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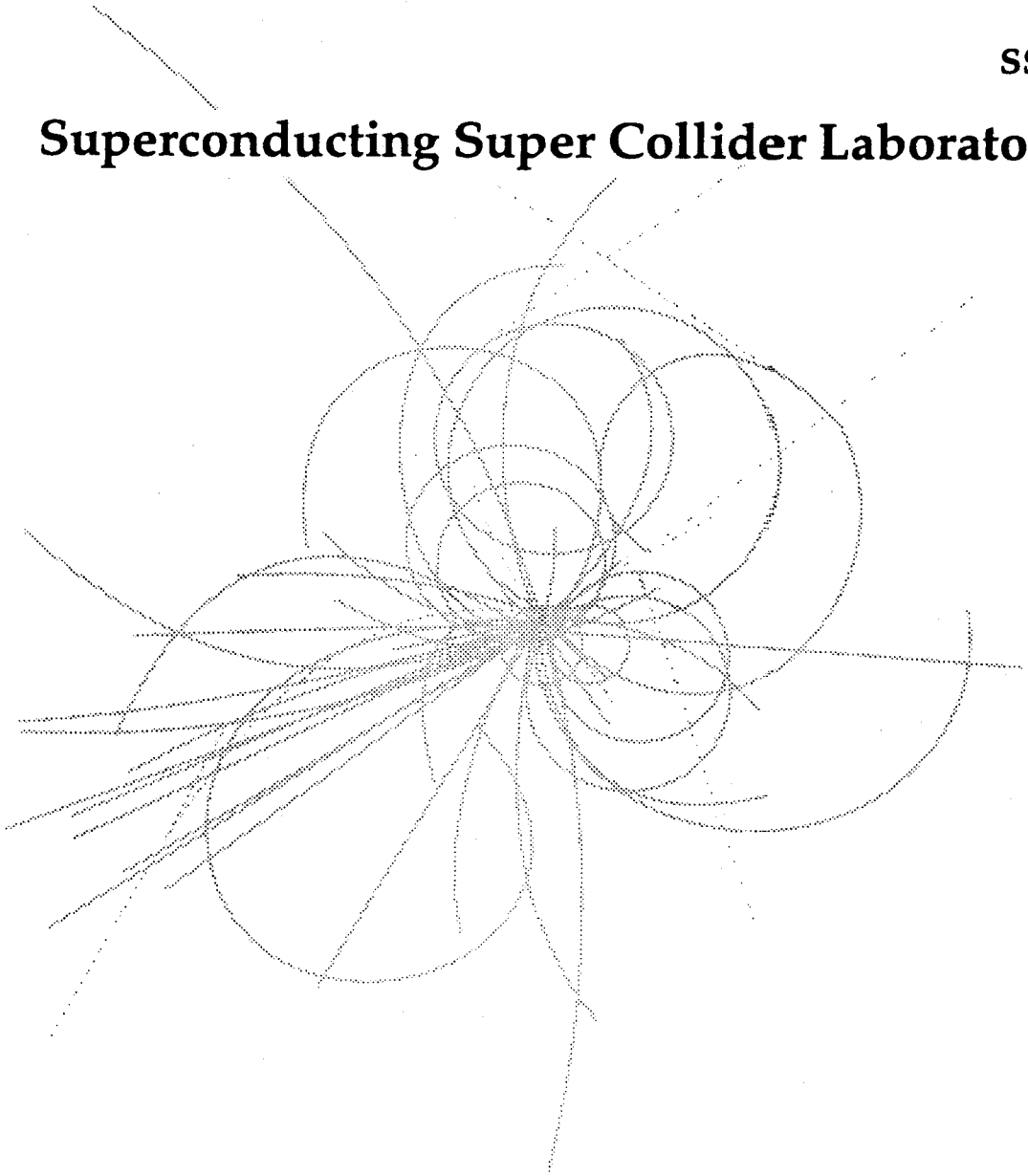

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NORMALIZATION OF THE PARAMETERIZED COURANT-SNYDER MATRIX FOR SYMPLECTIC FACTORIZATION OF A PARAMETERIZED TAYLOR MAP

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Abstract

The transverse motion of charged particles in a circular accelerator can be well represented by a one-turn high-order Taylor map. [1] For particles without energy deviation, the one-turn Taylor map is a 4-dimensional polynomials of four variables. The four variables are the transverse canonical coordinates and their conjugate momenta. To include the energy deviation (off-momentum) effects, the map has to be parameterized with a smallness factor representing the off-momentum and so the Taylor map becomes a 4-dimensional polynomials of five variables. It is for this type of parameterized Taylor map that a mehtod is presented for converting it into a parameterized Dragt-Finn facctorization map. Parameterized nonlinear normal form and parameterized kick factorization can thus be obtained with suitable modification of the existing technique.

