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COMMENT ON ROUND-OFF ERRORS AND ON ONE-TURN TAYLOR MAPS

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Abstract

Round-off errors in long-term tracking for the Superconducting Super Collider (SSC) with 64-bit precision (single precision in Cray or normal double precision) are found to be insignificant. One-turn Taylor maps are useful for studying the beam dynamics of the alternating gradient synchrotrons and can potentially be used for fast long-term tracking up to 100 million turns for the SSC.

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During the workshop on nonlinear problems in future particle accelerators at Capri, many colleagues expressed concern about round-off errors in long-term tracking and the suitability of Taylor maps for accelerator study, especially for the study of the long-term behavior of particles in an alternating gradient synchrotron such as the SSC. Given the fact that the nonlinear effects in an alternating gradient synchrotron are due to chromaticities, magnet errors and orbit errors and their associated corrections, which must be small enough compared to the leading linear effects in the domain of interest, a one-turn Taylor map with a modest order should be valid. Therefore, the concern about a one-turn Taylor map is actually not whether it works, but whether an efficient numerical method and a powerful computer are available, that allows a practical computation of a one-turn Taylor map at an order high enough to ensure reliability.

There is no doubt that a one-turn Taylor map such as for the SSC and its associated analysis via such methods as Dragt-Finn factorization[1] or normal forms[2,3] would be valid for the study of short-term behavior of particles. The map can be used to calculate the smear, tune shifts, resonances and other quantities of interest. However, it has always been a controversial issue as to whether a one-turn Taylor map is valid for the study of the long-term behaviour of particles in an alternating gradient synchrotron such as the study of the SSC dynamic aperture.[4]

Recently, an "IMSL"-style differential-algebra library "ZLIB"[5] has been developed for mapping studies that could clarify this issue. "ZLIB" uses dynamic memory and is both vectorized and parallelized (multi-tasked) besides being scalarly optimized to allow fast performance for map extraction, tracking, and analysis. Associated with "ZLIB" is a program "Zmap"[6] for the extraction of a differential-algebra map in a beam line. In particular, "Zmap" can extract a one-turn map from a post-Teapot[7] tracking program "Ztrack"[4]. Extraction of a six-dimensional one-turn map for the SSC up to the 16th order is practically possible. The cpu time needed in each processor to extract a one-turn map for the SSC injection lattice on a eight-processor Cray-2 (C machine in NERSC) is about 30 hours for a 16th-order map and about 1.5 hours for an 11th-order map.

Shown in Figure 1 are two survival plots, in the same frame, comparing tracking data for approximately a million turns from "Ztrack" element-by-element tracking (shown in " " symbol) for an SSC injection lattice and from its associated 11th-order one-turn Taylor map tracking (shown in " " symbol) using routines in "ZLIB". Although the map tracking result and the element-by-element tracking result are not identical, they agree in the global behaviour, as well as in many details up to one million turns. Including the time for extracting the one-turn Taylor map, to obtain a survival plot with resonable detail up to

10 million turns (the required proton-coasting time of the SSC injection lattice), an 11th-order Taylor map tracking is about 40 times faster than the corresponding "Ztrack" element-by-element tracking for the SSC injection lattice. It would be reduced to only about 8 times faster if a 16th-order map is used.

For particles with small (relative to the dynamic aperture) initial transverse and longitudinal amplitudes, there is little doubt that an 11th-order Taylor map tracking for the SSC is as accurate as the element-by-element tracking for all practical purposes. However, interest is in those particles with reasonable longitudinal amplitudes and with transverse amplitudes approximately equal to the radius of the dynamic aperture, if the ultimate goal is to determine the dynamic aperture. To be precise, Figure 1 shows only that the 11th-order Taylor map tracking is valid up to a million turns in this particular case. However, considering the general property of the alternating gradient synchrotron and that increasing an order higher for the Taylor map will generally increase its one-turn-tracking accuracy by 5 to 10 times (nearly one more digit accuracy) for particles under interest (see Figure 2), it seems reasonable to expect that a one-turn Taylor map with an order higher than the 11th but less than the 16th will be valid for the SSC long-term tracking for more than 100 million turns. The one-turn-tracking accuracy for particles under interest is about 7 to 8 digits for an 11th-order Taylor map, due to truncation of higher orders, while it is about 11 digits for an element-by-element tracking using "Ztrack" due to round-off errors with 64-bit precision. Therefore, Figure 1 also shows that the effects due to round-off errors for the SSC long-term tracking are negligible with 64-bit precision. Furthermore, round-off errors will eventually dominate the errors from the truncation of higher orders for a Taylor map tracking, if one increases the order of the map by a few orders over the 11th. Such a Taylor map should be regarded as symplectic from a numerical view point if the extraction of the map is from a symplectic tracking program, which happens to be true for the "Zmap". However, a truncated Taylor map is not symplectic in general.

