SSCL-302

NORMALIZATION OF THE PARAMETERIZED COURANT-SNYDER MATRIX

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Abstract

An explicit normal form of the parameterized Courant-Snyder matrix is obtained by symplectic factorization. The explicit calculation of the δ- (off-momentum-) dependent tune $\mu(\delta)$ and Twiss parameters $\beta(\delta), \gamma(\delta), \alpha(\delta)$ may have applications in accelerator lattice design. Furthermore, being able to symplectify a truncated parameterized Courant-Snyder matrix is a key step to fully parameterized map studies in nonlinear dynamics.

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1. INTRODUCTION

An obtainable Courant-Snyder matrix parameterized for off-momentum transverse propagation relative to the off-momentum closed orbit in an alternating synchrotron can be expanded in the form

$$M(\delta) = M_0 + \delta M_1 + \delta^2 M_2 + \cdots + \delta^n M_n + \sigma(\delta^{n+1}) \ ,$$

where $\delta = \Delta P/P_0$ is a parameter for the off-momentum. The linear periodic evolution of the transverse phase space coordinates are given by

$$\vec{x}^{\,i+1} = M(\delta)\vec{x}^{\,i} \ ,$$

where the transverse phase space coordinates, $\vec{x}^{\,i}$, are relative to the periodic closed orbit which is a polynomial of δ. In other words, the dispersion due to off-momentum is contained in the δ-dependent closed orbit. Note that $\vec{x}^{\,i} = \left[x^i, p_x^i, y^i, p_y^i\right]$ (or $\vec{x}^{\,i} = \left[x^i, p_x^i\right]$ for non-coupled beam dynamics). The matrix M_0 is symplectic, but $M(\delta)$ is generally not. It is symplectic only up to the truncated order δ^n, that is, $M^T(\delta)SM(\delta) = S + \sigma(\delta^{n+1})$, where S is the symplectic identity given by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{for } 2\times 2 \text{ matrices} \ .$$

In this paper, a method for symplectifying the truncated parameterized Courant-Snyder Matrix is presented. This method will simultaneously convert a parameterized Courant-Snyder matrix into a parameterized normal form. A single-parameter, 2×2 parameterized Courant-Snyder Matrix is considered. Extension to multi-parameter, $2n \times 2n$ parameterized Courant-Snyder Matrix is possible with the use of computer. The main task is to make a series of canonical

transformations such that

$$R_n(\delta) = \begin{pmatrix} \cos\mu(\delta) & \sin\mu(\delta) \\ -\sin\mu(\delta) & \cos\mu(\delta) \end{pmatrix}$$

$$= A^{-1}(\delta)M(\delta)A(\delta) + \sigma(\delta^{n+1})$$

so that neglecting terms with order higher than δ^n, a parameterized Courant-Snyder Matrix M(δ) can be represented by a symplectic matrix given by

$$M_s(\delta) = A(\delta)R_n(\delta)A^{-1}(\delta)$$

$$= \begin{pmatrix} \cos\mu(\delta) + \alpha(\delta)\sin\mu(\delta) & \beta(\delta)\sin\mu(\delta) \\ -\gamma(\delta)\sin\mu(\delta) & \cos\mu(\delta) - \alpha(\delta)\sin\mu(\delta) \end{pmatrix} ,$$

where the series of canonical generation matrices have been concatenated into a single symplectic matrix A(δ), and

$$\mu(\delta) = \mu_0 + \delta\mu_1 + \delta^2\mu_2 + \cdots + \delta^n\mu_n ,$$

$$\beta(\delta) = \beta_0 + \delta\beta_1 + \delta^2\beta_2 + \cdots + \delta^n\beta_n + \ldots + \delta^{n(n+1)}\beta_{n(n+1)} ,$$

$$\gamma(\delta) = \gamma_0 + \delta\gamma_1 + \delta^2\gamma_2 + \cdots + \delta^n\gamma_n + \ldots + \delta^{n(n+1)}\gamma_{n(n+1)} ,$$

$$\alpha(\delta) = \alpha_0 + \delta\alpha_1 + \delta^2\alpha_2 + \cdots + \delta^n\alpha_n + \ldots + \delta^{n(n+1)}\alpha_{n(n+1)} ,$$

with

$$1 + \alpha^2(\delta) = \beta(\delta)\gamma(\delta).$$

The parameterized tune $\mu(\delta)$ and Twiss parameters $\beta(\delta), \alpha(\delta)$, and $\gamma(\delta)$ can be explicitly calculated through the order-by-order normalization of the parameterized Courant-Snyder matrix.

