Quantum Variances for Transverse Betatron Dynamics

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Quantum variances for the transverse dynamics associated with the injection phase of the Superconducting Super Collider are derived using squeezed state methods. During the injection phase, protons injected at large amplitudes display bounded chaotic motion. Small differences in the injection conditions quickly lead to diverse chaotic trajectories. Quantum variances set bounds for the precision to which these differences can be known. The equations for transverse motion are obtained from an invariant action.

I. INTRODUCTION

During the injection phase into the Superconducting Super Collider (SSC) it has been observed that chaotic dynamical behavior occurs for particles near the edge of the dynamic aperture [1]. With an injection amplitude difference as small as 10^{-10} mm, chaotic divergences for the x and y transverse betatron motions begin to occur as soon as 15×10^3 turns around the SSC. It is of practical interest to see what size uncertainties quantum physics places on the precision to which the injection transverse amplitude can be known. It is the purpose of this paper to show that these limits are indeed significant for the SSC, and that they are in fact larger than the small differences already mentioned which lead quickly to chaotic dynamics. As an example of the type of chaotic motion which can occur near the edge of the dynamic aperture during the injection period of the SSC, numerical results are given in Figure 1 as a function of turn number for the modulus $(|I_x| + |I_y|)^{1/2}$, where I_x and I_y are the Courant-Snyder [2] invariants for transverse betatron motion. For this example, the same set of multipole errors used in Run 68 of [1] were used to track nine particles injected on a grid with transverse positions $x = y$ and zero injection angles starting at $x = y = 3.5$ mm and stepped inwards by an amount of 10^{-10} mm. It is clearly seen in Figure 1 that the particle trajectories diverge shortly after 15×10^3 turns. In Figure 2, the same set of nine particles is shown for 7×10^4, and it is seen that the chaotic motion remains bounded. This type of bounded motion has been seen to persist for this set of particles for more than 3×10^6 turns. To illustrate this behavior, two representative particles from this set of nine are shown in Figure 3.

This paper is organized so that in Section II an invariant formulation is introduced to describe transverse betatron dynamics. In this formulation, relativistic invariance is manifest, and the transformation to curvilinear coordinates is presented

as a transformation preserving the invariant action. This formulation, described in greater detail in Appendix A, is used to illustrate the approximations that are made in obtaining the usual Courant-Snyder equations. It is the quantum nature of this set of equations in the linear approximation which are studied. The description of the quantization procedure for the Courant-Snyder system is given in Section III, where it is seen that the time-dependent coherent states appropriate for this system are obtained from a complete set of squeezed eigenstates of the Courant-Snyder invariant. As discussed in Appendix B, the quantum state nearest to the classical state for a time-independent simple harmonic oscillator is a coherent state. The generalization of this state appropriate for a time-dependent simple harmonic oscillator is used to evaluate the uncertainties in position and angle for protons executing transverse betatron motion. Numerical results are given for the uncertainties which are appropriate for the SSC. It is seen that these can be larger than the differences introduced for the injection values of the transverse amplitude near the edge of the dynamic aperture which lead to the rapid onset of chaotic motion. Concluding remarks are found in Section IV, and additional mathematical discussions are given in the appendices.

II. BETATRON EQUATIONS

The equations of motion for transverse betatron oscillations can be found from a Lorentz invariant Hamiltonian [3]. This Hamiltonian, which is equal to the rest energy of the proton, is also invariant under general curvilinear coordinate transformations [4], and this suggests the introduction of an invariant action. This allows one to represent the special relativistic dynamical equations for betatron motion in the coordinate system related to the curved ideal orbit of a proton in the SSC. In addition, this formulation allows one to explicitly see the various approximations which

are made to obtain the usual Courant-Snyder equations. For this invariant Hamiltonian, the Minkowski space contravariant coordinates are defined as the four-vector $q^i = (q^o, q^1, q^2, q^3) = (ct, x, y, z)$, where c is the speed of light. The contravariant components of the four-velocity are defined to be

$$u^i = dq^i/ds, \tag{2.1}$$

and those of the four-momentum for a particle of rest mass m are

$$p^i = mcu^i, \tag{2.2}$$

where ds is the invariant measure

$$ds = \sqrt{dq_i dq^i}, \tag{2.3}$$

where use is made of the Einstein summation convention. The covariant differential components of a four-vector are found from

$$dq_i = g_{ij} dq^j. \tag{2.4}$$

The non-zero Minkowski space components of the covariant metric tensor g_{ij} are

$$(g_{oo}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1). \tag{2.5}$$

The contravariant components of the metric tensor are found from the relation

$$g^{ij} g_{jk} = \delta^i_k \tag{2.6}$$

to be

$$(g^{oo}, g^{11}, g^{22}, g^{33}) = (1, -1. - 1, -1). \tag{2.7}$$

The invariant Hamiltonian is defined as

$$H(p^i, q^i)/c = p_i u^i - L(u^i, q^i) + mc, \tag{2.8}$$

where the invariant Lagrangian for the motion of a particle of mass m and electric charge e in an electromagnetic field with four-potential $A(q)^i$ is defined as

$$L(u^i, q^i) = mc\sqrt{u_i u^i} + (e/c)A(q)_i u^i. \tag{2.9}$$

The conjugate four-momentum is

$$p_i = \frac{\partial L}{\partial u^i} = \frac{mcu_i}{\sqrt{u_i u^i}} + (e/c)A(q)_i, \tag{2.10}$$

and the Lagrangian (2.9) becomes

$$L(u^i, q^i) = p_i u^i. \tag{2.11}$$

For this Lagrangian, the action which is invariant under curvilinear coordinate transformations is defined as

$$S = \int L(u^i, q^i) ds. \tag{2.12}$$

Using the method of stationary action and the approximations described in Appendix A, one finds equations (A.41) and (A.42), which describe transverse betatron motion.

As shown in Appendix A, the approximate Hamiltonian for this system may be written in terms of the transverse coordinates x and y and the local radius of curvature ρ of the ideal orbit as

$$\bar{p}_3 \approx \left(-(e/c)\frac{A^z}{|\vec{p}|} - 1 + \frac{(p^x)^2 + (p^y)^2}{2}\right) \times (1 + x/\rho) \tag{2.13}$$

with

$$ReA(x, y) = Re \sum_{n=0} \frac{eB_o}{c|\vec{p}|} \frac{(b_n(\bar{s}) + ia_n(\bar{s}))}{(n+1)} (x + iy)^{n+1} \tag{2.14}$$

and with

$$A^z(x,y) = -ReA(x,y), \tag{2.15}$$

where B_o is the dipole bending field, $a_n(\bar{s})$ and $b_n(\bar{s})$ are, respectively, the normal and skew multipole coefficients, and $|\vec{p}|$ is found in (A.33). In addition, the transverse magnetic fields are found from

$$B_y(x,y) + iB_x(x,y) = \bar{\mathcal{D}}ReA(x,y) \tag{2.16}$$

with

$$\bar{\mathcal{D}} = \partial_x - i\partial_y. \tag{2.17}$$

In the linear approximation, the Hamiltonian for (A.41) and (A.42) for a particle with ideal orbit momentum $|\vec{p}_o|$ becomes

$$H_{approx} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{K_x(s)}{2}x^2 - \frac{K_y(s)}{2}y^2. \tag{2.18}$$

III. QUANTUM LIMITS ON INJECTION

As a first approximation for finding the quantum limits associated with the Hamiltonian (2.18) for transverse betatron oscillations, one neglects the non-linear multipole contributions and considers for each degree of freedom a time-dependent harmonic oscillator with Hamiltonian

$$\mathbf{H} = \frac{\mathbf{p}^2 + K(\bar{s})\mathbf{q}^2}{2}, \tag{3.1}$$

where $\bar{s}$, the arc length along the ideal orbit, takes the role of the time coordinate. The dynamical evolution of the conjugate quantum operators $\mathbf{p} = \mathbf{q}'$ and $\mathbf{q}$ is determined

from the Heisenberg equations

$$\begin{aligned} \mathbf{q}' &= i[\mathbf{H}, \mathbf{q}] \\ \mathbf{p}' &= i[\mathbf{H}, \mathbf{p}], \end{aligned} \tag{3.2}$$

where $'$ denotes differentiation with respect to $\bar{s}$. The Courant-Snyder quantity

$$2\mathbf{I}(\bar{s}) = [(w\mathbf{p} - w'\mathbf{q})^2 + (\mathbf{q}/w)^2] \tag{3.3}$$

is an invariant. This follows from

$$\frac{d\mathbf{I}}{d\bar{s}} = i[\mathbf{H}, \mathbf{I}] + \frac{\partial \mathbf{I}}{\partial \bar{s}}, \tag{3.4}$$

along with the conditions

$$\begin{aligned} w'' + K(\bar{s})w - \frac{1}{w^3} &= 0 \\ \mathbf{q}'' + K(\bar{s})\mathbf{q} &= 0. \end{aligned} \tag{3.5}$$