Recently Irwin[8] and Dragt[9] have proposed methods of converting a Dragt-Finn factorization map[1,10] into kicks for particle tracking. Since a one-turn Taylor map is valid for long-term tracking, a factorization kick map should also be valid, provided: (a) the same order of Taylor map is used to extract the factorization kicks; and (b) high-order (larger than the order of the Taylor map) spurious terms are reasonably suppressed. Research in this direction is in progress. The method proposed by Irwin[8] has been programmed to arbitrary order for some preliminary studies[11] while the method proposed by Dragt[9] with less kicks is currently under implementation in the tracking program "Marylie".[12]

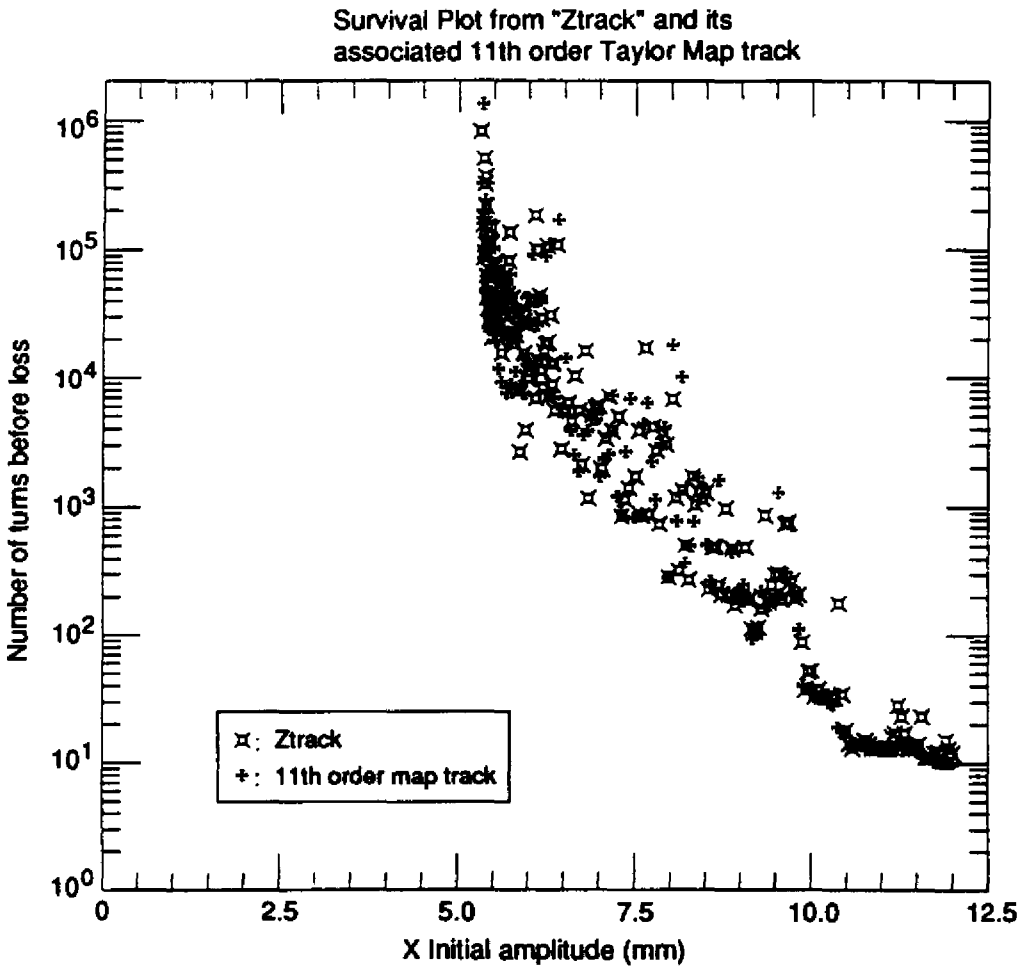

Figure 1. Million-turn survival plots for a 2-TeV injection lattice of the SSC, comparing the data from an 11th-order Taylor map (extracted with "Zmap") tracking with the data from its associated "Ztrack" element-by-element tracking. The two sets of data match quite well, showing that: (a) round-off errors (64-bit precision) for long-term tracking in the SSC is of no concern since "Ztrack" element-by-element tracking is accomplished at an accuracy of 11 digits in one turn, while the 11th-order Taylor map tracking is less accurate by 3 to 4 digits (7 to 8 digits of accuracy) in one turn for the selected region of interest, due to truncation of higher orders, and still generate the same result; and (b) increasing the order of the Taylor map by one or two would allow reliable fast tracking up to 10 million turns for the SSC (the required proton-coasting time of the SSC injection lattice) since increasing one order higher in the Taylor map will enhance nearly 10 times as much accuracy (about one more digit accuracy) in one turn tracking.

