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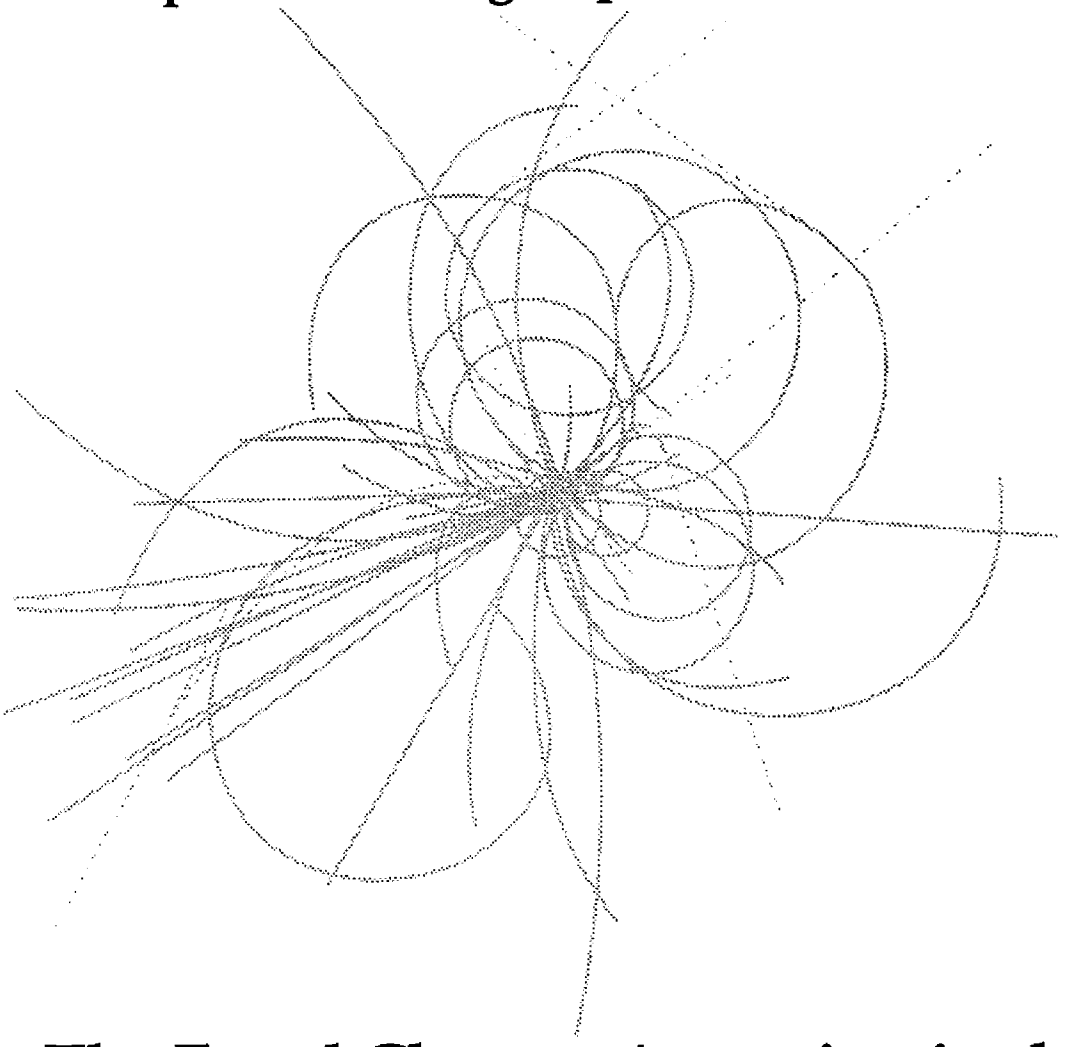

The Equal Charges Attraction in the 2+1 Topologically Massive Gauge Theories and Possible Vacuum Instability

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THE EQUAL CHARGES ATTRACTION IN THE $2+1$ TOPOLOGICALLY MASSIVE GAUGE THEORIES AND POSSIBLE VACUUM INSTABILITY

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Abstract

The particles interaction in the $2+1$ topologically massive gauge theories is given. It turns out, that contrary to the usual case, two equally charged particles (fermions or the non-Abelian topologically massive vector bosons) attract each other, not repulse. This attraction can lead in principle to vacuum instability, in which we will use some trial wave function techniques. Possible applications are briefly discussed.