I. Introduction

A one-turn Taylor map parameterized for off-momentum transverse propagation relative to the dispersed closed orbit in a circular alternating-gradient synchrotron [2] can be given in the form

$$\begin{aligned}\vec{x}' &= m(\vec{x},\delta):\vec{x}\\ &= M(\delta)\vec{x}+\sum_{i=0}^{n_2}\vec{U}_{2,i}(\vec{x})\delta^i+\sum_{i=0}^{n_3}\vec{U}_{3,i}(\vec{x})\delta^i+\cdots,\end{aligned}\tag{1}$$

where the transverse canonical phase space coordinates, $\vec{x}=[x,p_x,y,p_y]$ are deviations relative to the dispersed closed orbit; $\delta=\Delta P/P_0$ is a parameter for the off-momentum, which is usually a smallness factor in the domain of physical interest. Terms in each of the polynomials $\vec{U}_{k,i}(\vec{x})$ for $k=2,3,\cdots$ and $i=0,1,\cdots,n_k$ are all of the order k of $\vec{x}$. The parameterized Courant-Snyder matrix, $M(\delta)$, is truncated at an order of δ^n and given by

$$M(\delta)=M_0+\delta M_1+\delta^2M_2+\cdots+\delta^nM_n,\tag{2}$$

where $M_0,M_1,\cdots,M_n$ are all 4×4 and usually coupled matrices. Note that $n\geq n_2\geq n_3\geq\cdots\geq n_k\geq\cdots$ and that the dispersed closed orbit can be expressed as

$$\vec{x}_c=\vec{x}_{c,0}+\delta\vec{x}_{c,1}+\delta^2\vec{x}_{c,2}+\cdots,\tag{3}$$

which is a polynomial of δ with an order of n or higher.

If the Taylor map is extracted from a symplectic tracking program, then it is symplectic when it is expanded to the infinite order of $\vec{x}$ and δ. We assume the Taylor map given by Eq. 1 is such a type of Taylor map truncated at a finite order of δ and $\vec{x}$. Such a truncated Taylor map, although not symplectic because of the missing of the high-order effects, can be order-by-order converted into a Dragt-Finn factorization map. [3] However, the addition of the parameter δ makes such a factorization process trickier. In this paper, the author presents a method of factorizing such a perameterized Taylor map given by Eq. 1. Only a single parameter, δ, is considered. though extension to a multi-parameter case is analogous.

II. Normalization Of $M(\delta)$

The key to the Dragt-Finn factorization of a parameterized Taylor map given by Eq. 1 is the normalization of its associated parameterized Courant-Snyder matrix $M(\delta)$ given by Eq. 2. A method [4] has been presented recently to normalize $M(\delta)$ via order-by-order symplectic factorization such that

$$R(\delta)=A^{-1}(\delta)M(\delta)A(\delta)+\sigma(\delta^{n+1}),$$

where the normalized parameterized rotation $R(\delta)$ and the canonical generation matrix $A(\delta)$ are given as follows.

$$R(\delta)=\begin{pmatrix}\cos\mu_1(\delta) & \sin\mu_1(\delta) & 0 & 0\\ -\sin\mu_1(\delta) & \cos\mu_1(\delta) & 0 & 0\\ 0 & 0 & \cos\mu_2(\delta) & \sin\mu_2(\delta)\\ 0 & 0 & -\sin\mu_2(\delta) & \cos\mu_2(\delta)\end{pmatrix},$$

where

$$\mu_\alpha(\delta)=\mu_{\alpha,0}+\delta\mu_{\alpha,1}+\delta^2\mu_{\alpha,2}+\cdots+\delta^n\mu_{\alpha,n},$$

for $\alpha=1,2$. The inverse of $R(\delta)$ is also a decoupled rotation given by

$$R^{-1}(\delta)=\begin{pmatrix}\cos\mu_1(\delta) & -\sin\mu_1(\delta) & 0 & 0\\ \sin\mu_1(\delta) & \cos\mu_1(\delta) & 0 & 0\\ 0 & 0 & \cos\mu_2(\delta) & -\sin\mu_2(\delta)\\ 0 & 0 & \sin\mu_2(\delta) & \cos\mu_2(\delta)\end{pmatrix}.$$

