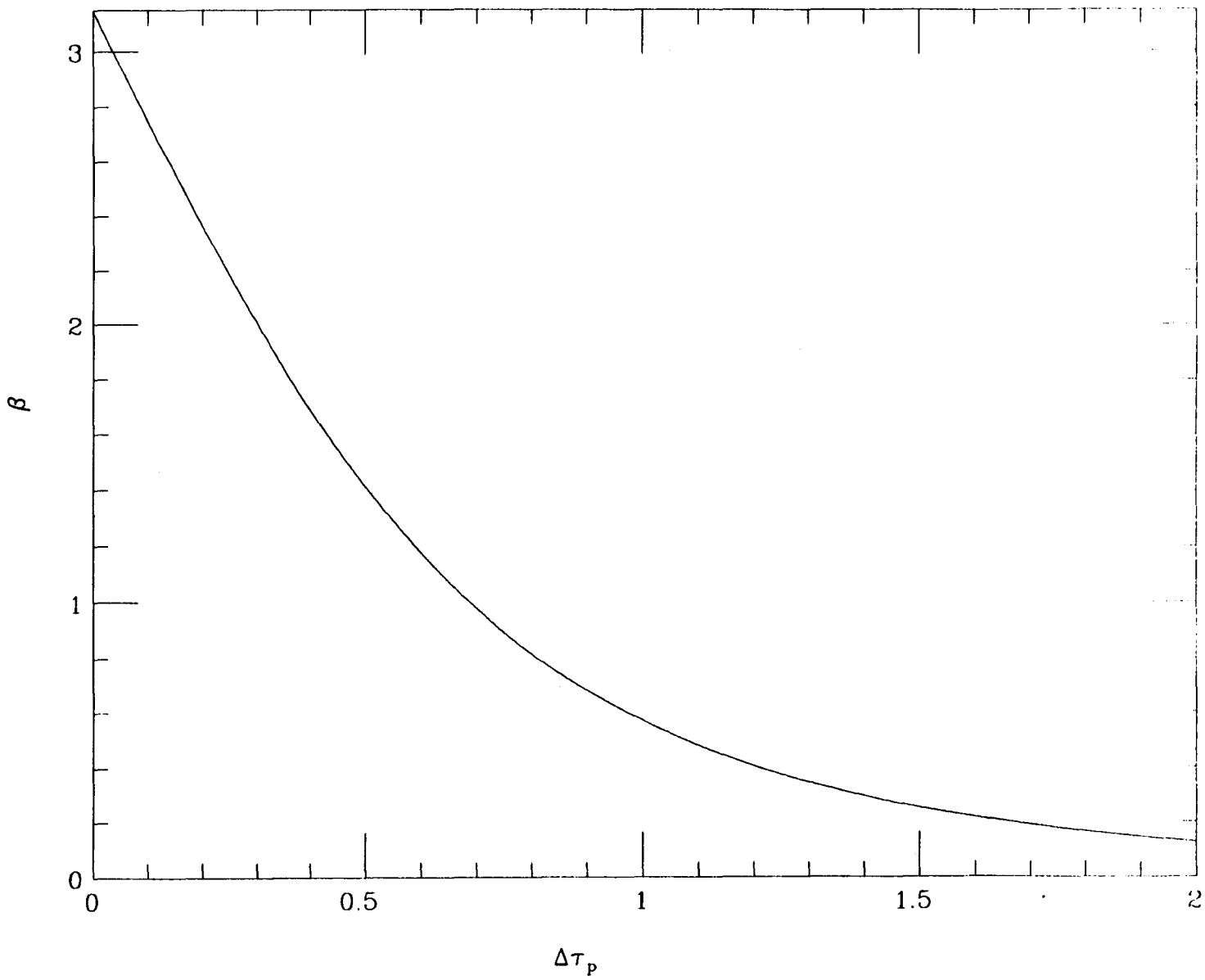

Fig. 1.