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1. INTRODUCTION

The Topologically Massive Gauge Theories (TMGT) introduced in [1, 2, 3] and their low-energy limit–the Chern-Simons (CS) theories are widely popular now, because of the possible applications for some condensed matter problems such as the Fractional Quantum Hall Effect (FQHE) [4] and may be the high T_c superconductivity (see for example reviews [5, 6]), as well as for the 2-D Conformal Field Theories [7, 8], and may be used for the three-dimensional formulation of the off-shell closed string theory–the Open Topological Membrane theory [9, 10]. Besides, the three-dimensional gauge theory has appeared naturally as the high temperature limit of the usual four-dimensional theory, and the effective CS term appears in the case of non-zero chemical potential. Thus, it can be seen that TMGT theories are appearing in different regions of physics and it can be obtained as new information about them, by studying the properties of the TMGT.

It turns out that the particles interactions in these theories are rather unusual. It was shown in [11] for the case of charged $U(1)$ fermions and in [12] for the case of the $SU(2)$ gluons, that the two equally charged particles attract each other, not repulse, as would be natural for any theory when the vector particles exchange in the t-channel. This is the crucial difference between the usual gauge theory and gravity. Due to the universal attraction in gravity, there is the instability of a sufficiently large amount of matter. In usual gauge theories the charged matter is stable and at real values of the coupling constants. Only at imaginary values of the last (or negative α) is there the famous Dyson instability [13], reflecting the asymptotic nature of the perturbation series. As far as it is known, there are no other examples of the attraction between equally charged particles in gauge theories, therefore the TMGT is the only example of a field theory where Dyson instability can be realized in principle.

The paper is arranged as follows: In Section 2, the case of the Abelian $U(1)$ theory will be discussed, extending the results in [11] to the many flavor case. In

Section 3, the case of the non-Abelian $SU(2)$ TMGT will be considered and shown that equally charged gluons on-shell will attract each other. In Sections 4 and 5, it can be seen that the effective quantum Hamiltonians, which are corresponding to the obtained scattering amplitudes, and use of the variational wave function, makes an estimate of the average energy of the state. In Section 6, some possible applications of the vacuum instability (if exists), the nature of the real-ground state is summarized and the results obtained.

2. THE $U(1)$ TOPOLOGICALLY MASSIVE GAUGE THEORY WITH N_f FLAVORS

Starting from the Lagrangian

$$L = -\frac{1}{4\gamma}F_{\mu\nu}^2 + \frac{k}{16\pi}\,\epsilon_{\mu\nu\lambda}F_{\mu\nu}A_\lambda + \sum_{i=1}^{N_f}\bar{\Psi}_i(iD_\mu\gamma^\mu - m_i)\Psi_i\,. \tag{1}$$

where γ has the dimension of mass and fermionic masses m_i are arbitrary. In the case of equal fermionic masses, the global $U(N_f)$ symmetry is obtained. For simplicity, consider the case of equal fermionic charges, and put them equal to 1, the real coupling constant is γ. To get the propagator of the gauge boson, it is necessary to fix gauge. In Landau gauge it equals

$$G_{\mu\nu}(p) = \frac{\gamma\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + iM\epsilon_{\mu\nu\lambda}\frac{p_\lambda}{p^2}\right)}{p^2 - M^2}\,, \tag{2}$$

where $M = k\gamma/4\pi$ is the topological mass of the gauge particle (it shall be called photon later). Integrating over the gauge field, the effective fermionic theory can be obtained with the induced non-local four-fermion interaction, described by the the interaction Lagrangian

$$L_{\text{int}} = \sum_{i,j}\int d^3x\, d^3y[\bar{\Psi}_i\gamma_\mu\Psi_i](x)\; G_{\mu\nu}(x-y)[\bar{\Psi}_j\gamma_\nu\Psi_j](y)\,. \tag{3}$$