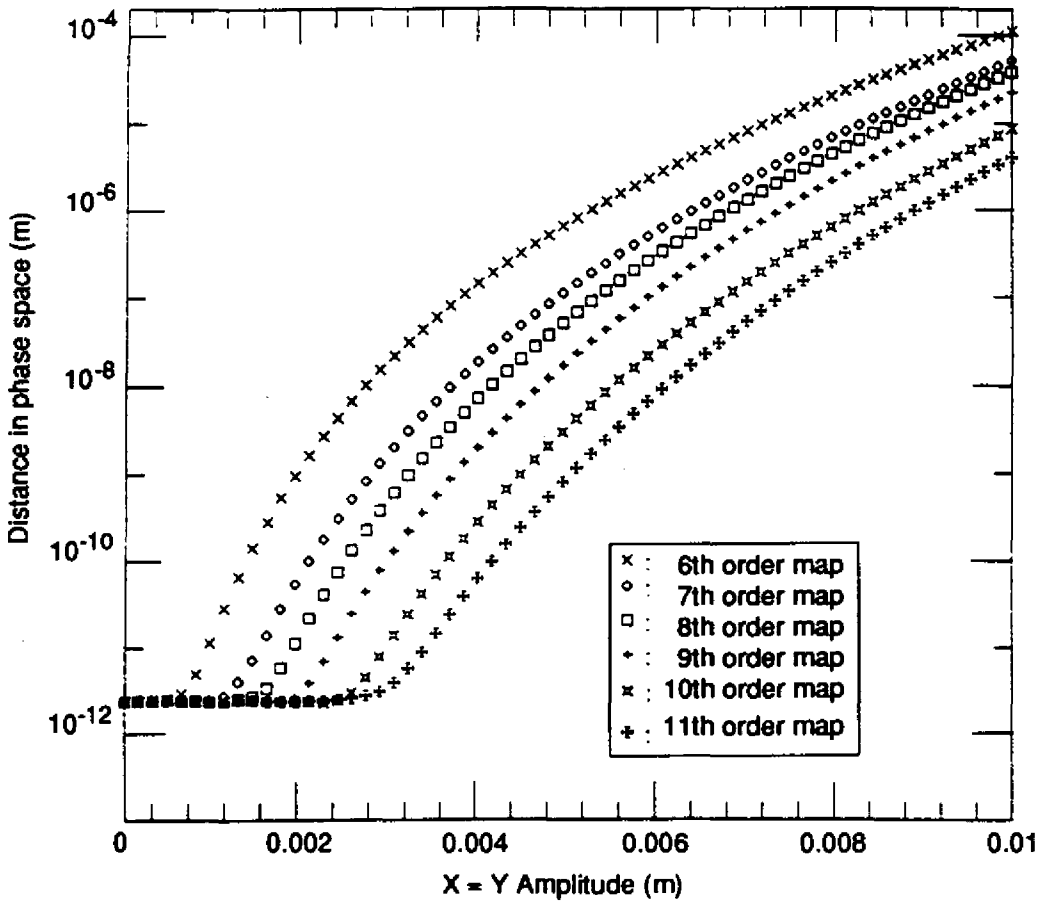

Figure 2. Error analysis of a one-turn Taylor map tracking for a 2-TeV injection lattice of the SSC (the same lattice as used for the survival plots in Figure 1). The horizontal axis represents the transverse amplitude of the particles while the vertical axis represents the weighted distance D between the positions in the four-dimensional transverse phase space, obtained with an element-by element tracking (Ztrack) and with a Taylor map tracking after one turn for each particle. $D = [(x^Z - x^M)^2 + \beta_x^2(p_x^Z - p_x^M)^2 + \frac{\beta_x}{\beta_y}(y^Z - y^M)^2 + \beta_x\beta_y(p_y^Z - p_y^M)^2]^{1/2}$, where $\beta_x' = 0.0 = \beta_y'$ at the measuring point. The deviation in energy for each of the particles is $\delta = 4.2\text{x}10^{-6}$. The distance at small amplitudes is dominated by the round-off errors. The region of interest for long-term tracking is around 0.005m in transverse amplitude where the above figure shows the accuracy of the 11th-order map to be between 7 to 8 digits after one turn.

A modest-order one-turn differential-algebra Taylor map being valid for long-term tracking also implies that the interpolation Taylor map proposed by Warnock[13] should also be valid provided that such an interpolation map is constructed up to a suitable order.

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References

1. A. J. Dragt and J. M. Finn, *Lie Series and Invariant Functions for Analytic Syplectic Maps*, J. Math. Phys. 20, 2649 (1979)
2. E. Forest, M. Berz, and J. Irwin, *Normal Form Methods for Complicated Periodic Systems*, Particle Accel. 24, 91 (1989)