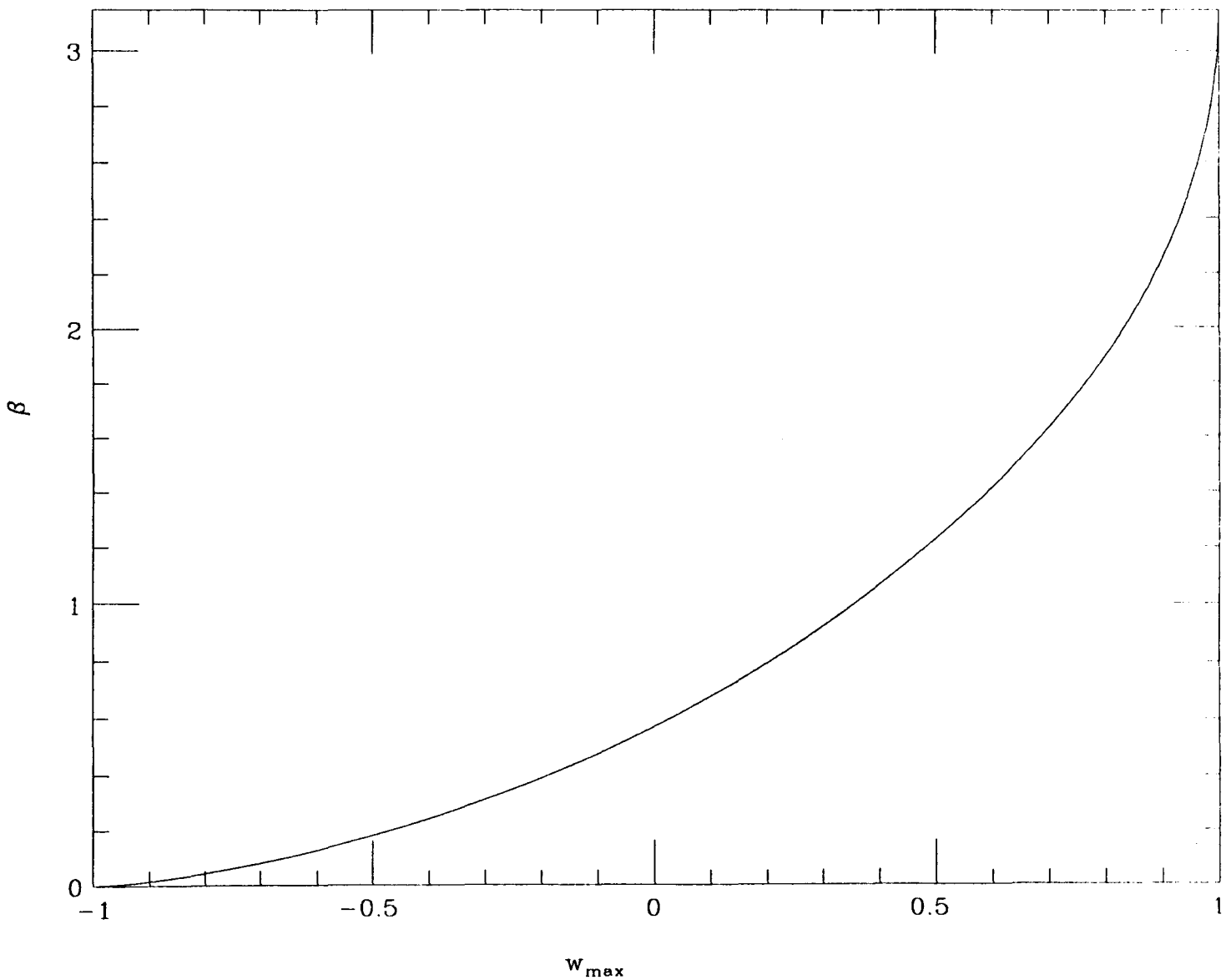

Fig. 2.