2. NORMALIZATION OF THE 0th ORDER

We begin with the normalization of the 0th order and will consider stable motion only. The main task is to find a symplectic generation matrix A_0 and its inverse A_0^{-1} (also symplectic) such that

$$R_0 = \begin{pmatrix} \cos\mu_0 & \sin\mu_0 \\ -\sin\mu_0 & \cos\mu_0 \end{pmatrix} = A_0^{-1} M_0 A_0 \ . \tag{1}$$

This has been well understood since the proposal of the alternating-gradient synchrotron. [1] Let

$$M_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} ,$$

with (a) $a_0 d_0 - b_0 c_0 = 1$ for M_0 to be symplectic and (b) $\left|\frac{1}{2}(a_0 + d_0)\right| < 1$ and $b_0 c_0 < 0$ for stable motion. Then

$$\cos\mu_0 = \frac{1}{2}\,\mathrm{Tr}(M_0) = \frac{1}{2}(a_0 + d_0) \ .$$

With a unique solution for μ_0 by choosing

$$\sin\mu_0 = -\frac{c_0}{|c_0|}\sqrt{1 - \cos^2\mu_0},$$

the generation matrix A_0 is generally given by

$$A_0 = \begin{pmatrix} \dfrac{y(a_0 - \cos\mu_0) - x\sin\mu_0}{c_0\sqrt{\beta_0}} & \dfrac{x(a_0 - \cos\mu_0) + y\sin\mu_0}{c_0\sqrt{\beta_0}} \\ \\ \dfrac{y}{\sqrt{\beta_0}} & \dfrac{x}{\sqrt{\beta_0}} \end{pmatrix}$$

where

$$\beta_0 = -\frac{1}{c_0}(x^2 + y^2)\sin\mu_0 > 0$$

and x and y can be any real numbers. By choosing $x = 1$ and $y = -(a_0 - \cos\mu_0)/(\sin\mu_0)$, we have the Courant-Snyder's choice that [1]

$$A_0 = \begin{pmatrix} \sqrt{\beta_0} & 0 \\ \frac{-\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} , \tag{2}$$

$$A_0^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix} ,$$

where

$$\beta_0 = \frac{b_0}{\sin\mu_0} ,$$

$$\alpha_0 = \frac{a_0 - d_0}{2\sin\mu_0} .$$

M_0 is then given by

$$M_0 = A_0 R_0 A_0^{-1} = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix} ,$$

where

$$\gamma_0 = \frac{1 + \alpha_0^2}{\beta_0} .$$

The above solution can be easily checked by substituting A_0 and M_0 into Eq. (1).

3. NORMALIZATION OF THE 1st ORDER

We now proceed for the normaliztion of the first order by first making a canonical transformation of $M(\delta)$ as follows:

$$\begin{aligned} {}_1M(\delta) = A_0^{-1} M(\delta) A_0 &= A_0^{-1} M_0 A_0 + \delta A_0^{-1} M_1 A_0 + \cdots + \delta^n A_0^{-1} M_n A_0 + \sigma(\delta^{n+1}) \\ &= R_0 + \delta\ {}_1M_1 + \delta^2\ {}_1M_2 + \cdots + \delta^n\ {}_1M_n + \sigma(\delta^{n+1}) . \end{aligned}$$

Since A_0 and A_0^{-1} are symplectic, ${}_1M(\delta)$ is symplectic up to order δ^n, the same as $M(\delta)$, that is,

$$ {}_1M^T(\delta) S \, {}_1M(\delta) = S + \sigma(\delta^{n+1}) \, . $$

This, together with R_0 being symplectic ($R_0^T S R_0 = S$), yields the following necessary conditions:

$$ \begin{aligned} \delta^1 &: R_0^T S \, {}_1M_1 + {}_1M_1^T S R_0 = 0 \, , \\ &\vdots \\ \delta^n &: R_0^T S \, {}_1M_n + {}_1M_n^T S R_0 + \cdots + {}_1\bar{S}_n = 0 \end{aligned} \tag{3} $$

where

$$ \begin{cases} {}_1\bar{S}_n = {}_1M_{n/2}^T S \, {}_1M_{n/2} & \text{if } n \text{ is even,} \\ {}_1\bar{S}_n = {}_1M_{(n-1)/2}^T S \, {}_1M_{(n+1)/2} + {}_1M_{(n+1)/2}^T S \, {}_1M_{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases} $$