The symplectic matrix $A(\delta)$ is of order $\delta^{n(n+1)/2}$ which is a concatenation of a series order-by-order canonical generation matrices given by

$$A(\delta)=A_0(I+\delta^1A_1)(I+\delta^2A_2)\cdots(I+\delta^nA_n).$$

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Its inverse is given by

$$A^{-1}(\delta) = (I - \delta^n A_n)\cdots(I - \delta^2 A_2)(I - \delta A_1)A_0^{-1}.$$

Note that A_0 and its inverse A_0^{-1} are the familiar generation matrices for normalization of M_0 such that [5]

$$\begin{aligned} R_0 &= A_0^{-1} M_0 A_0 \\ &= \begin{pmatrix} \cos\mu_{1,0} & \sin\mu_{1,0} & 0 & 0 \\ -\sin\mu_{1,0} & \cos\mu_{1,0} & 0 & 0 \\ 0 & 0 & \cos\mu_{2,0} & \sin\mu_{2,0} \\ 0 & 0 & -\sin\mu_{2,0} & \cos\mu_{2,0} \end{pmatrix}. \end{aligned}$$

For the detailed solution of $\mu_{\alpha,i}$ and A_i for $i = 1, 2, \cdots, n$ and $\alpha = 1, 2$, please refer to Reference [4].

III. Dragt-Finn Factorization

For Dragt-Finn factorization of the parameterized Taylor map given by Eq. 1, one first makes a similarity transformation on the map such that

$$\begin{aligned} {}_1m(\vec{x},\delta):\vec{x} &= \mathcal{A}(\vec{x},\delta)m(\vec{x},\delta)\mathcal{A}^{-1}(\vec{x},\delta):\vec{x} \\ &= R(\delta)\vec{x} + \sum_{i=0}^{n_2} {}_1\vec{U}_{2,i}(\vec{x})\delta^i + \sum_{i=0}^{n_3} {}_1\vec{U}_{3,i}(\vec{x})\delta^i + \cdots, \end{aligned}$$

where $\mathcal{A}(\vec{x},\delta)$ is the global form of $A(\delta)$. One then make a concatenation of the transformed map ${}_1m(\vec{x},\delta)$ with $\mathcal{R}^{-1}(\vec{x},\delta)$ such that

$$\begin{aligned} {}_2m(\vec{x},\delta):\vec{x} &= \mathcal{R}^{-1}(\vec{x},\delta){}_1m(\vec{x},\delta):\vec{x} \\ &= \vec{x} + \sum_{i=0}^{n_2} {}_2\vec{U}_{2,i}(\vec{x})\delta^i + \sum_{i=0}^{n_3} {}_2\vec{U}_{3,i}(\vec{x})\delta^i + \cdots, \end{aligned} \quad (4)$$

where $\mathcal{R}^{-1}(\vec{x},\delta)$ is the global form of the inverse of the rotation matrix $R(\delta)$. Due to symplecticity of the Taylor map (when it is expanded to the infite order), for each of the ${}_2\vec{U}_{2,i}(\vec{x})$ for $i = 0, 1, \cdots, n_2$ there exists a third-order (every term is third order) polynomial of $\vec{x}$, $f_{3,i}(\vec{x})$ such that

$$[f_{3,i}(\vec{x}), \vec{x}] = {}_2\vec{U}_{2,i}(\vec{x}). \quad (5)$$

Therefore, from Eq. 4, one obtains

$${}_2m(\vec{x},\delta):\vec{x} = exp(:\sum_{i=0}^{n_2} f_{3,i}(\vec{x})\delta^i:)\vec{x} + \sum_{i=0}^{n_3} {}_2\vec{U'}_{3,i}(\vec{x})\delta^i + \cdots,$$

where

$${}_2\vec{U'}_{3,i}(\vec{x}) = {}_2\vec{U}_{3,i}(\vec{x}) - \frac{1}{2}[f_{3,i}(\vec{x}), [f_{3,i}(\vec{x}), \vec{x}]]$$

Concatenating ${}_2m(\vec{x},\delta)$ with

$$exp(-:\sum_{i=0}^{n_2} f_{3,i}(\vec{x})\delta^i:),$$

one obtains

$$\begin{aligned} {}_3m(\vec{x},\delta):\vec{x} &= exp(-:\sum_{i=0}^{n_2} f_{3,i}(\vec{x})\delta^i:){}_2m(\vec{x},\delta):\vec{x} \\ &= \vec{x} + \sum_{i=0}^{n_3} {}_3\vec{U}_{3,i}(\vec{x})\delta^i + \sum_{i=0}^{n_4} {}_3\vec{U}_{4,i}(\vec{x})\delta^i + \cdots. \end{aligned}$$