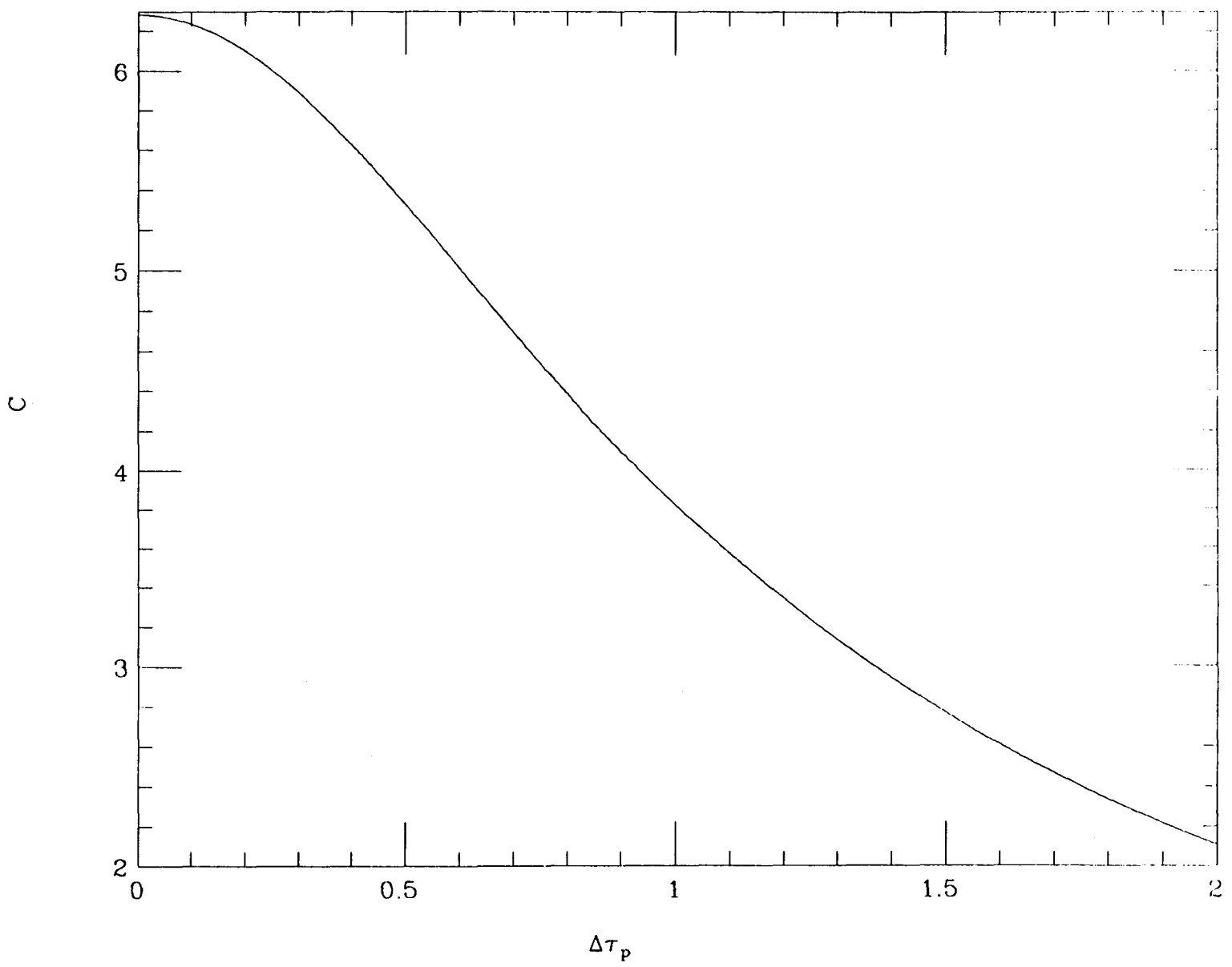

Fig. 3.