The main task is then to find a δ-dependent symplectic generation matrix $\bar{A}_1(\delta) = I + \delta A_1$ such that

$$ \begin{aligned} R_1(\delta) &= \begin{pmatrix} \cos(\mu_0 + \delta\mu_1) & \sin(\mu_0 + \delta\mu_1) \\ -\sin(\mu_0 + \delta\mu_1) & \cos(\mu_0 + \delta\mu_1) \end{pmatrix} \\ &= \bar{A}_1^{-1}(\delta)(R_0 + \delta \, {}_1M_1)\bar{A}_1(\delta) + \sigma(\delta^2) \, , \end{aligned} \tag{4} $$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix and A_1 is independent of δ.

There are two necessary conditions for $\bar{A}_1(\delta)$ to be symplectic, i.e. $\bar{A}_1^T(\delta) S \bar{A}_1(\delta) = S$, which are

$$ A_1^T S + S A_1 = 0 \Leftrightarrow \mathrm{Tr}(A_1) = 0 $$

and

$$A_1^T S A_1 = 0 \Leftrightarrow \det(A_1) = 0$$

and so $A_1 A_1 = 0$. We will call such a matrix (A_1) as a nil-potent matrix. Therefore, the inverse matrix of $\tilde{A}_1(\delta)$ is simply

$$\bar{A}_1(\delta) = I - \delta A_1 \ ,$$

which is also symplectic. Let

$${}_1M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

with a constraint that $(a_1 + d_1)\cos\mu_0 = (c_1 - b_1)\sin\mu_0$ since $R_0^T S \ {}_1M_1 + {}_1M_1^T S R_0 = 0$. Expanding Eq. (4) up to order δ, we obtain

$$\delta\mu_1 \begin{pmatrix} -\sin\mu_0 & \cos\mu_0 \\ -\cos\mu_0 & -\sin\mu_0 \end{pmatrix} = \delta(R_0 A_1 - A_1 R_0 + {}_1M_1) + \sigma(\delta^2) \ ,$$

that is,

$$\mu_1 \begin{pmatrix} -\sin\mu_0 & \cos\mu_0 \\ -\cos\mu_0 & -\sin\mu_0 \end{pmatrix} = R_0 A_1 - A_1 R_0 + {}_1M_1 \ . \tag{5}$$

With the given constraints on A_1 : $\mathrm{Tr}(A_1) = 0$ and $\det(A_1) = 0$ and on ${}_1M_1$: $(a_1 + d_1)\cos\mu_0 = (c_1 - b_1)\sin\mu_0$, we obtain

$$\mu_1 = -\frac{\mathrm{Tr}({}_1M_1)}{2\sin\mu_0} = -\frac{a_1 + d_1}{2\sin\mu_0} \ ,$$

and

$$A_1 = \frac{1}{4\sin\mu_0} \begin{pmatrix} b_1 + c_1 & d_1 - a_1 \pm \sqrt{h_1} \\ d_1 - a_1 \mp \sqrt{h_1} & -(b_1 + c_1) \end{pmatrix} \ ,$$

where

$$h_1 = (a_1 - d_1)^2 + (b_1 + c_1)^2$$

$$= \left[\frac{\mathrm{Tr}({}_1M_1)}{\sin\mu_0}\right]^2 - 4\det({}_1M_1)\ .$$

The above solutions can be easily checked by substituting $\bar{A}_1 = I + \delta A_1$ and ${}_1M_1$ into Eq. (2). Note that the constraint that A_1 is a nil-potent matrix can also be easily checked.

