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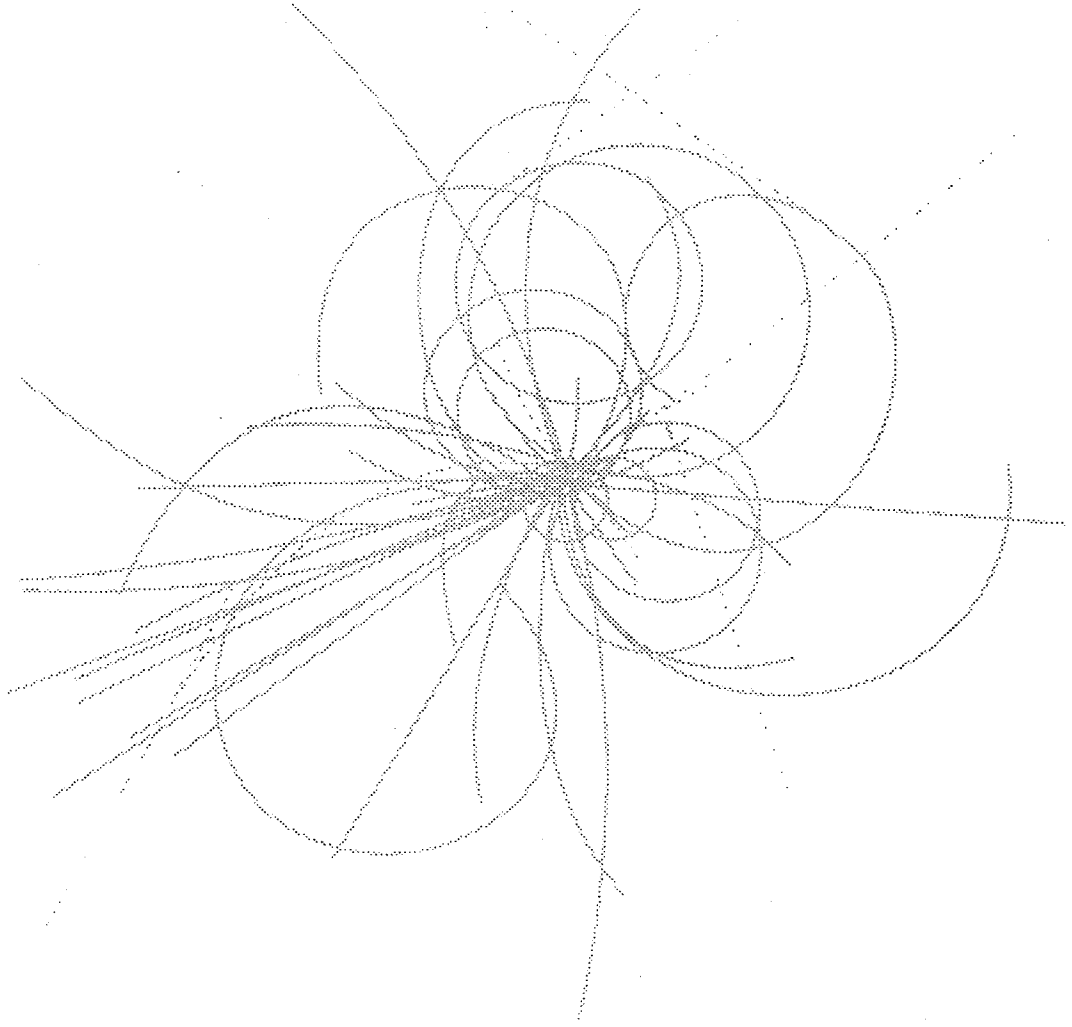

Constant of Motion and Dynamic Equations for One-Dimensional Autonomous System and Radiation Damping

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CONSTANT OF MOTION AND DYNAMIC EQUATIONS FOR ONE-DIMENSIONAL AUTONOMOUS SYSTEM AND RADIATION DAMPING

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* Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC02-89ER40486.

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Abstract

The dynamic equations for a one-dimensional autonomous system along with their dynamic transformations are expressed in terms of the Constant of Motion of the system. The relation between Liouville's theorem and Hamilton's equations with the Constant of Motion is studied. This approach is applied to the radiation damping suffered by a charged particle inside the beam circulating around a collider-ring accelerator.