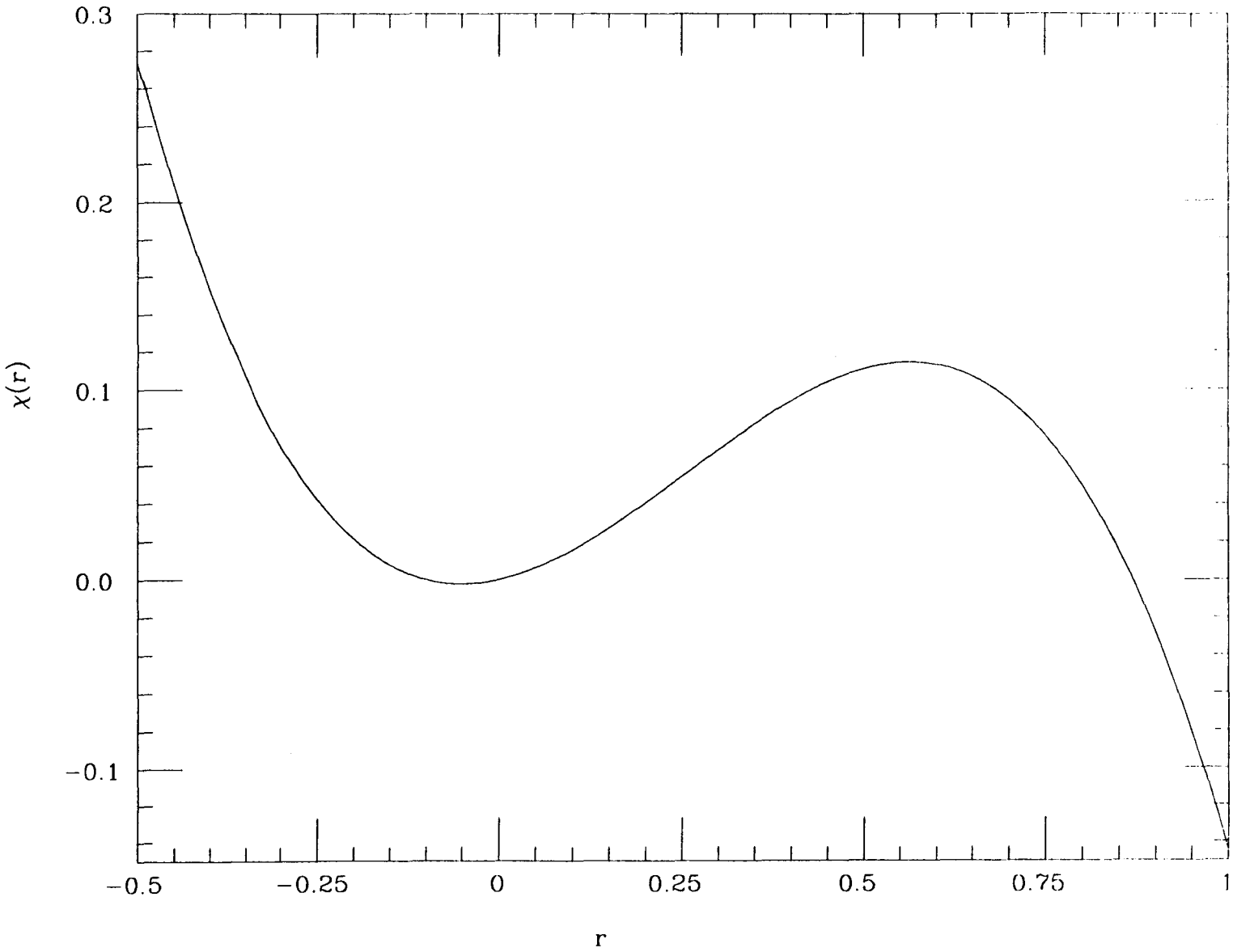

Fig. 4.