4. NORMALIZATION OF THE 2nd ORDER

Normalization of the second order begins with a canonical transformation of ${}_1M(\delta)$ as follows:

$$\begin{aligned}
{}_2M(\delta) &= \bar{A}_1^{-1}(\delta)\ {}_1M(\delta)\bar{A}_1(\delta) \\
&= (I - \delta A_1)\ {}_1M(I + \delta A_1) \\
&= (I - \delta A_1)(R_0 + \delta\ {}_1M_1)(I + \delta A_1) + \delta^2(I - \delta A_1)\ {}_1M_2(I + \delta A_1) + \cdots \\
&\quad + \delta^n(I - \delta A_1)\ {}_1M_n(I + \delta A_1) + \sigma(\delta^{n+1})\ ,
\end{aligned}$$

and so

$${}_2M(\delta) = R_1(\delta) + \delta^2{}_2M_2 + \delta^3{}_2M_3 + \cdots + \delta^n{}_2M_n + \sigma(\delta^{n+1})\ ,$$

where $R_1(\delta)$ is given in Eq. (4). Similar to ${}_1M$, ${}_2M$ is symplectic up to order δ^n, since $\bar{A}_1(\delta)$ and its inverse $\bar{A}_1^{-1}(\delta)$ are both symplectic, that is,

$${}_2M^T S\ {}_2M = S + \sigma(\delta^{n+1})\ .$$

This, together with $R_1^T(\delta)S\ R_1(\delta) = S$, yields one of the necessary conditions:

$$\delta^2 : R_0^T S\ {}_2M_2 + {}_2M_2^T S\ R_0^T = 0 \tag{6}$$

Let

$$ {}_2M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} . $$

Similar to the constraint of ${}_1M_1$, Eq. (6) yields the following constraint for ${}_2M_2$:

$$ (a_2 + d_2)\cos\mu_0 = (c_2 - b_2)\sin\mu_0 . $$

The main task is then to find a symplectic generation matrix $\bar{A}_2(\delta) = (I + \delta^2 A_2)$ such that

$$ \begin{aligned} R_2(\delta) &= \begin{pmatrix} \cos(\mu_0 + \delta\mu_1 + \delta^2\mu_2) & \sin(\mu_0 + \delta\mu_1 + \delta^2\mu_2) \\ -\sin(\mu_0 + \delta\mu_1 + \delta^2\mu_2) & \cos(\mu_0 + \delta\mu_1 + \delta^2\mu_2) \end{pmatrix} \\ &= \bar{A}_2^{-1}(\delta)(R_1 + \delta^2\, {}_2M_2)\bar{A}_2(\delta) + \sigma(\delta^3) . \end{aligned} \tag{7} $$

Similar to A_1, two conditions for A_2 given by

$$ A_2^T S + S A_2 = 0 \Leftrightarrow \mathrm{Tr}(A_2) = 0 $$

and

$$ A_2^T S A_2 = 0 \Leftrightarrow \det(A_2) = 0 $$

are necessary for $\bar{A}_2(\delta) = I + \delta^2 A_2$ to be symplectic ($\bar{A}_2^T S \bar{A}_2 = S$). Therefore, A_2 is a nil-potent matrix, i.e. $A_2 A_2 = 0$, and so the inverse of $\bar{A}_2(\delta)$ is simply

$$ \bar{A}_2^{-1}(\delta) = I - \delta^2 A_2 , $$

which is also symplectic.

Expanding Eq. (7) up to order δ^2, we obtain

$$\delta^2 \mu_2 \begin{pmatrix} -\sin\mu_0 & \cos\mu_0 \\ -\cos\mu_0 & -\sin\mu_0 \end{pmatrix} = \delta^2 (R_0 A_2 - A_2 R_0 + {}_2M_2) + \sigma(\delta^3) \,,$$

that is,

$$\mu_2 \begin{pmatrix} -\sin\mu_0 & \cos\mu_0 \\ -\cos\mu_0 & -\sin\mu_0 \end{pmatrix} = R_0 A_2 - A_2 R_0 + {}_2M_2 \,. \tag{8}$$