Expressed in the usual Courant-Synder parameters, one finds

$$\begin{aligned} q_{1,2} &= we^{\pm i\psi} \\ \psi' &= \frac{1}{w^2} \\ 2\mathbf{I} &= \gamma(\bar{s})\mathbf{q}^2 + \alpha(\bar{s})(\mathbf{qp} + \mathbf{pq}) + \beta(\bar{s})\mathbf{p}^2, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} \alpha(\bar{s}) &= -ww' \\ \beta(\bar{s}) &= w^2 \\ \gamma(\bar{s}) &= \frac{1 + \alpha^2(\bar{s})}{\beta(\bar{s})}. \end{aligned} \tag{3.7}$$

The quantum states for this system can be constructed with the aid of the squeezing operator defined as

$$S = e^{\frac{1}{2}(\xi^* \mathbf{a}^2 - \xi \mathbf{a}^{\dagger 2})}, \tag{3.8}$$

with complex $\xi = |\xi| exp(i\phi)$ and boson operators $\mathbf{a}$ and $\mathbf{a}^\dagger$. Additional properties of $\mathbf{S}(\xi)$ and the associated squeezed state [7] are discussed in Appendix D. The time-independent rationalized Hamiltonian is

$$\mathbf{H}_o = \frac{\mathbf{p}^2 + \mathbf{q}^2}{2}, \tag{3.9}$$

where the boson operators are found from

$$\mathbf{q} = \frac{\mathbf{a} + \mathbf{a}^\dagger}{\sqrt{2}} \qquad \mathbf{p} = \frac{\mathbf{a} - \mathbf{a}^\dagger}{\sqrt{2}i}, \tag{3.10}$$

with the commutation relations

$$[\mathbf{q}, \mathbf{p}] = i \quad \Longrightarrow \quad [\mathbf{a}, \mathbf{a}^\dagger] = 1. \tag{3.11}$$

The Courant-Synder invariant (3.3) can be found from the time-independent Hamiltonian using the squeezing operator (3.8) to write

$$2I(\bar{s}) = \mathbf{S}\mathbf{H}_o\mathbf{S}^\dagger = \left(\mathbf{b}^\dagger\mathbf{b} + \frac{1}{2}\right), \tag{3.12}$$

where

$$\mathbf{b}(\bar{s}) = \mathbf{S}e^{i\theta}\mathbf{a}\mathbf{S}^\dagger = \frac{1}{2}\left(\frac{1}{w} + w - iw'\right)\mathbf{a} + \frac{1}{2}\left(\frac{1}{w} - w - iw'\right)\mathbf{a}^\dagger, \tag{3.13a}$$

with

$$\begin{aligned} \cosh|\xi| &= \frac{1}{2}\sqrt{(1/w + w)^2 + w'} \\ \tan\theta &= -\frac{ww'}{1 + w^2} \\ \tan(\theta + \phi) &= -\frac{ww'}{1 - w^2}. \end{aligned} \tag{3.13b}$$

The eigenstates of $I(\bar{s})$ satisfy the eigenvalue equation

$$\begin{aligned} \mathbf{I}(\bar{s})|n, \bar{s}) &= \left(n + \frac{1}{2}\right) | \, n, \bar{s}) \\ |n, \bar{s}) &= \frac{(\mathbf{b}^\dagger)^n}{\sqrt{n!}}|0). \end{aligned} \tag{3.14}$$

The units of $I(\bar{s})$ are $(\hbar/|\vec{p}|)$ rad, where $|\vec{p}|$ is defined by (A.33). These are not Schrödinger states, for they are not solutions of the equation

$$i\frac{\partial}{\partial \bar{s}}|n,\bar{s})_s = \mathbf{H}(\bar{s})|n,\bar{s})_s. \tag{3.15}$$

However, the Schrödinger states are of the form [5]

$$|n,\bar{s})_s = e^{i\alpha_n(\bar{s})}|n,\bar{s}), \tag{3.16}$$

where the phase, as shown in Appendix C, is

$$\alpha_n(\bar{s}) = -\left(n+\frac{1}{2}\right)\int^{\bar{s}} \frac{d\bar{s}'}{w^2(\bar{s}')}. \tag{3.17}$$

To evaluate the uncertainties associated with the quantities $\mathbf{q}(\bar{s})$ and $\mathbf{p}(\bar{s})$ determined from (3.1) and (3.2), one must use the appropriate coherent state associated with $\mathbf{H}(\bar{s})$. The coherent state used is the time-dependent generalization of the coherent state obtained from the eigenstates of the time-independent Hamiltonian [6] which is the nearest quantum state to the classical state of the simple harmonic oscillator. The coherent state for a time-dependent simple harmonic oscillator can be generated from the squeezed ground state as

$$|\beta,\bar{s})_s = \mathbf{D}(\beta)|0,\bar{s})_s, \tag{3.18}$$

where the coherent state operator $\mathbf{D}(\beta)$ is defined as

$$\mathbf{D}(\beta) = e^{\beta\mathbf{b}^\dagger(\bar{s})-\beta^*\mathbf{b}(\bar{s})}, \tag{3.19a}$$

and where β^* is the complex conjugate of the parameter β, which is the eigenvalue of the operator $\mathbf{b}(\bar{s})$. This parameter is related to the classical value of the invariant $I(\bar{s})$ since

$${}_s(\beta,\bar{s}|\mathbf{I}(\bar{s})|\beta,\bar{s})_s = (|\beta|^2+1/2)(\hbar/|\vec{p}|) \text{ rad.} \tag{3.19b}$$

The variance of an operator $\mathbf{Q}$ is defined as

$$\sigma^2(Q) = {}_s(\beta, \bar{s}|(\mathbf{Q} - \bar{Q})^2|\beta, \bar{s})_s, \tag{3.20}$$

where the mean value of the operator $\mathbf{Q}$ is

$$\bar{Q} = {}_s(\beta, \bar{s}|\mathbf{Q}|\beta, \bar{s})_s. \tag{3.21}$$

Time-independent Hamiltonian (3.9) results found using the coherent state $|\alpha)$ of (B.20) are

$$\sigma(q) = \sigma(p) = \sqrt{\frac{\hbar}{2}}$$

and

$$\sigma(p)\sigma(q) = \frac{\hbar}{2}, \tag{3.22}$$

which yield the minimum value for the uncertainty product. For the Hamiltonian (3.9), the coherent state (B.20) represents the quantum state nearest to the classical state, $\hbar \to 0$, where $\sigma(q)$, $\sigma(p)$, and the uncertainty product are zero.