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1. INTRODUCTION

It is well known that the motion of a single particle moving in one-dimensional space can be described by the Newton's equation

$$\frac{d^2x}{dt^2} = F \quad , \tag{1}$$

where the mass of the particle has been included in the definition of the external force, F, acting on the particle, "x" is the position of the particle, and "t" represents the time. If F does not depend explicitly on time, it is said that the system is "autonomous," and equation (1) can be written as the following "Autonomous Dynamical System" (ADS):

$$\frac{dv}{dt} = F(x, v) \tag{2a}$$

and

$$\frac{dx}{dt} = v \quad , \tag{2b}$$

where v is the velocity of the particle.

One of the principal objectives in physics is to find the first integral of motion of this system, because it represents the first constant of motion which is associated with the total energy of the particle, and is in close relation with the Hamiltonian and Lagrangian of the system [1] which, in turn, represents the motion of the particle in the phase space. This space is the natural space in accelerator physics to see what happens with a beam of charged particles (or a single charged particle) traveling in the accelerator. [2] Many dissipated systems (in particular synchrotron radiation damping) can be expressed in the form (2), these systems represent a challenge for a consistent Hamiltonian and Lagrangian formulation and have interest in Classical Mechanic, Electrical Network Theory, [3] Statistic Mechanics and in Quantum Mechanics as well. In order to understand

the problem presented by these types of systems, a new approach will be formulated on the basis of the Constant of Motion for the ADS (2). In this paper, the constant of motion is given with the criterion to select it, having this constant of motion, the dynamics equations, the Liouville-like's theorem, Hamilton-like's equations are given in terms of it. The changes in these expressions because of a "dynamical transformation" is also studied in conjunction with the requirements to have an action-angle-like transformation. Finally, this approach is used to study the radiation damping suffered by a charged particle in a beam that is circulating around a collider-ring accelerator.

2. CONSTANT OF MOTION

A constant of motion is a function, K, defined in the space $\{x, v\}$ which satisfies the following expression:

$$\frac{dK}{dt} = 0 \quad , \tag{3}$$

or according to (2), this means that the following Partial Differential Equation of First Order (PDFO) is satisfied:

$$v\frac{\partial K}{\partial x} + F(x,v)\frac{\partial K}{\partial v} = 0 \quad . \tag{4}$$

This equation can be solved by the characteristics method, [4] where the equations for the characteristic curves are given by

$$\frac{dx}{v} = \frac{dv}{F(x,v)} = \frac{dK}{0} \quad . \tag{5}$$

The last term in (5) means that the function K is an arbitrary function of the characteristic curve obtained from the solution of the two first terms of (5). This functionality may be selected in such a way that the usual energy expression can be obtained, if the function $F(x,v)$ reduces to function $F(x)$ when one parameter reaches a specific value. [1] This last statement will be clear when the radiation damping example can be seen.

3. DYNAMIC EQUATIONS USING THE CONSTANT OF MOTION, K

Let ρ be an arbitrary function defined in the space $\{x, v\}$ and assume K is the constant of motion of the system (2) then, the time variation of ρ is given by

$$\frac{d\rho}{dt} = v\frac{\partial \rho}{\partial x} + F(x,v)\frac{\partial \rho}{\partial v} \quad , \tag{6}$$

but from equation (4), the function F can be expressed in terms of the partial derivatives of the constant of motion K as

$$F(x,v) = -\frac{\left(\frac{\partial K}{\partial x}\right)}{\left(\frac{\partial K}{\partial v}\right)} \quad , \tag{7}$$

where

$$\frac{\partial K}{\partial v} \neq 0 \quad , \tag{8}$$

otherwise the constant of motion would be a simple real number. Using (7) in (6), the variation of ρ with respect to time is expressed by

$$\frac{d\rho}{dt} = \frac{v}{\left(\frac{\partial K}{\partial v}\right)}\{\rho, K\}_{x,v} \quad , \tag{9}$$

where $\{\rho, K\}_{x,v}$ is the following Poisson-like bracket

$$\{\rho, K\}_{x,v} = \frac{\partial \rho}{\partial x}\frac{\partial K}{\partial v} - \frac{\partial \rho}{\partial v}\frac{\partial K}{\partial x} \quad . \tag{10}$$

Taking ρ as the functions x and v respectively in (9), the dynamic equations in terms of the constant of motion are

$$\frac{dx}{dt} = v \tag{11a}$$

and

$$\frac{dv}{dt} = -\frac{v}{\left(\frac{\partial K}{\partial v}\right)}\left(\frac{\partial K}{\partial x}\right) \quad . \tag{11b}$$

4. HAMILTON-LIKE AND LIOUVILLE-LIKE'S EQUATIONS

The Hamilton-like's equations can be obtained from (9) and (11) if the following condition is chosen:

$$\frac{v}{\left(\frac{\partial K}{\partial v}\right)} = 1 \quad , \tag{12}$$

but this means that the constant of motion must be of the form

$$K = \frac{v^2}{2} + V(x) \quad , \tag{13}$$

where $V(x)$ is an arbitrary function. This expression is the usual total energy of the system when the force does not depend on the velocity so, for this particular case the concept of constant of motion, total energy, and Hamiltonian can be used undistinguishly.