Note that Eq. (8) is similar to Eq. (5) and that A_2 and ${}_2M_2$ have similar constraints as A_1 and ${}_1M_1$, respectively. Therefore the solution for μ_2 and A_2 should be similar to μ_1 and A_1, respectively. They are given by

$$\mu_2 = -\frac{\mathrm{Tr}({}_2M_2)}{2\sin\mu_0} = -\frac{a_2 + d_2}{2\sin\mu_0} \,,$$

and

$$A_2 = \frac{1}{4\sin\mu_0} \begin{pmatrix} b_2 + c_2 & d_2 - a_2 \pm \sqrt{h_2} \\ d_2 - a_2 \mp \sqrt{h_2} & -(b_2 + c_2) \end{pmatrix} \,,$$

where

$$h_2 = (a_2 - d_2)^2 + (b_2 + c_2)^2 = \left[\frac{\mathrm{Tr}({}_2M_2)}{\sin\mu_0}\right]^2 - 4\det({}_2M_2) \,.$$

The above solution can be checked by substituting $\bar{A}_2(\delta) = (I + \delta^2 A_2)$ and ${}_2M_2$ into Eq. (7).

5. ITERATION FOR NORMALIZATION UP TO Nth ORDER

The process (in section 4) of 2nd–order normalization being similar to the process (in section 3) of the 1st–order normalization gives us little doubt that we could follow the same process to normalize higher orders by iteration. The solution can always be checked afterwards.

Let us assume that an $(i-1)^{\text{th}}$-order generation matrix $\bar{A}_{i-1}(\delta) = I + \delta^{i-1}A_{i-1}$ has been obtained and a canonical transformation has been made such that

$$\begin{aligned} {}_iM(\delta) &= \bar{A}_{i-1}^{-1}(\delta)\ {}_{i-1}M(\delta)\bar{A}_{i-1}(\delta) \\ &= R_{i-1}(\delta) + \delta^i{}_iM_i + \delta^{i+1}{}_iM_{i+1} + \cdots + \delta^n{}_iM_n + \sigma(\delta^{n+1}) \end{aligned}$$

where $R_{i-1}(\delta)$ is a rotation given by

$$R_{i-1}(\delta) = \begin{pmatrix} \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^{i-1}\mu_{i-1}) & \sin(\mu_0 + \delta\mu_1 + \cdots + \delta^{i-1}\mu_{i-1}) \\ -\sin(\mu_0 + \delta\mu_1 + \cdots + \delta^{i-1}\mu_{i-1}) & \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^{i-1}\mu_{i-1}) \end{pmatrix} ,$$

and

$${}_iM_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} ,$$

with a constraint that $(a_i + d_i)\cos\mu_0 = (c_i - b_i)\sin\mu_0$, which is one of the necessary conditions from the property that ${}_iM$ is symplectic up to n^{th} order. The ith order symplectic generation matrix $\bar{A}_i(\delta)$ is then given by

$$\bar{A}_i(\delta) = I + \delta^i A_i ,$$

where

$$A_i = \frac{1}{4\sin\mu_0}\begin{pmatrix} b_i + c_i & d_i - a_i \pm \sqrt{h_i} \\ d_i - a_i \mp \sqrt{h_i} & -(b_i + c_i) \end{pmatrix} , \tag{9}$$

and

$$h_i = (a_i - d_i)^2 + (b_i + c_i)^2 = \left[\frac{\mathrm{Tr}({}_iM_i)}{\sin\mu_0}\right]^2 - 4\det({}_iM_i) .$$

A canonical transformation can then be made to obtain ${}_{i+1}M$ as follows:

$$\begin{aligned} {}_{i+1}M(\delta) &= (I - \delta^i A_i){}_iM(\delta)(I + \delta^i A_i) \\ &= (I - \delta^i A_i)(R_{i-1}(\delta) + \delta^i{}_iM_i)(I + \delta^i A_i) \\ &\quad + \delta^{i+1}(I - \delta^i A_i){}_iM_{i+1}(I + \delta^i A_i) \\ &\quad + \cdots + \delta^n(I - \delta^i A_i){}_iM_n(I + \delta^i A_i) + \sigma(\delta^{n+1}) \\ &= R_i(\delta) + \delta^{i+1}{}_{i+1}M_{i+1} + \delta^{i+2}{}_{i+1}M_{i+2} + \cdots + \delta^n{}_{i+1}M_n + \sigma(\delta^{n+1}) , \end{aligned}$$

where the rotation $R_i(\delta)$ is given by

$$R_i(\delta) = \begin{pmatrix} \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^i\mu_i) & \sin(\mu_0 + \delta\mu_1 + \cdots + \delta^i\mu_i) \\ -\sin(\mu_0 + \delta\mu_1 + \cdots + \delta^i\mu_i) & \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^i\mu_i) \end{pmatrix} ,$$

and

$$\mu_i = -\frac{\mathrm{Tr}({}_iM_i)}{2\sin\mu_0} = -\frac{a_i + d_i}{2\sin\mu_0} .$$