One can now use the states (3.16) and the definition of the variance (3.20) to obtain results appropriate for the SSC. Using the time-dependent coherent state (3.18) and the appropriate scaling transformations used to find (2.18)

$$\begin{aligned} P &\to \frac{p}{|\vec{p}|} \\ H &\to \frac{H}{|\vec{p}|} \\ \hbar &\to \frac{\hbar}{|\vec{p}|}, \end{aligned} \tag{3.23}$$

where the momentum $|\vec{p}|$ defined in (A.33) is

$$|\vec{p}| \approx \frac{\mathcal{E}}{c}, \tag{3.24}$$

with the proton energy $\mathcal{E}$ given by (A.32), one finds that the variances and the uncertainty product, represented in terms of the Courant-Snyder parameters (3.7), are

$$\begin{aligned} \sigma(q) &= \sqrt{\frac{\hbar\beta(\bar{s})}{2|\vec{p}|}} \\ \sigma\left(\frac{p}{\vec{p}}\right) &= \sqrt{\frac{\hbar\gamma(\bar{s})}{2|\vec{p}|}} \\ \frac{p}{|\vec{p}|} &= \frac{dq}{d\bar{s}} \\ \sigma(q)\sigma\left(\frac{dq}{d\bar{s}}\right) &= \frac{\hbar\sqrt{\beta(\bar{s})\gamma(\bar{s})}}{2|\vec{p}|}. \end{aligned} \tag{3.25}$$

Writing the amplitude as $q_{amp} = \sqrt{\epsilon/\pi\beta(\bar{s})}$ with the emittance $\epsilon = 2\pi I(\bar{s})$, one finds the results

$$\begin{aligned} \frac{\sigma(q_{amp})}{q_{amp}} &= \left(\frac{\epsilon_q}{\epsilon}\right)^{1/2} \\ \epsilon_q/\pi &= \frac{\hbar}{2|\vec{p}|} \approx \frac{\hbar c}{2\mathcal{E}} \approx \frac{\lambda_{proton}}{2}, \end{aligned} \tag{3.26}$$

where ϵ_q/π, the quantum emittance, represents half the resolution distance of a proton in the beam. With the approximations

$$\begin{aligned} \hbar c &\approx 2 \times 10^{-19} \text{ TeV m} \\ \mathcal{E} &\approx 2 \text{ TeV}, \end{aligned} \tag{3.27}$$

one finds

$$\epsilon_q/\pi \approx 5 \times 10^{-20} \text{ m}. \tag{3.28}$$

For a typical SSCTRK tracking result showing chaotic motion with $\beta(\bar{s}) \approx 300$ m and with $x_{amp} \approx 3.5$ mm, one finds

$$\epsilon/\pi \approx 4 \times 10^{-8} \text{ m}$$

and

$$\sigma(x_{amp}) \approx 3.9 \times 10^{-6} \text{ mm}. \tag{3.29}$$

Similarly, the angular uncertainty is

$$\sigma\left(\frac{dx}{d\bar{s}}\right) \approx 1.3 \times 10^{-11}\,\text{rad}. \tag{3.30}$$

Here one should note that $\sigma(x_{amp}) \gg 10^{-10}$ mm, which is the difference in injection position used for study of chaotic motion in [1].

IV. CONCLUSIONS

In conclusion one can see that a completely invariant Lagrangian leads to an invariant Hamiltonian which gives the usual equations of motion for a proton in an electromagnetic field. Using the invariant action associated with this formalism, one can obtain with a curvilinear coordinate transformation the transverse betatron equations of motion. In their form with non-linear multipole errors, these equations lead to chaotic dynamical behavior for particles injected into the SSC near the edge of the dynamic aperture. As described in [1] and in Section I, differences in the injection amplitude as small as 10^{-10} mm quickly lead to chaotic dynamical behavior. It is of interest to see what size uncertainties quantum physics places on the amplitudes.

It has been shown in Section III of this paper that the linearized form of the betatron equations of motion can be quantized and that the time-dependent quantum states appropriate for evaluating the uncertainties of the position and momentum operators are coherent states which are a coherent expansion in a complete set of time-dependent Schrödinger squeezed states. The variances for position and angle associated with these states have been evaluated, and one can see that they depend on the Courant-Snyder parameters $\beta(\bar{s})$ and $\gamma(\bar{s})$. This makes these variances different from the usual coherent state variances associated with a time-independent simple harmonic oscillator. The maximum value of $\beta(\bar{s})$ in the SSC arcs during injection is

300 m, and this value was used to obtain the variance given in (3.25). However larger variances can occur during the injection phase in the interaction regions where $\beta(\bar{s}) >$ 1800 m. For values of $\beta(\bar{s})$ appropriate for the injection phase of the SSC, it is seen that these quantum variances can be large enough so that they must be considered as uncertainties when asking questions about the regions in (q, q') phase space where chaotic behavior is seen. In obtaining these variances, a time-dependent coherent state has been used. The justification for this state is that it reproduces the correct result for the uncertainty inequality for $\mathbf{q}$ and $\mathbf{p}$ in (3.25) since, as seen from (3.7), $\beta(\bar{s})\gamma(\bar{s}) \geq 1$. At values of $\bar{s}$ where $\alpha(\bar{s}) = 0$, these state become minimum uncertainty states since $\beta(\bar{s})\gamma(\bar{s}) = 1$. Furthermore, these states become the correct minimum uncertainty coherent states discussed in Appendix C when $w(\bar{s})$ is a constant. It is of interest to note that the emittance defined as $\epsilon = 2\pi I(\bar{s})$, which is the area of the (q, q') phase space ellipse, is quantized. This is seen from (3.14) which gives the coherent state result (3.19b). Using (3.20) and the method used to arrive at (3.19b), one finds the variance

$$\sigma(\epsilon) = \pi|\beta|(\hbar/|\vec{p}|) \text{ rad.}$$

Although the quantum discussion has been confined to the linear oscillator equations, estimates of the variances associated with non-linear contributions can be made using quantum perturbation theory. The description of this method, along with the associated results and a discussion of the quantum corrections to resonant frequencies, will appear in a future publication.

APPENDIX A: THE BETATRON EQUATIONS

Considering (2.8) and the identity

$$\frac{d}{ds}H(p^i,q^i)/c = \frac{1}{c}\frac{\partial H(p^i,q^i)}{\partial p_i}\frac{dp_i}{ds} + \frac{1}{c}\frac{\partial H(p^i,q^i)}{\partial q^i}\frac{dq^i}{ds}$$

$$= \frac{d}{ds}(p_i u^i) - \frac{\partial L(u^i,q^i)}{\partial q^i}u^i - \frac{\partial L(u^i,q^i)}{\partial u^i}\frac{du^i}{ds} = 0, \qquad (A.1)$$

one can infer for Minkowski space the equations of motion

$$\begin{aligned} c\frac{dq^i}{ds} &= \frac{\partial H(p^i,q^i)}{\partial p_i} \\ c\frac{dp_i}{ds} &= -\frac{\partial H(p^i,q^i)}{\partial q^i}, \end{aligned} \qquad (A.2)$$

where the invariant Hamiltonian becomes

$$H(p^i,q^i) = mc^2 = c\sqrt{(p-(e/c)A(q))_i(p-(e/c)A(q))^i}. \qquad (A.3)$$

Using the above, along with

$$\frac{d}{ds}A(q)_i = u^j\partial_j A(q)_i, \qquad (A.4)$$

with $\partial_j = \partial/\partial x^j$, one finds the equation of motion

$$\frac{d}{ds}mcu_i = (e/c)u^j F_{ij}, \qquad (A.5)$$

where the electromagnetic field tensor is

$$F_{ij} = \partial_i A(q)_j - \partial_j A(q)_i. \qquad (A.6)$$

Since

$$F^{oj} = (0, -\vec{E}), \qquad (A.7)$$

with the electric field defined as

$$\vec{E} = \frac{\partial}{\partial ct}\vec{A}(q) + \vec{\nabla}A(q)^o, \tag{A.8}$$

and since for the magnetic field $\vec{B}$ satisfies

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B^\gamma, \tag{A.9}$$

with $\alpha, \beta = 1, 2,$ or 3 and $\epsilon_{123} = 1$ for the completely antisymmetric tensor, it follows that

$$\frac{d}{dt}\gamma mc^2 = \frac{d\mathcal{E}}{dt} = \vec{E}\cdot\vec{v} \tag{A.10}$$

$$\frac{d}{dt}\gamma m\vec{v} = e\vec{E} + (e/c)\vec{v}\times\vec{B}. \tag{A.11}$$

The first equation gives the energy $\mathcal{E}$ gain or loss of the particle of rest mass m and velocity $\vec{v}$, and the second is the Lorentz equation. In obtaining these results, use has been made of $u^o = \gamma, \vec{u} = \gamma\vec{\beta}$, $ds = cdt/\gamma$, $\vec{\beta} = \vec{v}/c$, and $\gamma = (1 - \vec{\beta}\cdot\vec{\beta})^{-1/2}$.