Now, it is clear from (2), (4) and (6) that if there exists an autonomous system of N noninteracting particles moving in one dimension under the same external force, F, and choosing ρ (density) as an arbitrary function of the coordinates and velocities of these particles, the evolution with respect to time of this density is given by

$$\frac{d\rho}{dt} = \sum_{i=1}^{N} \frac{v_i}{\left(\frac{\partial K}{\partial v_i}\right)} \{\rho, K\}_{x_i, v_i} \quad , \tag{14}$$

where the constant of motion, K, is given by

$$K = \sum_{i=1}^{N} k(x_i, v_i) \tag{15}$$

and $k(x_i, v_i)$ satisfies equation (4) for each i-coordinate. The density ρ will be a constant of motion if and only if

$$\{\rho, K\}_{x_i, v_i} = 0 \qquad i = 1, \ldots, N \tag{16}$$

and this, in turn, implies that

$$\rho = \rho(K) \quad . \tag{17}$$

This is the Liouville-like's theorem expressed in terms of the constant of motion.

5. DYNAMICAL TRANSFORMATIONS

The problem to address now is the dynamical transformations and their effect in expressions (2), (9), and (4). A dynamical transformation is a change of the variables which transforms the old variables, $\{x, v\}$, to new ones, $\{Q, P\}$, such that the transformation is invertible, that is, the variables x and v are given by

$$x = x(Q, P) \tag{18a}$$

and

$$v = v(Q, P) \quad , \tag{18b}$$

and the Jacobian of the transformation, $J_{x,v}$ is given by

$$\begin{aligned} J_{x,v} = \det\left(\frac{\partial(x,v)}{\partial(Q,P)}\right) &= \det\begin{pmatrix} \frac{\partial x}{\partial Q} & \frac{\partial x}{\partial P} \\ \frac{\partial v}{\partial Q} & \frac{\partial v}{\partial P} \end{pmatrix} \\ &= \left(\frac{\partial x}{\partial Q}\right)\left(\frac{\partial v}{\partial P}\right) - \left(\frac{\partial x}{\partial P}\right)\left(\frac{\partial v}{\partial Q}\right) \neq 0 \quad . \end{aligned} \tag{19}$$

Using the following relations:

$$\frac{dx}{dt} = \frac{\partial x}{\partial Q}\frac{dQ}{dt} + \frac{\partial x}{\partial P}\frac{dP}{dt} \quad , \tag{20a}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial Q}\frac{dQ}{dt} + \frac{\partial v}{\partial P}\frac{dP}{dt} \quad , \tag{20b}$$

and condition (19), the dynamical system (2) is expressed in terms of the variables Q and P, in the following form:

$$\frac{dQ}{dt} = \left\{ \left(\frac{\partial x}{\partial P}\right) f(Q,P) - \left(\frac{\partial v}{\partial P}\right) v \right\} \Big/ J_{x,v} \tag{21a}$$

and

$$\frac{dP}{dt} = \left\{ -\left(\frac{\partial x}{\partial Q}\right) f(Q,P) + \left(\frac{\partial v}{\partial Q}\right) v \right\} \Big/ J_{x,v} \quad , \tag{21b}$$

where the function f, has been defined as

$$f(Q,P) = F(x(Q,P), v(Q,P)) \quad . \tag{22}$$

To see the changes in the dynamic equations, use the relations

$$\frac{\partial}{\partial x} = \frac{\partial Q}{\partial x}\frac{\partial}{\partial Q} + \frac{\partial P}{\partial x}\frac{\partial}{\partial P} \quad , \tag{23a}$$

$$\frac{\partial}{\partial v} = \frac{\partial Q}{\partial v}\frac{\partial}{\partial Q} + \frac{\partial P}{\partial v}\frac{\partial}{\partial P} \quad , \tag{23b}$$