The above process is then iterated until we obtain the nth order symplectic generation matrix $\bar{A}_n(\delta) = I + \delta^n A_n$ and make the $(n+1)^{\text{th}}$ canonical transformation to obtain

$$(I - \delta^n A_n)_n M (I + \delta^n A_n) = R_n(\delta) + \sigma(\delta^{n+1}) .$$

6. THE PARAMETERIZED TWISS PARAMETERS

Tracing back the n+1 canonical transformations, we have

$$\begin{aligned} M(\delta) = {} & A_0(I + \delta A_1)(I + \delta^2 A_2)\cdots(I + \delta^n A_n)R_n(\delta)(I - \delta^n A_n) \\ & \cdots(I - \delta^2 A_2)(I - \delta A_1)A_0^{-1} + \sigma(\delta^{n+1}) , \end{aligned}$$

where

$$R_n(\delta) = \begin{pmatrix} \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^n\mu_n) & \sin(\mu_0 + \delta\mu_1 + \cdots + \delta^n\mu_n) \\ -\sin(\mu_0 + \delta\mu_1 + \cdots + \delta^n\mu_n) & \cos(\mu_0 + \delta\mu_1 + \cdots + \delta^n\mu_n) \end{pmatrix}$$

is a rotation (a rotation is always symplectic); A_0 and its inverse A_0^{-1} are the usual Courant-Snyder generation matrix obtained from M_0 (note: $M_0^T S M_0 = S$); $(I + \delta^i A_i)$ and their inverses $(I - \delta^i A_i)$ for $i = 1, 2, \ldots, n$, are the order-by-order generation matrices, which are all symplectic.

In a practical case, $\sigma(\delta^{n+1})$ is usually negligible if δ is small and n is large enough, so that $M(\delta)$ can be represented by the symplectic matrix

$$\begin{aligned} M_s(\delta) = {} & A_0(I + \delta A_1)(I + \delta^2 A_2)\cdots(I + \delta^n A_n)R_n(\delta)(I - \delta^n A_n) \\ & \cdots(I - \delta^2 A_2)(I - \delta A_1)A_0^{-1} . \end{aligned} \tag{10}$$

One of the remaining interesting studies would be to find the parameterized Twiss parameters, that is, to find $\beta(\delta)$, $\gamma(\delta)$, and $\alpha(\delta)$ in the form of power series

expansion of δ to their required complete order $\delta^{n(n+1)}$ such that

$$\beta(\delta) = \beta_0 + \delta\beta_1 + \delta^2\beta_2 + \cdots + \delta^{n(n+1)}\beta_{n(n+1)},$$

$$\gamma(\delta) = \gamma_0 + \delta\gamma_1 + \delta^2\gamma_2 + \cdots + \delta^{n(n+1)}\gamma_{n(n+1)},$$

$$\alpha(\delta) = \alpha_0 + \delta\alpha_1 + \delta^2\alpha_2 + \cdots + \delta^{n(n+1)}\alpha_{n(n+1)},$$

and

$$1 + \alpha^2(\delta) = \beta(\delta)\gamma(\delta).$$

The parameterized Courant-Snyder matrix can then be expressed as:

$$\begin{aligned} M_s &= \begin{pmatrix} \cos\mu(\delta) + \alpha(\delta)\sin\mu(\delta) & \beta(\delta)\sin\mu(\delta) \\ -\gamma(\delta)\sin\mu(\delta) & \cos\mu(\delta) - \alpha(\delta)\sin\mu(\delta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\mu(\delta) + \alpha_0\sin\mu(\delta) & \beta_0\sin\mu(\delta) \\ -\gamma_0\sin\mu(\delta) & \cos\mu(\delta) - \alpha_0\sin\mu(\delta) \end{pmatrix} \\ &\quad + \delta\sin\mu(\delta)\begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} + \delta^2\sin\mu(\delta)\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} + \cdots \\ &\quad + \delta^{n(n+1)}\sin\mu(\delta)\begin{pmatrix} \alpha_{n(n+1)} & \beta_{n(n+1)} \\ -\gamma_{n(n+1)} & -\alpha_{n(n+1)} \end{pmatrix}. \end{aligned} \tag{11}$$