It is usual to express the equations of motion for betatron dynamics in curvilinear coordinates. The curvilinear coordinates normally used in accelerator physics are x, y, and $\bar{s}$, where x is the horizontal position of the particle relative to the ideal orbit, y is the vertical position coordinate relative to the orbital plane, and $\bar{s}$ is the arc length along the ideal orbit. The position vector of the particle is

$$\vec{r} = (\rho + x)\mathbf{e}_1 + y\mathbf{e}_2. \tag{A.12}$$

The origin of this vector is at the instantaneous center of curvature of the ideal orbit. The coordinates relative to the system with origin at $\vec{r}_o = \rho\hat{e}_1$ are x, y, and z measured respectively along the right-handed orthonormal basis $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$. For motion relative to the torsion-free ideal orbit, the differential of the position vector is

$$d\vec{r} \approx dx\hat{e}_1 + dy\hat{e}_2 + (1 + x/\rho)d\bar{s}\hat{e}_3 + O(d\theta^2), \tag{A.13}$$

where

$$d\bar{s} = \rho d\theta. \tag{A.14}$$

It follows from (A.13) that

$$(dr)^2 \approx (dx)^2 + (dy)^2 + (1 + x/\rho)^2 (d\bar{s})^2. \tag{A.15}$$

The contravariant coordinates in the curvilinear system are

$$(\bar{q}^o, \bar{q}^1, \bar{q}^2, \bar{q}^3) = (q^o, q^1, q^2, \bar{s}), \tag{A.16}$$

and it follows from (A.15) that

$$d\bar{q}^3 = dq^3/(1 + x/\rho). \qquad (A.17).$$

The invariant space-time measure is

$$ds = \sqrt{\bar{g}_{ij} d\bar{q}^i d\bar{q}^j}, \tag{A.18}$$

and the components of the covariant metric tensor $\bar{g}_{ij}$ are

$$(\bar{g}_{oo}, \bar{g}_{11}, \bar{g}_{22}, \bar{g}_{33}) = (1, -1, -1, -(1 + x/\rho)^2). \tag{A.19}$$

The curvilinear contravariant components of the four-momentum are

$$(\bar{p}^o, \bar{p}^1, \bar{p}^2, \bar{p}^3) = (p^o, p^1, p^2, p^3/(1 + x/\rho)), \tag{A.20}$$

and the covariant components are

$$(\bar{p}_o, \bar{p}_1, \bar{p}_2, \bar{p}_3) = (p_o, p_1, p_2, p_3(1 + x/\rho)). \tag{A.21}$$

Similar relations are obtained for the coordinate differential four-vector dq^i and the vector-potential $A(q)^i$.

The betatron equations of motion are found in the curvilinear coordinate system using the fact that the action S is invariant under Lorentz transformations and orthogonal spatial coordinate transformations. The action (2.12) is an invariant under the coordinate transformation $q^i \to \bar{q}^i$ so that

$$S = \int p_i u^i ds = \int \bar{p}_i \bar{u}^i ds. \tag{A.22}$$

A variation of this action gives

$$\delta S = \int (\delta \bar{p}_i \bar{u}^i + \bar{p}_i \delta \bar{u}^i) ds. \tag{A.23}$$

Using

$$\delta \bar{u}^i = \frac{d\delta \bar{q}^i}{ds}, \tag{A.24}$$

and

$$\delta \bar{p}_3 = \frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha} \delta \bar{p}_\alpha + \frac{\partial \bar{p}_3}{\partial \bar{q}^i} \delta \bar{q}^i, \quad \alpha \neq 3, \tag{A.25}$$

one finds for δS

$$\delta S = \int \left((\frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha} \bar{u}^3 + \bar{u}^\alpha) \delta \bar{p}_\alpha + (\frac{\partial \bar{p}_3}{\partial \bar{q}^i} \bar{u}^3 - \frac{d\bar{p}_i}{ds}) \delta \bar{q}^i + \frac{d(\bar{p}_i \delta \bar{q}^i)}{ds}) \right) ds. \tag{A.26}$$

In the usual manner, this yields the dynamical equations of motion

$$\begin{aligned} \frac{d\bar{q}^\alpha}{ds} &= -\bar{u}^3 \frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha} \\ \frac{d\bar{p}_i}{ds} &= \bar{u}^3 \frac{\partial \bar{p}_3}{\partial \bar{q}^i}. \end{aligned} \tag{A.27}$$

Using the representation of the invariant measure

$$ds = \sqrt{d\bar{q}_i d\bar{q}^i} = \sqrt{\frac{d\bar{q}_i}{d\bar{q}^3} \frac{d\bar{q}^i}{d\bar{q}^3}} d\bar{q}^3, \tag{A.28}$$

and the coordinates $(\bar{q}^o, \bar{q}^1, \bar{q}^2, \bar{q}^3) = (q^o, x, y, \bar{s})$ along with

$$\bar{u}^3 = 1/\sqrt{\frac{d\bar{q}_i}{d\bar{q}^3}\frac{d\bar{q}^i}{d\bar{q}^3}}, \tag{A.29}$$

one finds, using the notation $x' = dx/d\bar{s}$ etc., the dynamical equations

$$q^{o\prime} = -\frac{\partial \bar{p}_3}{\partial p^o} \qquad p^{o\prime} = \frac{\partial \bar{p}_3}{\partial \bar{q}^o}$$

$$x' = \frac{\partial \bar{p}_3}{\partial p^x} \qquad p^{x\prime} = -\frac{\partial \bar{p}_3}{\partial x}$$

$$y' = \frac{\partial \bar{p}_3}{\partial p^y} \qquad p^{y\prime} = -\frac{\partial \bar{p}_3}{\partial y}$$

$$(\bar{p}_3)' = \frac{\partial \bar{p}_3}{\partial \bar{s}}. \tag{A.30}$$

The effective Hamiltonian $\bar{p}_3$ for this system is found from the invariant

$$(mc)^2 = (p-(e/c)A(q))_i(p-(e/c)A(q))^i = (\bar{p}-(e/c)\bar{A}(q))_i(\bar{p}-(e/c)\bar{A}(q))^i. \tag{A.31}$$

Defining the energy $\mathcal{E}$ from

$$\mathcal{E}/c = p^o - (e/c)A(q)^o = mc\gamma \tag{A.32}$$

and the scalar three-momentum as

$$|\vec{p}| = \sqrt{\mathcal{E}/c)^2 - (mc)^2}, \tag{A.33}$$

one can use

$$\bar{g}^{33} = -(1 + x/\rho)^{-2} \tag{A.34}$$

and

$$\bar{A}_3 = -(1 + x/\rho)A^z \tag{A.35}$$

to obtain

$$\frac{\bar{p}_3}{|\vec{p}|} = -(e/c)(1+x/\rho)\frac{A^z}{|\vec{p}|} - \sqrt{1-\left(\frac{p^x-(e/c)A^x}{|\vec{p}|}\right)^2-\left(\frac{p^y-(e/c)A^y}{|\vec{p}|}\right)^2} \times (1+x/\rho). \tag{A.36}$$

For constant $|\vec{p}|$, one can introduce the scaled variables for the four-momentum $p^i/|\vec{p}| \to p^i$ in the dynamical equation (A.30) and in (A.36). These equations simplify for the case of a purely transverse magnetic field, where $A^x = A^y = 0$ and for the approximation

$$\sqrt{1-(p^x)^2-(p^y)^2} \approx 1 - \frac{(p^x)^2+(p^y)^2}{2}, \tag{A.37}$$

which is valid for large $|\vec{p}|$. For this case, one finds that (A.36) becomes

$$\bar{p}_3 \approx \left(-(e/c)\frac{A^z}{|\vec{p}|} - 1 + \frac{(p^x)^2+(p^y)^2}{2}\right) \times (1+x/\rho), \tag{A.38}$$

and, neglecting terms of order p^{x2}/ρ, (A.30) gives

$$x' = p^x(1+x/\rho) \qquad y' = p^y(1+x/\rho) \tag{A.39}$$

$$\frac{d}{d\bar{s}}\left(\frac{x'}{1+x/\rho}\right) = \frac{\partial}{\partial x}\left[(e/c)(1+x/\rho)\frac{A^z}{|\vec{p}|} + 1 + x/\rho\right]$$

$$\frac{d}{d\bar{s}}\left(\frac{y'}{1+x/\rho}\right) = \frac{\partial}{\partial y}\left[(e/c)(1+x/\rho)\frac{A^z}{|\vec{p}|} + 1 + x/\rho\right]. \tag{A.40}$$