and the condition

$$\begin{aligned} J_{Q,P} = \det\left(\frac{\partial(Q,P)}{\partial(x,v)}\right) &= \det\begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial v} \\ \frac{\partial P}{\partial x} & \frac{\partial P}{\partial v} \end{pmatrix} \\ &= \left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial P}{\partial v}\right) - \left(\frac{\partial Q}{\partial v}\right)\left(\frac{\partial P}{\partial x}\right) \neq 0 \end{aligned} \tag{24}$$

in (9) and (10). After making some rearrangements, the result is

$$\frac{d\tilde{\rho}}{dt} = \frac{v J_{Q,P}}{\frac{\partial Q}{\partial v}\frac{\partial \widetilde{K}}{\partial Q} + \frac{\partial P}{\partial v}\frac{\partial \widetilde{K}}{\partial P}} \left\{\tilde{\rho}, \widetilde{K}\right\}_{Q,P} \tag{25}$$

and

$$\left[v\frac{\partial Q}{\partial x} + f\frac{\partial Q}{\partial v}\right]\frac{\partial \widetilde{K}}{\partial Q} + \left[v\frac{\partial P}{\partial x} + f\frac{\partial P}{\partial v}\right]\frac{\partial \widetilde{K}}{\partial P} = 0 \quad , \tag{26}$$

where the functions $\tilde{\rho}$ and $\widetilde{K}$, have been defined as

$$\tilde{\rho}(Q,P) = \rho(x(Q,P), v(Q,P)) \quad , \tag{27}$$

and

$$\widetilde{K}(Q,P) = K(x(Q,P), v(Q,P)) \quad . \tag{28}$$

Expression (25) expresses the fact that Constant of Motion, as well as the Liouville-like's theorem, are invariant under dynamical transformations. It must be noticed that the constant of motion, $\widetilde{K}$, can also be obtained from the following PDFO:

$$\left(\frac{dQ}{dt}\right)\frac{\partial \widetilde{K}}{\partial Q} + \left(\frac{dP}{dt}\right)\frac{\partial \widetilde{K}}{\partial P} = 0 \tag{29}$$

with dQ/dt and dP/dt given by (21a) and (21b), but it can be demonstrated that equations (26) and (29) are equivalents by using (22) in (29), equaling coefficients with (26), and arriving at the following identities:

$$\frac{\partial v}{\partial Q}\frac{\partial Q}{\partial v} + \frac{\partial v}{\partial P}\frac{\partial P}{\partial v} = 1 \quad , \tag{30a}$$

$$\frac{\partial v}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial v}{\partial P}\frac{\partial P}{\partial x} = 0 \quad , \tag{30b}$$

$$\frac{\partial x}{\partial Q}\frac{\partial Q}{\partial v} + \frac{\partial x}{\partial P}\frac{\partial P}{\partial v} = 0 \quad , \tag{30c}$$

and

$$\frac{\partial x}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial x}{\partial P}\frac{\partial P}{\partial x} = 1 \quad . \tag{30d}$$

In this way the results obtained are consistent.

If the the constant of motion is of the form (13), it has been observed that in this case a Hamilton-like's equations can be obtained to describe the dynamics of the system, and from relation (25), it is clear that $J_{Q,P} = 1$, in order to have invariance of the Hamilton-like's equations under dynamical transformations. The symplectic transformation is a stronger condition which comes from the invariance of the symplectic metric, [5] characterized by the matrix S, given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \tag{31}$$

that appears in the definition of the Poisson-like bracket

$$\{\rho, K\}_{x,v} = \left(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial v}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial v} \end{pmatrix} \quad . \tag{32}$$

6. ACTION-ANGLE-LIKE TRANSFORMATION

The next problem to address is the conditions required to obtain the following type of transformation:

$$\frac{dQ}{dt} = \omega = constant \tag{33a}$$

and

$$\frac{dP}{dt} = 0 \tag{33b}$$

which are the "action-angle-like transformation" for system (2). Using these conditions in (9), (20), and (24), the following results can be obtained:

$$P = P(K) \quad , \tag{34}$$

$$v = \omega\left(\frac{\partial x}{\partial Q}\right) \quad , \tag{35a}$$

$$f = \omega\Big(\frac{\partial v}{\partial P}\Big) \quad , \tag{35b}$$

and

$$J_{Q,P} = \{Q, K\}_{x,v} \frac{\partial P}{\partial K} \neq 0 \quad . \tag{36}$$