The next interest is the case where all fermions are on-shell, thus using the Gordon representation for current

$$\bar{\Psi}(p_2)\gamma_\mu\Psi(p_1) = \frac{(p_1+p_2)_\mu}{2m}\cdot\bar{\Psi}(p_2)\Psi(p_1)$$

$$+\frac{(p_2-p_1)_\nu}{2m}\,\bar{\Psi}\,(p_2)\frac{[\gamma_\nu,\gamma_\mu]}{2}\,\Psi(p_1)$$

$$=\frac{(p_1+p_2)_\mu}{2m}\,\bar{\Psi}(p_2)\Psi(p_1)$$

$$+\,i\epsilon_{\mu\nu\lambda}\,\frac{(p_2-p_1)_\nu}{2m}\,\bar{\Psi}(p_2)\gamma_\lambda\Psi(p_1)\,. \tag{4}$$

Substituting (4) into (3), the two-into-two scattering amplitude (see Figure 1) is obtained, which is the sum of two terms: diagonal and antisymmetric. Later, the interest shall be in the case of the nonrelativistic fermions, when all the momenta are small when comparing the fermionic masses m. In this case, the space component of current is small concerning the zero component, and the main contribution, to the interaction, is derived from the zero component of the currents. However, there is the infrared antisymmetric singular term (P-odd) in the photon propagator (2), which can can seal the small momentum factor in the space component of the current, thus leading to the same magnitude of interaction J_iJ_0 as in the J_0J_0. This is nothing but the Pauli spin-magnetic interaction as was demonstrated in[11], and this is the key for unusual interaction between particles–attraction instead of repulsion. Substituting (2) and (4) into (3), and using the nonrelativistic limit the scattering amplitude between i and j fermions:

$$A(\vec{p_1},\vec{p_2}\to\vec{p_1}+\vec{q},\vec{p_2}-\vec{q}) = \Psi_i^+(\vec{p_1}+\vec{q})\Psi_i(\vec{p_1})\Psi_j^+(\vec{p_2}-\vec{q})\Psi_j(\vec{p_1})$$

$$\times\frac{\gamma}{\vec{q}^2+M^2}\left[\left(-1+\frac{M}{m}\right)-\frac{iM}{m}\right.$$

$$\left.\times\,\epsilon_{k\ell}\frac{(p_1-p_2)_k q_\ell}{\vec{q}^2}+0(\vec{p},\vec{q})\right]. \tag{5}$$

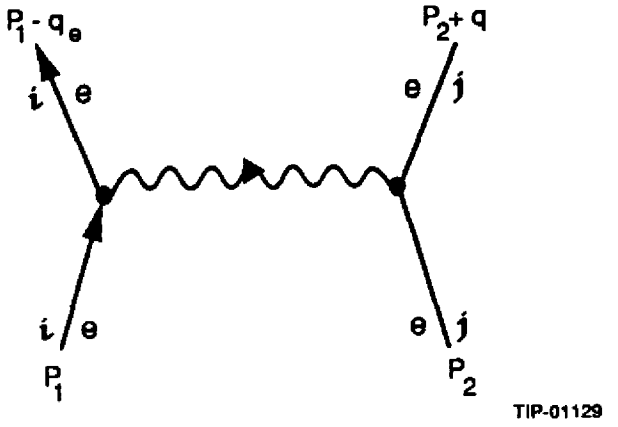

Figure 1. Two Scattering Amplitude

Here m is the reduced mass for the two masses m_i and m_j and neglects the differences between $\bar{\Psi}\gamma_o\Psi$ and $\bar{\Psi}\Psi$ due to the nonrelativistic approximation and there is no summation over i or j. Due to the fact that all fermions are on-shell there is no energy transfer–the elastic process only is dealt with here. The real part of amplitude describes the local interaction in the limit of q small comparing to M when the imaginary part describes the Aahoronov-Bohm type interaction (see[2, 14]). It is easy to show[11], that this amplitude generates the correct potential in the two-body Schrödinger equation for the nonrelativistic spinor particles.