Neglecting terms of order $(x/\rho)^2$, x'^2/ρ, and skew multipole errors, one finds using $eB_o(\bar{s})\rho = c|\vec{p}_o|$

$$x'' + \frac{|\vec{p}_o|}{|\vec{p}|}\left(\frac{1}{\rho^2} + \frac{eB_1(\bar{s})}{c|\vec{p}_o|}\right)x = \frac{|\vec{p}| - |\vec{p}_o|}{\rho|\vec{p}|} \sim 0 \tag{A.41}$$

$$y'' - \frac{|\vec{p}_o|}{|\vec{p}|}\frac{eB_1(\bar{s})}{c|\vec{p}_o|}y = 0, \tag{A.42}$$

where $|\vec{p}_o|$ is the three-momentum along the ideal orbit and where

$$\begin{aligned} B^y(x,y) &\approx B_o(\bar{s}) + B_1(\bar{s})x + \ldots \\ B^x(x,y) &\approx B_1(\bar{s})y + \ldots. \end{aligned} \tag{A.43}$$

APPENDIX B: COHERENT STATE PROPERTIES

In this appendix it is explained why a coherent state is a minimum uncertainty state. One begins by considering the functional

$$\begin{aligned} J(\lambda) &= \|\lambda \mathbf{d}(A) - i\mathbf{d}(B)\| \geq 0 \\ J(\lambda) &= (\psi|\lambda \mathbf{d}^\dagger(A) + i\mathbf{d}^\dagger(B), \lambda \mathbf{d}(A) - i\mathbf{d}(B)|\psi) \geq 0 \end{aligned} \tag{B.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are Hermitian operators and where the mean values $\bar{A}$, and $\bar{B}$ are real. In (B.1) the deviation of an operator $\mathbf{Q}$ is defined as

$$\mathbf{d}(Q) = \mathbf{Q} - \bar{Q}, \tag{B.2}$$

where the mean value of $\mathbf{Q}$ for a state $|\psi)$ is defined as

$$\bar{Q} = (\psi|\mathbf{Q}|\psi).$$

If one has the commutation relation for the Hermitian operators $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$

$$[\mathbf{A}, \mathbf{B}] = i\mathbf{C}, \tag{B.3}$$

then the deviations satisfy the commutation relation

$$[\mathbf{d}(A), \mathbf{d}(B)] = i\mathbf{C}. \tag{B.4}$$

One then finds from (B.1) that

$$J(\lambda) = \lambda^2\sigma^2(A) + \lambda\bar{C} + \sigma^2(B) \geq 0, \tag{B.5}$$

where the variance for an operator $\mathbf{Q}$ is

$$\sigma(Q) = \sqrt{(\psi|\mathbf{Q}^2|\psi) - (\psi|\mathbf{Q}|\psi)^2}. \tag{B.6}$$

If $J(\lambda)$ is to be a minimum for all λ, then

$$\sigma(A)\sigma(B) \geq \frac{|\bar{C}|}{2}. \tag{B.7}$$

This is seen if one writes (B.1) as

$$J(\lambda) = \left(\lambda\sigma(A) + \frac{\bar{C}}{2\sigma(A)}\right)^2 - \left(\frac{\bar{C}}{2\sigma(A)}\right)^2 + \sigma^2(B) \geq 0. \tag{B.8}$$

Since $J(\lambda)$ must be ≥ 0 for all values of λ, it follows that

$$\sigma(A)\sigma(B) \geq \frac{|\bar{C}|}{2}. \tag{B.9}$$

An additional condition is required to assign a value to λ. One is seeking a state $|\psi)$ which will produce equality in (B.9), the minimum uncertainty value. To achieve this, one requires $J(\lambda) = 0$, and this occurs for

$$\lambda = \frac{-\bar{C}}{2\sigma(A)\sigma(A)}. \tag{B.10}$$

This minimum value of $J(\lambda)$ is valid if and only if

$$(\lambda \mathbf{d}(A) - i\mathbf{d}(B))|\psi) = 0. \tag{B.11}$$

As an example, one considers the time-independent Hamiltonian for the simple harmonic oscillator

$$\mathbf{H} = \frac{\mathbf{p}^2 + \mathbf{q}^2}{2}. \tag{B.12}$$

Using $\mathbf{A} = \mathbf{q}$ and $\mathbf{B} = \mathbf{p}$, one finds the equality

$$\sigma(p)\sigma(q) = n + 1/2 \tag{B.13}$$

and the moments

$$(n|\mathbf{p}^2|n) = (n|\mathbf{q}^2|n) = n + \frac{1}{2}, \tag{B.14}$$

which are valid for the number state $|n)$. The minimum uncertainty value occurs for the ground state $|0)$, and $\sigma(q) = \sigma(p) = 1/\sqrt{2}$ since $\bar{q} = \bar{p} = 0$. Also (B.10) and (B.14) imply $\lambda = -1$ and

$$(\mathbf{q} + i\mathbf{p})|\psi) = (\bar{q} + i\bar{p})|\psi). \tag{B.15}$$

This condition can now be used to find the coherent state $|\alpha)$ as a coherent expansion in the eigenstates $|n)$ of (B.12). Using the usual boson operators

$$\mathbf{a} = \frac{\mathbf{q} + i\mathbf{p}}{\sqrt{2}} \qquad \mathbf{a}^\dagger = \frac{q - i\mathbf{p}}{\sqrt{2}} \tag{B.16}$$

and the notations $|\psi) \sim |\alpha)$ and $\alpha = (\bar{q} + i\bar{p})/\sqrt{2}$, one sees that (B.15) implies

$$\mathbf{a}|\alpha) = \alpha|\alpha). \tag{B.17}$$

The state $|\alpha)$ is found from the following argument:

$$\begin{aligned}
\mathbf{a}|n) &= \sqrt{n}\,|n-1) \\
\mathbf{a}^\dagger|n) &= \sqrt{n+1}\,|n+1) \\
|\alpha) &= \sum_{n=0}^{\infty} C_n|n) \\
\mathbf{a}|\alpha) &= \sum_{n=0}^{\infty} C_n\sqrt{n}\,|n-1) = \sum_{n=0}^{\infty} C_n\alpha|n) \\
C_{n+1}\sqrt{n+1} &= \alpha\, C_n \\
\text{therefore } C_n &= \frac{\alpha^n}{\sqrt{n!}}\, C_0.
\end{aligned} \tag{B.18}$$

With the normalization $(\alpha|\alpha) = 1$, one finds for real C_0

$$C_0 = e^{-\frac{|\alpha|^2}{2}} \tag{B.19}$$

and

$$|\alpha) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n). \tag{B.20}$$

This state may be generated from the ground state $|0)$ of (B.12) as

$$|\alpha) = \mathbf{D}(\alpha)|0), \tag{B.21}$$

with the coherent state operator defined by

$$\mathbf{D}(\alpha) = e^{\alpha \mathbf{a}^\dagger - \alpha^* \mathbf{a}}, \tag{B.22}$$

where α^* is the complex conjugate of α. This can be seen using the identity

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}} e^{-\frac{1}{2}[\mathbf{A},\mathbf{B}]} \tag{B.23}$$

to write

$$\mathbf{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \mathbf{a}^\dagger} \left(1 - \bar{\alpha}\mathbf{a} + \frac{a^{*2}\mathbf{a}^2}{2} + \ldots\right)|0). \tag{B.24}$$

Since $\mathbf{a}|0) = 0$, one finds

$$\begin{aligned} |\alpha) &= \mathbf{D}(\alpha)|0) \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n (\mathbf{a}^\dagger)^n}{n!}|0), \end{aligned} \tag{B.25}$$

which is the same as (B.20) since $\mathbf{a}^{\dagger n}|0) = \sqrt{(n!)}|n)$.