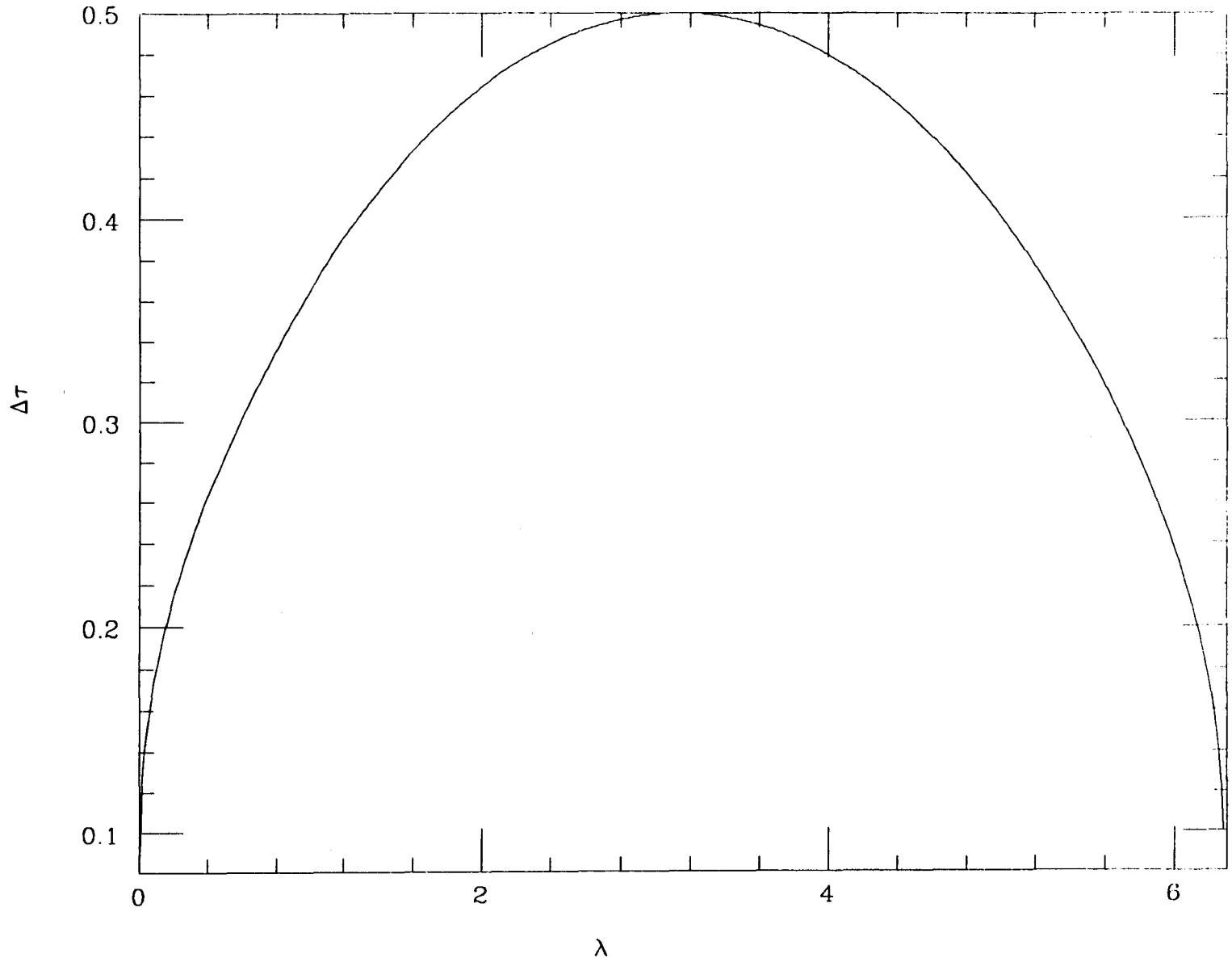

Fig. 5.