To find β_i, γ_i, and α_i for i = 1, 2, $\cdots$, n(n+1), one first reform Eq. (10) as follows:

$$\begin{aligned} M_s =& A_0R_n(\delta)A_0^{-1} + \delta A_0[A_1R_n(\delta) - R_n(\delta)A_1]A_0^{-1} \\ &+ \delta^2 A_0[A_2R_n(\delta) - R_n(\delta)A_2 - A_1R_n(\delta)A_1]A_0^{-1} + \cdots . \end{aligned} \tag{12}$$

One then substitutes Eq. (2) for A_0 and A_0^{-1} and Eq. (9) for A_i, i=1,2, $\cdots$, n, into Eq. (12) and then compares it with Eq. (11). While β_i, γ_i, and α_i for i = 3, $\cdots$,

n(n+1), are lengthy but staightforward and so left as an exercise for interested readers, the first-order and the second-order perturbation Twiss parameters, are given as follows:

$$\beta_1 = \frac{\beta_0(b_1 + c_1)}{2\sin\mu_0},$$

$$\alpha_1 = \frac{\alpha_0(b_1 + c_1) + (a_1 - d_1)}{2\sin\mu_0},$$

$$\gamma_1 = \frac{\alpha_0^2(b_1 + c_1) + 2\alpha_0(a_1 - d_1) - (b_1 + c_1)}{2\beta_0\sin\mu_0},$$

$$\beta_2 = \frac{\beta_0(b_2 + c_2)}{2\sin\mu_0} + \frac{\beta_0[h_1 \mp (a_1 - d_1)\sqrt{h_1}]}{8\sin\mu_0},$$

$$\alpha_2 = \frac{\alpha_0(b_2 + c_2) + (a_2 - d_2)}{2\sin\mu_0} + \frac{\pm(b_1 + c_1)\sqrt{h_1} + \alpha_0 h_1 \mp \alpha_0(a_1 - d_1)\sqrt{h_1}}{8\sin\mu_0},$$

$$\gamma_2 = \frac{\alpha_0^2(b_2 + c_2) + 2\alpha_0(a_2 - d_2) - (b_2 + c_2)}{2\beta_0\sin\mu_0}$$

$$+ \frac{(1 + \alpha_0^2)h_1 \pm (1 - \alpha_0^2)(a_1 - d_1)\sqrt{h_1} \pm 2\alpha_0(b_1 + c_1)\sqrt{h_1}}{8\sin\mu_0}.$$

Note that $a_i = b_i = c_i = d_i = h_i = 0$ for $i > n$.

7. SUMMARY AND DISCUSSION

Given a parameterized truncated Courant-Snyder matrix

$$M(\delta) = M_0 + \delta M_1 + \delta^2 M_2 + \cdots + \delta^n M_n ,$$

a method has been proposed to normalize and symplectically factorize the matrix $M(\delta)$ ($M^T(\delta)SM(\delta) = S + \sigma(\delta^{n+1})$) through $n+1$ canonical transformations so that neglecting orders higher than δ^n, $M(\delta)$ can be represented by a symplectic parameterized matrix $M_s(\delta)$ given by Eq. (10) or Eq. (11).

This proposed method of symplectically factorizing a truncated parameterized Courant-Snyder Matrix together with a parameterized closed-orbit Taylor map (obtainable with the program Zmap [2]) will allow a fully parameterized Dragt-Finn factorization [3], which can be performed with the software available in ZLIB [4]. Note that previously Dragt-Finn factorization was semi-parameterized in a sense that an extra phase-space dimension was included to represent the canonical conjugate of the parameter (the off-momentum), during the factorization process and that only the nonlinear part was parameterized while the linear part was left non-parameterized. [5]

Note that the dispersion due to the parameter δ is contained in the parameterized closed orbit while the chromaticity due to the parameter δ is explicitly expressed in the normal form $R_n(\delta)$, that is, $\delta\mu_1 + \delta^2\mu_2 + \cdots + \delta^n\mu_n$ measures the chromatic condition of the system, where μ_1 is the "chromaticity".

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