It is useful to see how the coherent state produces the classical limit for an operator. For example, one can consider the operator $\mathbf{q} = (\mathbf{a} + \mathbf{a}^\dagger)/\sqrt{2}$ at $t = 0$. For the Hamiltonian (B.12), the time-developed operator is

$$\begin{aligned} \mathbf{q}(t) &= e^{i\mathbf{H}t} \mathbf{q} e^{-i\mathbf{H}t} \\ &= \mathbf{a}e^{-it} + \mathbf{a}^\dagger e^{it} \\ &= \frac{\mathbf{q} + i\mathbf{p}}{\sqrt{2}} e^{-it} + \frac{\mathbf{q} - i\mathbf{p}}{\sqrt{2}i} e^{it} \\ &= \mathbf{q}\cos t + \mathbf{p}\sin t, \end{aligned} \tag{B.26}$$

and the classical solution of the equation of motion is

$$q_{classical}(t) = \langle\alpha|\mathbf{q}|\alpha\rangle = q_c \cos t + p_c \sin t, \tag{B.27}$$

where the classical values of $\mathbf{q}$ and $\mathbf{p}$ at $t = 0$ are

$$\begin{aligned} q_c &= \langle\alpha|\mathbf{q}|\alpha\rangle = (0|\mathbf{D}^\dagger(\alpha)\mathbf{q}\mathbf{D}(\alpha)|0) \\ &= \frac{\alpha + \alpha^*}{\sqrt{2}} \\ p_c &= \frac{\alpha - \alpha^*}{\sqrt{2}i}. \end{aligned} \tag{B.28}$$

In deriving these results, use is made of the fact that $\mathbf{D}(\alpha)$ is a displacement operator with the properties

$$\begin{aligned} \mathbf{D}^\dagger(\alpha)\mathbf{a}\mathbf{D}(\alpha) &= \mathbf{a} + \alpha \\ \mathbf{D}^\dagger(\alpha)\mathbf{a}^\dagger\mathbf{D}(\alpha) &= \mathbf{a}^\dagger + \alpha^* \end{aligned} \tag{B.29}$$

so that the matrix elements are evaluated using the ground state $|0)$.

It is instructive to note additional properties associated with the coherent state (B.20). The eigenvalues of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ are distributed according to the Poisson distribution. This follows from the representation of the state $|\alpha)$ given by (B.20) and from the orthonormality of the states $|n)$ which are used to show

$$\begin{aligned} (\alpha|\mathbf{N}|\alpha) &= \sum_{n=0}^{\infty} P_n(\alpha)_{Poisson}\, n \\ P_n(\alpha)_{Poisson} &= \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}. \end{aligned} \tag{B.30}$$

The meaning of the parameter α can be seen if one uses (B.21) to evaluate the matrix element of (B.12)

$$(\alpha|\mathbf{H}|\alpha) = |\alpha|^2 + 1/2, \tag{B.31}$$

where this result represents the classical value of the energy plus the ground state energy. In addition one finds from (B.21) and (B.29) the variance $\sigma(H) = \sigma(N) = |\alpha|$. The Schrödinger states associated with $|\alpha)$ and the Hamiltonian (B.12) are found from

$$|\alpha, t) = e^{-i\mathbf{H}t}|\alpha). \tag{B.32}$$

It follows from (B.20) that

$$|\alpha, t) = e^{-it/2}|e^{-it}\alpha), \tag{B.33}$$

so that the time developed state remains a coherent state with $\alpha \rightarrow e^{-it}\alpha$. Also one notes that the state $|\alpha)$ produces the values $\sigma(q) = \sigma(p) = 1/\sqrt{2}$, known as the standard quantum limits, and the equality $\sigma(q)\sigma(p) = 1/2$. This is why this state is considered to be the nearest quantum state to the classical state for the time-independent simple harmonic oscillator. The q-space wave function associated with $|\alpha)$ is found by forming the matrix element

$$\Psi(q)_\alpha = (q|\alpha) = (q|\mathbf{D}(\alpha)|0). \tag{B.34}$$

Using (B.16), (B.23), and (B.28), one finds

$$\begin{aligned}\Psi(q)_\alpha &= e^{-ip_c q_c/2} e^{ip_c q}(q|e^{-iq_c \mathbf{P}}|0) \\ &= e^{-ip_c q_c/2} e^{ip_c q}\Psi_o(q - q_c),\end{aligned} \tag{B.35}$$

where $\Psi_o(q)$ is the ground state wave function of the simple harmonic oscillator.

Further mathematical properties are useful for evaluating matrix elements using the coherent state (B.21). The resolution of the identity operator is found to be

$$\frac{1}{\pi}\int |\alpha)(\alpha| d^2\alpha = 1, \tag{B.36}$$

with $d^2\alpha = dRe\,\alpha dIm\,\alpha$. This is seen using the expansion (B.20) to write

$$\begin{aligned}\frac{1}{\pi}\int d^2\alpha|\alpha)(\alpha| &= \frac{1}{\pi}\int_0^\infty |\alpha|d|\alpha| \times \int_0^\infty d\phi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{i(m-n)\phi} \\ &\times e^{-|\alpha|^2}\frac{|\alpha|^{m+n}}{\sqrt{n!m!}}|m)(n| \\ &= \sum_{n=0}^{\infty}\frac{|n)(n|}{n!}\int_0^\infty |\alpha|^{2n}e^{-|\alpha|^2}d|\alpha|^2 \\ &= \sum_{n=0}^{\infty}|n)(n| = 1.\end{aligned} \tag{B.37}$$

It can also be seen from (B.20) that the these state are not orthogonal, for one can show

$$|(\alpha|\beta)| = e^{-|\alpha-\beta|^2/2}, \tag{B.38}$$

which approaches orthogonality for large values of $|\alpha-\beta|$. For this reason the states $|\alpha)$ are said to be over complete. A particularly useful property is the relation

$$(\alpha|F(\mathbf{a},\mathbf{a}^\dagger)|\alpha) = (0|F(\mathbf{a}+\alpha,\mathbf{a}^\dagger+\alpha^*)|0), \tag{B.39}$$

which holds for normal ordered functions that have an expansion in operator products of the form

$$\mathbf{a}^{\dagger m}\mathbf{a}^n. \tag{B.40}$$

This result is easily established using $\mathbf{D}^\dagger(\alpha)\mathbf{D}(\alpha) = 1$, (B.17) along with its adjoint equation, and (B.29) to write

$$(\alpha|\mathbf{a}^{\dagger m}\mathbf{a}^n|\alpha) = \alpha^{*m}\alpha^n(\alpha|\alpha) = (0|(\mathbf{a}^\dagger+\alpha^*)^m(\mathbf{a}+\alpha)^n|0). \tag{B.41}$$

In addition, the operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ have a differential operator representation for the coherent state $|\alpha)$. Using (B.21) to write

$$\frac{\partial}{\partial\alpha}|\alpha) = \frac{\partial}{\partial\alpha}e^{\alpha\mathbf{a}^\dagger-\alpha^*\alpha/2}|0) = (\mathbf{a}^\dagger-\alpha^*/2)|\alpha), \tag{B.42}$$

one finds

$$\mathbf{a}^\dagger|\alpha) = (\frac{\partial}{\partial\alpha} + \alpha^*/2)|\alpha), \tag{B.43}$$

which along with (B.17) gives the desired representation.