Assume now that $J_{Q,P} \neq 1$, then using (33) in (25),

$$\frac{\partial \widetilde{K}}{\partial Q} = 0 \tag{37a}$$

and

$$\omega = -\frac{v J_{Q,P}}{\Big(\frac{\partial K}{\partial v}\Big)}\Big(\frac{\partial \widetilde{K}}{\partial P}\Big) \quad . \tag{37b}$$

Making use of (36) in (37b) and rearranging terms, it follows:

$$\omega\Big(\frac{\partial K}{\partial v}\Big) + v\{Q, P\}_{x,v}\Big(\frac{\partial P}{\partial K}\Big)\Big(\frac{\partial \widetilde{K}}{\partial P}\Big) = 0 \quad , \tag{38a}$$

using now equation (9) as well as expression (33a), the above Poisson-like bracket can be written as

$$\{Q, P\}_{x,v} = \frac{\omega}{v}\Big(\frac{\partial K}{\partial v}\Big) \quad , \tag{38b}$$

which can be substituted in (38b) to obtain the expression

$$\Big(\frac{\partial K}{\partial v}\Big)\left[1 + \Big(\frac{\partial P}{\partial K}\Big)\Big(\frac{\partial \widetilde{K}}{\partial P}\Big)\right] = 0 \quad , \tag{39}$$

but this implies that $\Big(\frac{\partial K}{\partial v}\Big) = 0$!, thus, the assumption that the Jacobian is different from the unit is wrong. Assume that $J_{Q,P} = 1$ and use the expression

$$\frac{\partial K}{\partial v} = \frac{\partial Q}{\partial v}\frac{\partial \widetilde{K}}{\partial Q} + \frac{\partial P}{\partial v}\frac{\partial \widetilde{K}}{\partial P} = \Big(\frac{\partial P}{\partial v}\Big)\frac{\partial \widetilde{K}}{\partial P} \tag{40}$$

in (37b) to obtain, after some rearrangements, the following expression:

$$\left[\omega\frac{\partial P}{\partial v} - v\right]\frac{\partial \widetilde{K}}{\partial P} = 0 \quad , \tag{41}$$

so, the term inside the square bracket must be zero, and the resulting equation can be solved by the following expression:

$$P = \frac{1}{\omega}\Big(\frac{1}{2}v^2 + V(x)\Big) \quad . \tag{42}$$

This result means that the action-angle-like transformations are only allowed in the Hamilton-like systems; furthermore, if the constant of motion of the autonomous system (2) is not of the form (13), an action-angle-like transformation is not a correct approach (which is evident if the motion is not periodic). The approach presented above will be applied to the radiation damping in the following section.

7. RADIATION DAMPING

When a beam of charged particles (electrons or protons) is moving around its orbit, in a stored ring accelerator, it suffers many electromagnetic perturbations produced by external sources (magnets, rf-cavities). These perturbations shape, transport, and accelerate the beam along its orbit, also induce oscillation in its motion (betatron, synchrotron oscillations). The charged particles emit electromagnetic radiation any time they are accelerated (tangentially or orthogonally to its motion). This emitted radiation reduces the energy of the beam and that of the individual charged particles in it. The energy lost by the beam can be compensated using a little bit more rf-power to maintain the beam in its designed orbit. The motion of the individual charged particles of the beam is damped relative to the synchronous particle in the beam [6] bringing about a reduction of the beam size and in its emittance which is called "*radiation damping.*" This damping effect is an important parameter to consider in high-energy accelerators. The "luminosity" [7] in these accelerators is inversely proportional to the

emittance of the beam, and represents a measure of the probability to have a specific event (creation of the desired particle) during the collision of two beams; thus, an increasing or decreasing in the luminosity parameter means a good or bad chance to "see" what is wanted in a collision experiment.

The damping effect in the charged particles of the beam can be studied at first approximation [6, 8] as a simple harmonic oscillation in their relative energy, $\mathbf{x} = \mathbf{E} - \mathbf{E_o}$, $\mathbf{E}$ being the energy of the charge and $\mathbf{E_o}$ the energy of the synchronous charge, damped by a friction term proportional to the rate of change in the relative energy-per-turn, $\mathbf{v} = \mathbf{dx/dt}$, that is, the equation which describes this phenomenon can be given by