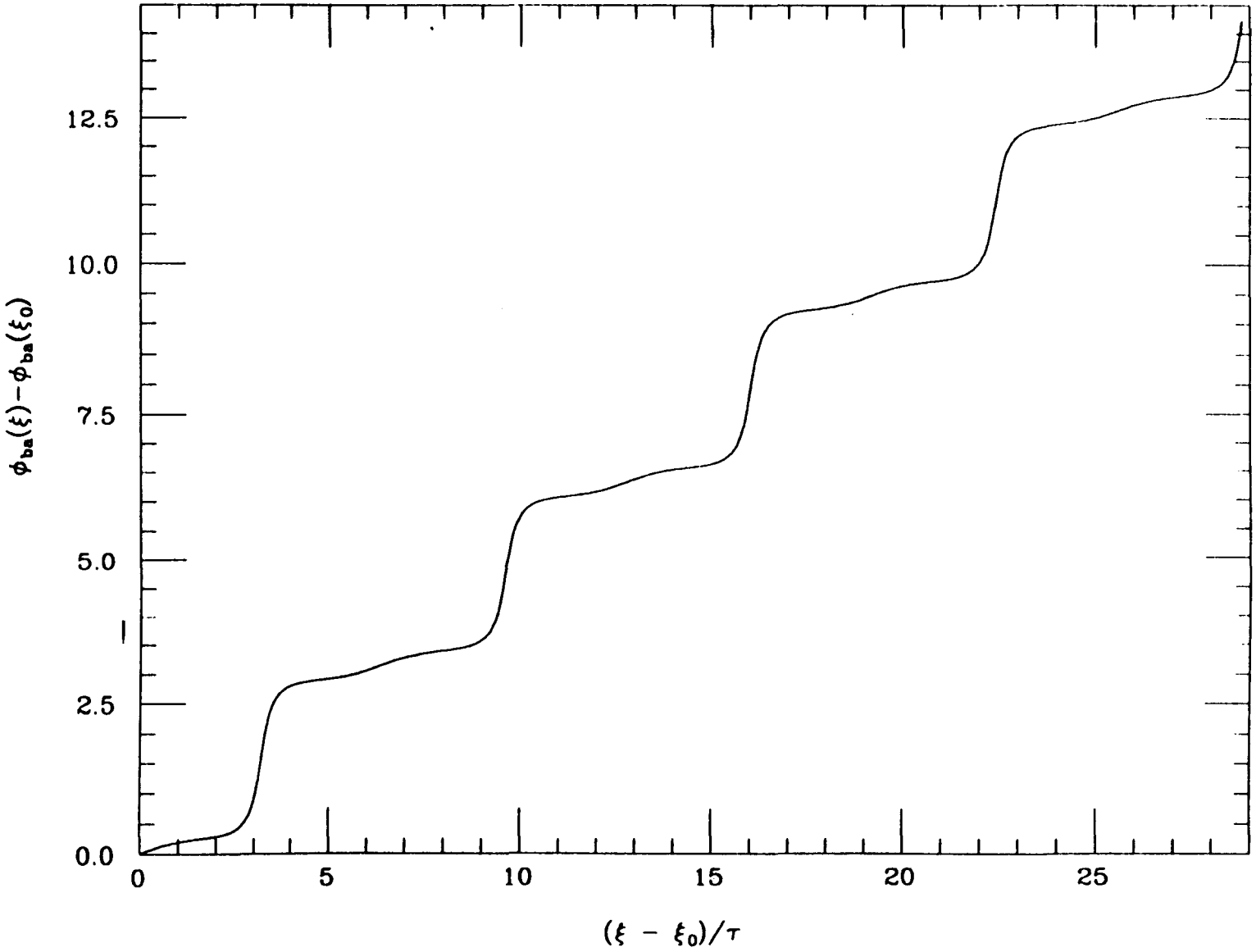

Fig. 6

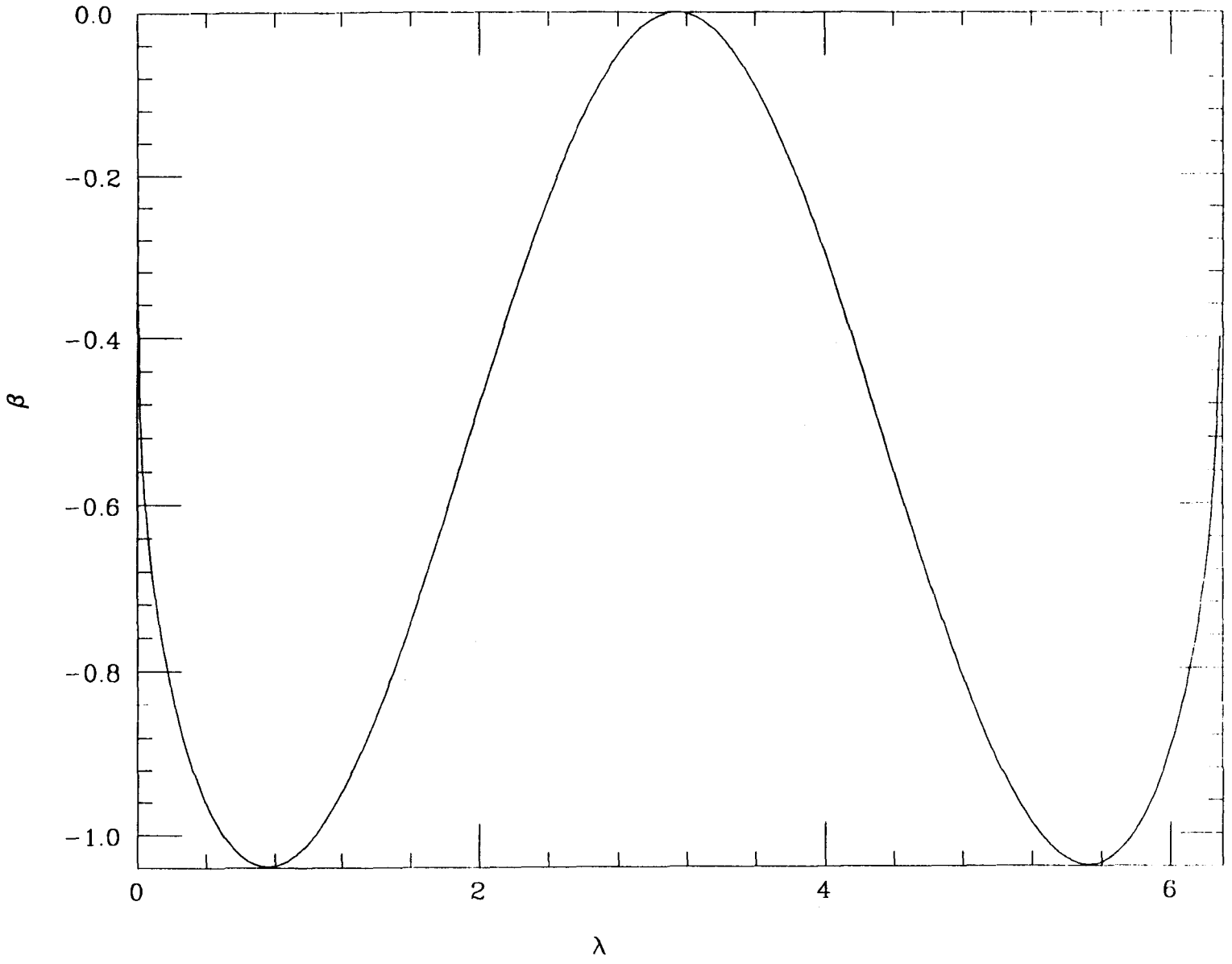

Fig. 7