APPENDIX C: THE SCHRÖDINGER STATE PHASE

The phase (3.17) can be found by first differentiating (3.16) with respect to $\bar{s}$, and then using (3.15) to write the matrix element

$$i\frac{\partial\alpha_n(\bar{s})}{\partial\bar{s}} + \left(n,\bar{s}\left|\frac{\partial}{\partial\bar{s}}\right|n,\bar{s}\right) = -i(n,\bar{s}|\mathbf{H}(\bar{s})|n,\bar{s})$$

$$= \frac{1}{2}(w'^2 + K(\bar{s})w^2 + 1/w^2)(n+1/2), \tag{C.1}$$

where (3.13) is used to express the Hamiltonian (3.1) as a function of $\mathbf{b}(\bar{s})$ and its adjoint. The matrix elements of this operator can be found from

$$\mathbf{b}^\dagger|n,\bar{s}) = \sqrt{n+1}\,|n+1,\bar{s}) \tag{C.2}$$

and

$$\mathbf{b}\,|n,\bar{s}) = \sqrt{n}\,|n-1,\bar{s}) \quad\Longrightarrow\quad (n,\bar{s}|\mathbf{b}^\dagger = \sqrt{n}(n-1,\bar{s}|. \tag{C.3}$$

Making the replacement $n \to n-1$ in (C.2), one can derive the identity

$$\left(n,\bar{s}\left|\frac{\partial\mathbf{b}^\dagger}{\partial\bar{s}}\right|n-1,\bar{s}\right) + \left(n,\bar{s}\left|\mathbf{b}^\dagger\frac{\partial}{\partial\bar{s}}\right|n-1,\bar{s}\right)$$

$$= \sqrt{n}\left(n,\bar{s}\left|\frac{\partial}{\partial\bar{s}}\right|n,\bar{s}\right), \tag{C.4}$$

where

$$\frac{\partial\mathbf{b}^\dagger}{\partial\bar{s}} = \frac{1}{2}(i(ww'' - w'^2) - 2w'/w)\mathbf{b} + i(ww'' - w'^2)\mathbf{b}^\dagger. \tag{C.5}$$

Therefore

$$\left(n,\bar{s}\left|\frac{\partial}{\partial\bar{s}}\right|n,\bar{s}\right)=\left(0,\bar{s}\left|\frac{\partial}{\partial\bar{s}}\right|0,\bar{s}\right)+i\,\frac{s}{2}(ww''-w'^2). \tag{C.6}$$

Choosing

$$\left(0,\bar{s}\left|\frac{\partial}{\partial\bar{s}}\right|0,\bar{s}\right)=\frac{i(ww''-w'^2)}{4}, \tag{C.7}$$

one finds from (3.5), (C.1), (C.6), and (C.7) the differential equation

$$\frac{d\alpha_n(\bar{s})}{d\bar{s}}=-\left(n+\frac{1}{2}\right)\frac{1}{w^2(\bar{s})}, \tag{C.8}$$

which has the solution (3.17).

APPENDIX D: SQUEEZED STATES AND CHARACTERISTIC FUNCTIONS

In this appendix the properties of a squeezed quantum state are described. It is seen in Appendix B that it is possible to have equality for the uncertainty product (B.9) and at the same time equality of the variances $\sigma(A)$ and $\sigma(B)$. This led to the construction of the coherent state (B.20). It is also possible to have a minimum uncertainty product without having equality of the variances, and this is the property of the squeezed state which is generated from the ground state $|0)$ of the Hamiltonian (3.9)as $|\xi)=\mathbf{S}(\xi)|0)$ using the operator

$$\mathbf{S}(\xi)=e^{(\xi^*\mathbf{a}^2-\xi\mathbf{a}^{\dagger 2})/2}, \tag{D.1}$$

with ξ a complex number. To demonstrate the nature of the squeezing operator, it is useful to introduce the Hermitian operators $\mathbf{Y}_1$ and $\mathbf{Y}_2$, defined by the equations

$$\begin{aligned}\mathbf{Y}_1+i\mathbf{Y}_2&=(\mathbf{q}+i\mathbf{p})e^{-i\phi/2}\\ \mathbf{Y}_1-i\mathbf{Y}_2&=(\mathbf{q}-i\mathbf{p})e^{i\phi/2},\end{aligned} \tag{D.2}$$

where ϕ is the phase of the complex number ξ. Using the state $|\psi) = \mathbf{S}(\xi)|0)$ in the expression for the variance (B.6), one obtains the results

$$\begin{aligned} \sigma(Y_1)_\xi &= e^{-|\xi|}/\sqrt{2} \\ \sigma(Y_2)_\xi &= e^{|\xi|}/\sqrt{2}. \end{aligned} \tag{D.3}$$

Here it is seen that the variance $\sigma(Y_1)$ is reduced (squeezed) below the coherent state standard quantum limit $1/\sqrt{2}$. However, $\sigma(Y_2)$ is increased above this value in such a manner that the uncertainty product remains an equality

$$\sigma(Y_1)\sigma(Y_2) = 1/2. \tag{D.4}$$

In deriving the results (D.3), one uses the transformation property of the operator $\mathbf{a}$

$$\mathbf{S}(\xi)\mathbf{a}\mathbf{S}^\dagger(\xi) = \mathbf{a}\cosh|\xi| + e^{i\phi}\sinh|\xi|\mathbf{a}^\dagger \tag{D.5}$$

and the adjoint of this expression.

The moments of the operators $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ for either the squeezed or the coherent state can be found through the use of the appropriate characteristic functions. Characteristic functions for these operators using thermal field methods [8] and a combined coherent squeezed thermal state have been derived in [9], and the special cases associated with zero temperature can be easily found from

$$C(\lambda, \xi, \alpha)_q = exp[i\lambda e^{-\xi} q_c - (e^{-\xi}\sigma(\beta))^2\lambda^2/2] \tag{D.6}$$

$$C(\lambda, \xi, \alpha)_p = exp[i\lambda e^{\xi} p_c - (e^{\xi}\sigma(\beta)^2\lambda^2/2], \tag{D.7}$$

where

$$\sigma(\beta) = \sqrt{\frac{1}{2}\coth(\beta/2)}, \tag{D.8}$$

with $\beta = \hbar\omega/k_B T$. The moments of the number operator for either the squeezed or the coherent state can be found from the characteristic function for the Hamiltonian

operator $\mathbf{H} = \mathbf{N} + 1/2$ using a coherent squeezed thermal state. This characteristic function for non-zero temperature is

$$C(\lambda, \xi, \alpha)_H = \sqrt{\frac{\omega}{ab}}\ e^{-(Ab+Ba)/ab}\ \frac{e^{x_a y_a/(y_a-1)}}{\sqrt{1-y_a}}\ \frac{e^{x_b y_b/(y_b-1)}}{\sqrt{1-y_b}}, \qquad (D.9)$$

with the notations

$$w = e^{i\lambda},$$

$$y_a = \tilde{a}w, \quad y_b = \tilde{b}w,$$

$$\tilde{a} = (a-1)/a, \quad \tilde{b} = (b-1)/b,$$

$$x_a = A/a(1-a), \quad x_b = B/b(1-b),$$

$$A = (q_c^2/2)e^{-2\xi}, \quad B = (p_c^2/2)e^{2\xi}, \qquad (D.10)$$

$$a = e^{-2\xi}(\bar{n}(\beta) + 1/2) + 1/2,$$

and

$$b = e^{2\xi}(\bar{n}(\beta) + 1/2) + 1/2.$$

In the above

$$\bar{n}(\beta) = (exp(\beta) - 1)^{-1}, \qquad (D.11)$$

and q_c and p_c are the classical values (B.28) of the operators $\mathbf{q}$ and $\mathbf{p}$. The nth moment of an operator $\mathbf{Q}$ is found from

$$\frac{\partial^n}{\partial\lambda^n} C(\lambda)_Q / i^n|_{\lambda=0}. \qquad (D.12)$$