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \Omega_o^2\, x = 0 \quad , \tag{43}$$

where Ω_o is the circular synchroton frequency that is given in terms of the revolution time, T_o, energy of the synchronous particle, E_o, change of the energy of the particle with respect the synchronous particle, edV/dT, and a constant, α_o, which depends on the magnetic guide field as

$$\Omega_o^2 = \frac{e}{T_o}\frac{dV}{dT}\frac{\alpha_o}{E_o} \quad ; \tag{44a}$$

and α is the damping rate coefficient that can be given in terms of the variation of energy lost per turn with respect of the energy of the particle, dU/dE, as

$$\alpha = \frac{1}{2T_o}\frac{dU}{dE} \quad . \tag{44b}$$

Equation (43) can be written as the following one-dimensional-autonomous dynamical system

$$\frac{dv}{dt} = -(\Omega_o^2\, x + 2\alpha v) \quad , \tag{45a}$$

and

$$\frac{dx}{dt} = v \quad . \tag{45b}$$

The constant of motion associated with this system can be found using equation (4) with the function $F(x,v)$ given by the right hand side of equation (45a). From the first two terms of (5) and after some arrangements, the following differential equation can be obtained for the characteristic curve:

$$\frac{dv}{dt} = -\left(\Omega_o^2 \frac{x}{v} + 2\alpha\right), \tag{46}$$

which can be set in an integrable expression by choosing a new variable, ξ, given by

$$\xi = \frac{x}{v} \quad . \tag{47}$$

The differential equation in terms of this variable is

$$\frac{d\xi}{dx} = \frac{\xi}{x}\left(1 + \Omega_o^2 \xi^2 + 2\alpha\xi\right) \quad , \tag{48}$$

and as the result of its integration, [9] the following characteristic curve is obtained:

$$C = \log\left(v^2 + \Omega_o^2 x^2 + 2\alpha x v\right) + 2\alpha G\left(\alpha, \Omega_o, \frac{x}{v}\right) \quad , \tag{49a}$$

where G has been defined as

$$G\left(\alpha, \Omega_o, \frac{x}{v}\right) = \begin{cases} \frac{1}{2\sqrt{\alpha^2-\Omega_o^2}} \log\left[\frac{\left(\alpha+\Omega_o^2\frac{x}{v}\right)-\sqrt{\alpha^2-\Omega_o^2}}{\left(\alpha+\Omega_o^2\frac{x}{v}\right)+\sqrt{\alpha^2-\Omega_o^2}}\right], & \text{if } \Omega_o^2 < \alpha^2; \\ -\frac{1}{\alpha+\frac{\Omega_o^2 x}{v}}, & \text{if } \Omega_o^2 = \alpha^2; \\ \frac{1}{\sqrt{\Omega_o^2-\alpha^2}} \arctan\left[\frac{\alpha+\Omega_o^2\frac{x}{v}}{\sqrt{\Omega_o^2-\alpha^2}}\right], & \text{if } \Omega_o^2 > \alpha^2. \end{cases} \tag{49b}$$

The constant of motion which has the correct limit (usual energy expression), when the dissipated coefficient α goes to zero, is given by [1]

$$K = \frac{1}{2}\exp(C) \quad , \tag{50}$$

and can be chosen as the constant of motion for the autonomous system (45) which is given in terms of the variables "x" and "v" as

$$K(x,v) = \frac{1}{2}\Big(v^2 + \Omega_o^2\, x^2 + 2\alpha x v\Big) \exp\left[2\alpha G\left(\alpha, \Omega_o, \frac{x}{v}\right)\right] \quad . \tag{51}$$

This constant of motion is divided into three cases, strong dissipation case ($\Omega^2 < \alpha^2$), critical dissipation case ($\Omega^2 = \alpha^2$), and weak dissipation case ($\Omega^2 > \alpha^2$). The qualitative behavior in the space $\{x, v\}$ of these cases can be seen in Figures 1, 2, and 3.

For very weak radiation levels,

$$\frac{\alpha^2}{\Omega_o^2} \ll 1 \quad , \tag{52}$$

and using the relations (51) and (49b), the constant of motion can be given at first order of approximation in α as

$$K(x,v) = \frac{1}{2}(v^2 + \Omega_o^2\, x^2) + 2\alpha\left[xv + \frac{1}{2\Omega_o}(v^2 + \Omega_o^2\, x^2)\arctan\left(\frac{\Omega_o x}{v}\right)\right] \quad . \tag{53}$$

Thus, it can be seen in sections 4 and 6, even with a very small dissipation term, the autonomous system (45) does not correspond to a Hamilton-like system and an action-angle-like procedure would be not consistent.