3. A. Bazzani, M. Giovannozzi, G. Servizi, G. Turchetti, and E. Todesco, *Normal Forms for Area Preserving Maps—Resonances and Dynamic Aperture*, CERN SPS/89-24 (1989).
4. Y. Yan, *Supercomputing for the Superconducting Super Collider*, SSCL-239, September 1989; also to be published in Energy Sciences Supercomputing by DOE National Energy Research Supercomputer Center (NERSC).
5. Y. Yan and C. Yan, *ZLIB—A Numerical Library for Differential Algebra*, Superconducting Super Collider Laboratory Report No. SSCL-300, December 1990.
6. Y. Yan, *Zmap: A Differential-Algebra Map Extraction Program for Teapot using ZLIB*, SSCL-299, 1990.
7. L. Schachinger and R. Talman, *Teapot: A Thin-Element Accelerator Program for Optics and Tracking*, Particle Accelerators, Vol. 22, 1987.
8. J. Irwin, *A Multi-Kick Factorization Algorithm for Nonlinear Maps*, SSC-228, 1989.
9. A. Dragt, *Method for Symplectic Tracking*, presented in the workshop on nonlinear problems in future particle accelerators, Capri, Italy (1990).
10. A. Dragt, et al., *Lie Algebraic Treatment of Linear and Nonlinear Beam Dynamics*, Ann. Rev. Nucl. Part. Sci., Vol 38, pp 455-496 (1988)
11. E. Forest and Y. Yan, private communication.
12. A. Dragt, et al., *Release of Marylie 3.0*, in Proceedings of the 1989 IEEE Particle Accelerator Conference, Vol. 2, p. 1283 (1989).
13. R. L. Warnock, *Construction of Symplectic Full-Turn Maps by Application of an Arbitrary Tracking Code*, in Proceedings of the 1989 IEEE Particle Accelerator Conference, Vol. 2, p. 1322 (1989).
14. M.Berz, *Differential Algebra—a new tool*, In Floyd Bennett and Joyce Kopta, editors, Proceedings of the 1989 IEEE Particle Accelerator Conference, March 20-23, 1989, IEEE Catalog Number 89CH2669-0.