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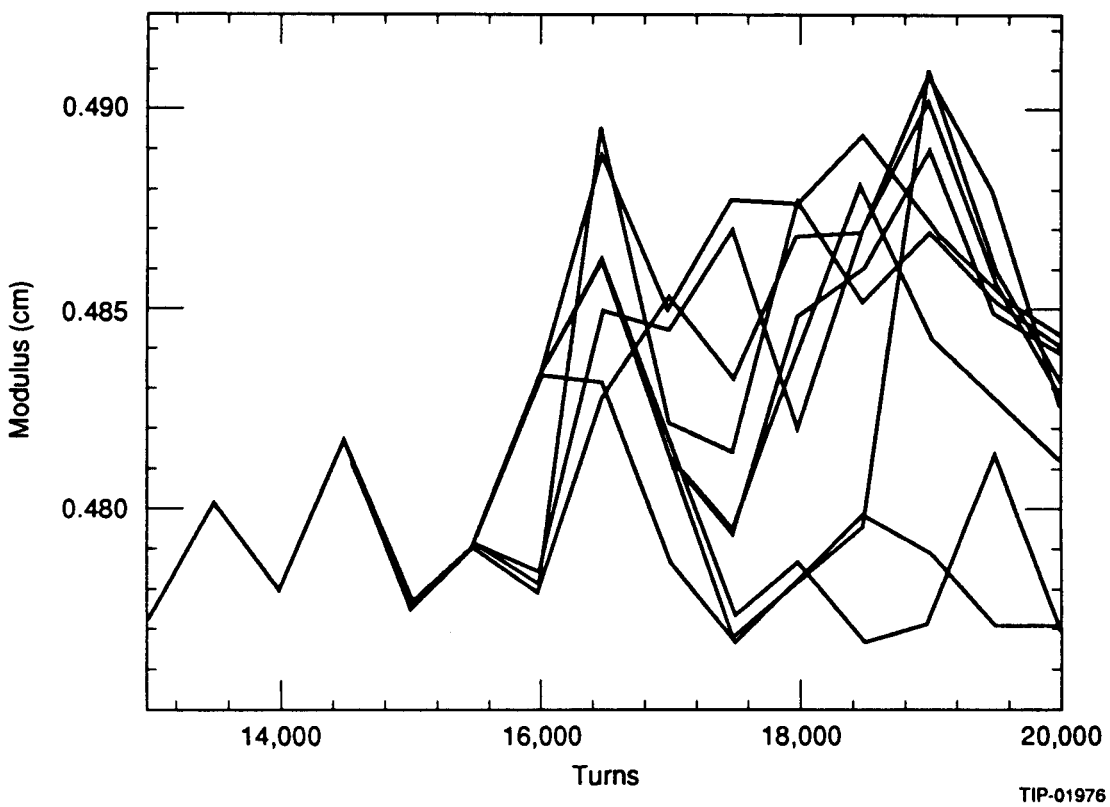

FIG. 1. Modulus as a function of turn number showing chaotic motion for a nine particle grid with $\Delta x = 10^{-10}$ mm.

Figure 1

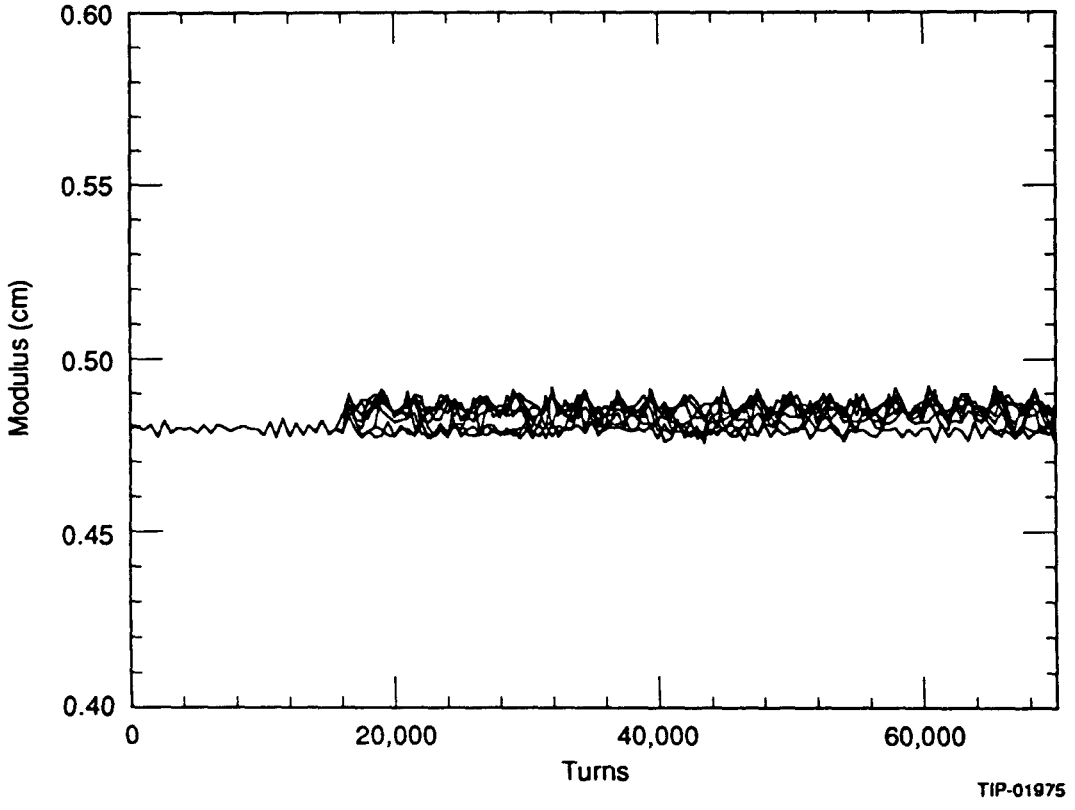

FIG. 2. Modulus as a function of turn number showing bounded chaotic motion for a nine particle grid with $\Delta x = 10^{-10}$ mm.

Figure 2

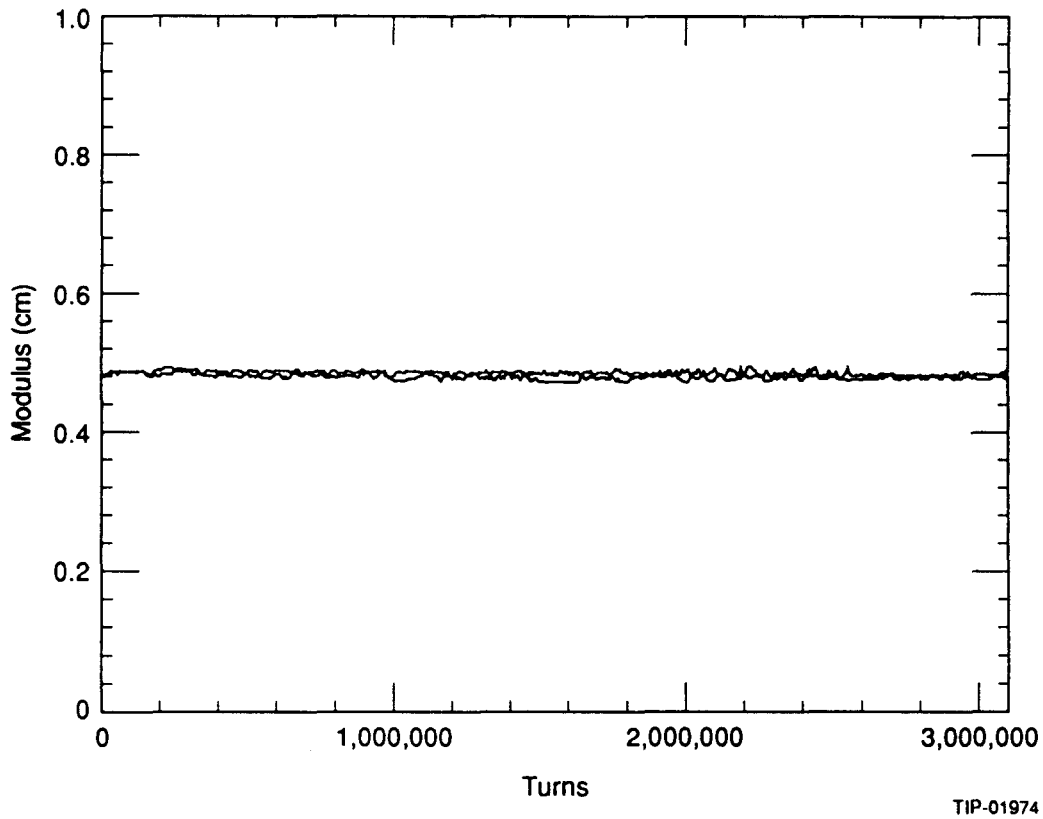

FIG. 3. Modulus as a function of turn number showing bounded chaotic motion for two particles with $\Delta x = 10^{-10}$ mm.

Figure 3