It can be seen from Figure 3, the curve, $K = \textit{constant}$, has a gap per cycle in the phase space when the velocity is zero, $v = 0$, this is due to the "arctan" function that appears in the constant of motion which brings about the spiral behavior in the phase space. The gap size is a measure of the energy lost per cycle by the charge and can be calculated taking the limit on the constant of motion when the velocity goes to zero from both sides. Estimate this size for the very weak radiation level case. Assuming $x > 0$, these limits are given by

$$K = \lim_{v \to 0^+} K(x,v) = \frac{1}{2}\,\Omega_o^2\, x_2^2 + \frac{\pi}{2}\,\alpha\,\Omega_o x_2^2 \tag{54a}$$

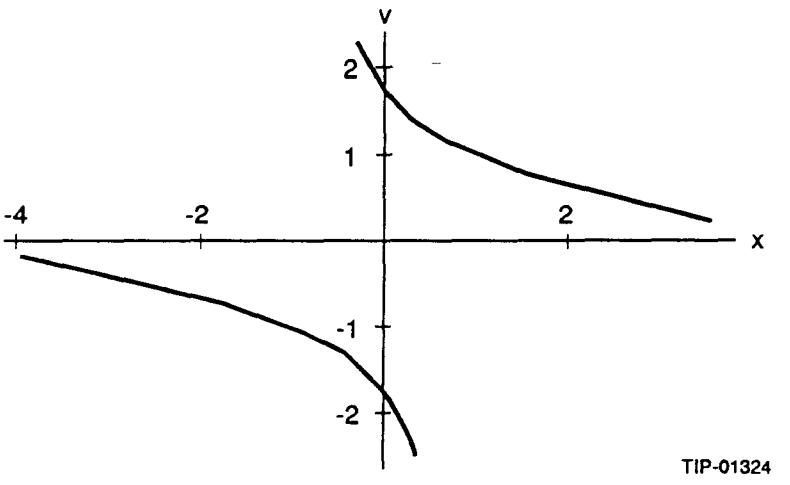

Figure 1. Strong Dissipated Case $\Omega_o^2/\alpha^2 = 1/3^2$, $K = 10$.

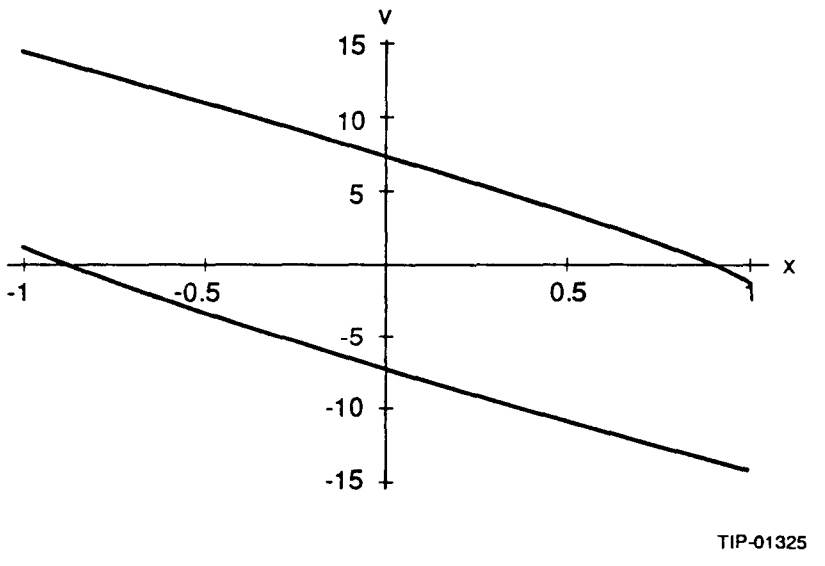

Figure 2. Critical Dissipated Case $\Omega_o^2 = \alpha^2 = 25$, $K = 10$.

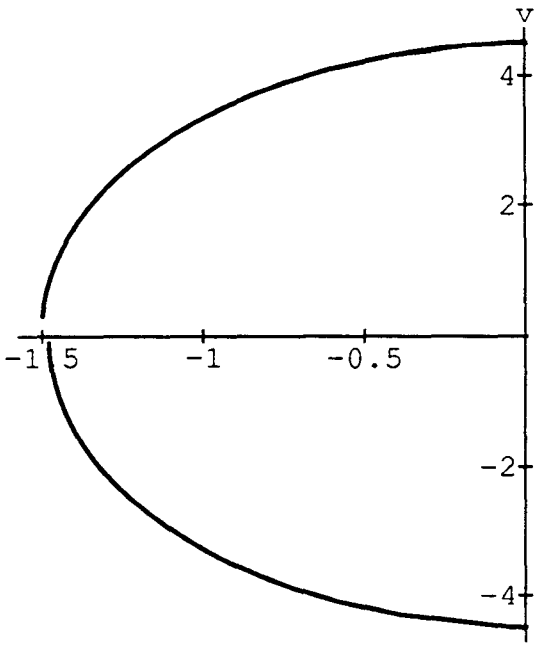

Figure 3. Weak Dissipated Case $\Omega_o^2/\alpha^2 = 3^2/(0.015)^2$, $K = 10$.