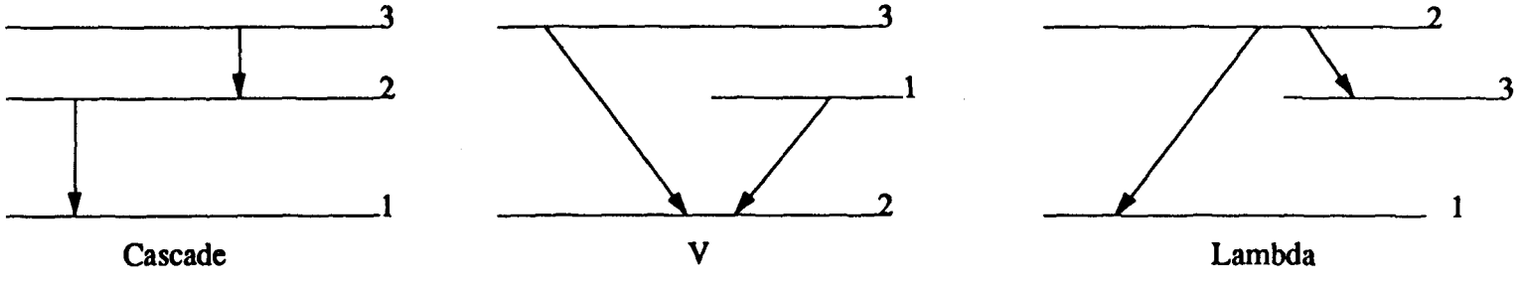

Fig. 8

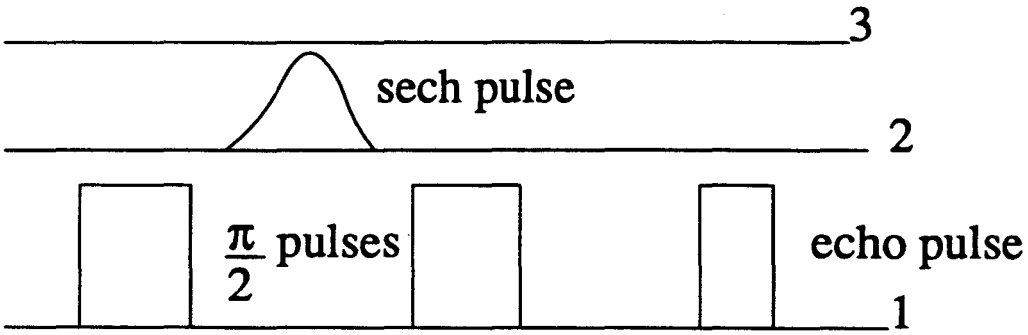

Fig. 9