Again, due to symplecticity, for each of the ${}_3\vec{U}_{3,i}(\vec{x})$ for $i = 0, 1, \cdots, n_3$, there exists a fourth-order (every term is fourth order) polynomial of $\vec{x}$, $f_{4,i}(\vec{x})$ such that

$$[f_{4,i}(\vec{x}), \vec{x}] = {}_3\vec{U}_{3,i}(\vec{x}). \quad (6)$$

One thus obtains the Dragt-Finn factorization of the fourth order given by

$$exp(:\sum_{i=0}^{n_3} f_{4,i}(\vec{x})\delta^i:).$$

Following the similar steps given above, one can proceed further to obtain the Dragt-Finn factorization of the higher orders given by

$$exp(:\sum_{i=0}^{n_{k-1}} f_{k,i}(\vec{x})\delta^i:),$$

for $k = 5, 6, \cdots$. The parameterized Taylor map given by Eq. 1 can then be represented by a symplectic map given in global form by

$$\begin{aligned} \vec{x'} &= m(\vec{x},\delta):\vec{x} \\ &= \mathcal{A}^{-1}(\vec{x},\delta)m_f(\vec{x},\delta)\mathcal{A}(\vec{x},\delta):\vec{x}, \end{aligned} \quad (7)$$

where

$$m_f(\vec{x},\delta) = \mathcal{R}(\vec{x},\delta)exp(:\sum_{i=0}^{n_2} f_{3,i}(\vec{x})\delta^i:)exp(:\sum_{i=0}^{n_3} f_{4,i}(\vec{x})\delta^i:)\cdots.$$

IV. Discussions

At first glance, one may wonder why Eq. 5 (or Eq. 6), which is one of the important steps for leading to the parameterized Dragt-Finn factorization of a dispersed-closed-orbit Taylor map, is true. Let us assume that the Taylor expansions are all infinite orders of δ, that is, $n = n_2 = n_3 = \cdots = \infty$. If a suitable small factor is substituted for δ into Eq. 1, then one obtains the associated regular closed-orbit non-parametrized Taylor map which is a familiar type of map for Dragt-Finn factorization. Now if we first obtain the linear parameterized normalized rotation $R(\delta)$ and its associated parametrized canonical generation matrix $A(\delta)$ up to the infinite order of δ, and then substitue the same small factor for δ, we would have obtained the same linear non-parameterized normalized rotation R and its associated non-parametrized canonical generation matrix A as those obtained for the associated

non-parameterized Dragt-Finn factorization. Therefore, it would make no differnece whether we first obtain Eq. 4 to infinite order of δ and then substitue the small factor for δ or we first substitue the small factor for δ in Eq. 1 to obtain a non-parameterized closed-orbit Taylor map and then obtain the associated non-parameterized equivalence of Eq. 4. Thus, due to symplecticity, there must exist a third-order (every term is third order) polynomial of $\vec{x}$, $f(\vec{x},\delta)$, such that

$$[f(\vec{x},\delta),\vec{x}] = \sum_{i=0}^{\infty} {}_2\vec{U}_{2,i}(\vec{x})\delta^i. \tag{8}$$

Since $f(\vec{x},\delta)$ can always be Taylor expanded as

$$f(\vec{x},\delta) = \sum_{i=0}^{\infty} f_i(\vec{x})\delta^i, \tag{9}$$

by comparing Eq. 8 and Eq. 9, one can conclude that

$$[f_i(\vec{x}),\vec{x}] = {}_2\vec{U}_{2,i}(\vec{x})$$

for each $i = 0, 1, 2, \cdots, \infty$. Furthermore, since terms with a certain order of δ can only be contributed through concatenation from terms with lower or equal orders of δ, truncation of higher-order terms will not change the outcome for the lower-order Lie operators. Therefore, $f_{3,i}(\vec{x}) = f_i(\vec{x})$ for $i = 0, 1, \cdots, n_2$.

In practice, differential algebra [6] can be used to obtain the Taylor map. Indeed, a truncated Taylor map of the type given by Eq. 1 and its associated dispersed closed orbit given by Eq. 3 can be obtained for the SSC lattice (and others) with the use of a post-Teapot [7] map extraction program Zmap. [8] Implementation of such a parameterized Taylor-map Dragt-Finn factorization can be achieved with some modification of the available non-parameterized Dragt-Finn factorization routine in Zlib [9] and others [10] [11]. Once the parameterized dragt-Finn factorization of the type given by Eq. 7 is obtained, paramterized normal form can be obtained with suitable modification of the existing technique [10] and parameterized kick factorization [12] [13] can be obtained for faster long-term symplectic kick map tracking.

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