and

$$K = \lim_{v \to 0^-} K(x, v) = \frac{1}{2}\Omega_o^2 x_1^2 - \frac{\pi}{2}\alpha\Omega_o x_1^2 \quad , \tag{54b}$$

where x_1 and x_2 are the points of intersection in the respective limits. Equaling both expressions and making arrangements, the following relation between both points is obtained:

$$x_2^2 - x_1^2 + \frac{\pi\alpha}{\Omega_o}\left(x_2^2 + x_1^2\right) = 0 \quad . \tag{55}$$

Assume x_2 is of the form

$$x_2 = x_1 + (\delta x_1) \quad , \tag{56}$$

and substitute this expression in (55) to obtain finally the algebraic equation

$$(\delta x_1)^2 + 2x_1(\delta x_1) + \frac{2\pi\alpha/\Omega_o}{1 + \pi\alpha/\Omega_o} x_1^2 = 0 \quad . \tag{57}$$

The roots of this polynomial are

$$(\delta x_1)^{(1)} = x_1 \left[\sqrt{1 - \frac{2\pi\alpha/\Omega_o}{1 + \pi\alpha/\Omega_o}} - 1 \right] \tag{58a}$$

and

$$(\delta x_1)^{(2)} = -x_1 \left[\sqrt{1 - \frac{2\pi\alpha/\Omega_o}{1 + \pi\alpha/\Omega_o}} + 1 \right] \quad , \tag{58b}$$

but only the first root, (58a), has the correct limits when the dissipated parameter, α, goes to zero (if $\alpha = 0$, the result will be $(\delta x_1) = 0$ too). Using condition (52) in (58a), the size of energy lost is given by

$$(\delta x_1) = -x_1 \left(\frac{\pi\alpha}{\Omega_o} \right) \quad , \tag{59a}$$

and it is not difficult to see that in the case $x_1 < 0$, the jump is given by

$$(\delta x_1) = x_1 \left(\frac{\pi\alpha}{\Omega_o} \right) \quad . \tag{59b}$$

For illustration, the numerical value of $\pi\alpha/\Omega_o$ for the pp-SSC accelerator at 20 Tev [10] is of the order of 10^{-8}, meanwhile for the $e^-e^+ - LEP$ accelerator at 55 Gev [8] is above 1.55×10^{-2}. This means it will take the order of 10^8 turns for a proton beam to shrink about 63% of its initial emittance in the SSC accelerator (the order of 65 turns for the electron beam (or positron beam) in the LEP accelerator). This shrinking effect in the beam can not continue long because the electrostatic repulsion of the charges and the quantum fluctuations of the radiation. [11] These effects will require a more elaborated study.

8. CONCLUSIONS

Dissipated systems (in particular, Synchrotron Radiation Damping) represent a challenge for a consistent Hamiltonian and Lagrangian formulation. In

order to understand the problems presented by these systems, a new approach has been formulated on the basis of the Constant of Motion for an Autonomous System. This approach is reduced to the "*standard*" formulation (Energy, Hamiltonian, Poisson bracket, canonical transformation and Liouville's theorem) when the constant of motion is of the form $\frac{1}{2}v^2 + V(x)$.

This approach was used to study the radiation damping of a proton in a bunch which is traveling in the orbit of the SSC accelerator. Guided by the *Nondissipated limit*, that is, the condition imposed in the solutions to have "the usual expressions" when the dissipated coefficient goes to zero, the approach brings about a constant of motion for the dissipated model, (52). This one can be simplified even more in the limit of very weak radiation levels, (53). Using this constant of motion, the relation was obtained (59) which represents the variation in the energy-per-cycle of the charged particle in the beam traveling in the accelerator ring. This dissipated system does not correspond to a Hamilton-like system and an action-angle-like transformation is not a consistent procedure. The change of the emittance with time and its relation with the luminosity require a more careful analysis since Luminosity depends on other parameters which change with time too. [12]

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