It can be seen from (5) that if $m < M$, the sign of the real part is changed. Because the first term in the parenthesis (which equals to 1) describes the usual repulsion between charges, it is seen that when the total sign is changed we get the attraction. This is realized if sufficient light fermions with the "good" sign of the mass is available. This is not very surprising, because in 2 dimensions the sign of fermionic mass is a sign of "spin," or what is better to say is the sign of the Pauli interaction, because there is no usual kinematical spin of the fermion in $2 + 1$ dimensions.

Here, the radiative corrections shall not be discussed, which may be very significant in the case when N_f/k is not the small parameter and can be seen later, this is the case when the possible vacuum instability can takes place. Take

note, that due to the radiative corrections, it is impossible to have very light fermions—due to corrections, the mass of the order of M/k will be generated.

3. THE NON-ABELIAN TOPOLOGICALLY MASSIVE GAUGE THEORY: THE $SU(2)$ TOY MODEL

In the preceding section it was noted, that due to the magnetic interaction, it was possible to get the attraction between two equally charged particles, but it depends on the sign of fermionic mass, comparing the sign of k or, which is the same, the sign of the photon mass M. In the non-Abelian theory there is no such free parameter–therefore either repulsion or attraction unambiguously can be obtained. It was shown in [12], there is indeed attraction on-shell. Here the result is in more detail. Consider the Lagrangian of the non-Abelian TMGT (work shall be in Landau gauge):

$$L = -\frac{1}{2\gamma}\,\mathrm{Tr}\,F_{\mu\nu}^2 + \frac{k}{4\pi}\,\epsilon_{\mu\nu\lambda}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)\,, \tag{6}$$

where

$$A_\mu = A_\mu^a\tau^a\,,\ [\tau^a,\tau^b] = f^{abc}\tau^c\,,\ \mathrm{tr}\ \tau^a\tau^b = -\frac{1}{2}\,\delta^{ab}\,. \tag{7}$$

Now, contrary to the Abelian case, the CS coefficient k is to be a integer, if to preserve the gauge invariance of the action under the large gauge transformations [3]. If normalization of the vector field is changed

$$A_\mu \to \sqrt{\gamma}\,A_\mu\,, \tag{8}$$

the Lagrangian in the form of [3, 15] is obtained:

$$L = -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}^2 + \frac{k\gamma}{4\pi}\,\epsilon_{\mu\nu\lambda}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}\sqrt{\gamma}A_\mu A_\nu A_\lambda\right)\,, \tag{9}$$

where, as in the Abelian case, the gauge particles masses equal to $M = k\gamma/4\pi$.

Later consider the case of the $SU(2)$ theory, the generalization to the case of the arbitrary gauge group G is straightforward. It is convenient to use the charged and neutral combination of the isotopical degree of freedom:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \pm iA_\mu^2\right)\,,\ Z = A_\mu^3\,. \tag{10}$$

Now consider the elastic amplitude of W^+W^+ scattering, which is shown in Figure 2. This amplitude, in general, is very complicated, because of the external vector indices, and depends on the polarization vectors of the scattering particles:

$$\begin{aligned} A(W^+W^+ \to W^+W^+) = {} & A_{\mu\nu\rho\sigma}(\vec{p_1},\vec{p_2}\,;\,\vec{p_1}+\vec{Q},\vec{p_2}-\vec{Q})e_{1\mu}(\vec{p_1})e_{2\nu}(\vec{p_2}) \\ & \times \bar{e_{3\sigma}}(\vec{p_1}+\vec{Q})\bar{e_{4\rho}}(\vec{p_2}-\vec{Q})\,. \end{aligned} \tag{11}$$

But if required, it only on-shell (and only then this amplitude is gauge invariant and has sense) the massive gauge boson is described by the one degree of freedom, the polarization vector $e_\mu(p)$ equals to[14, 15]:

$$e_\mu(\vec{p}) = \frac{1}{\sqrt{2}\,|\vec{p}|M}\begin{pmatrix} \vec{p}^{\,2} \\ Ep_1 + ip_2M \\ Ep_2 - ip_1M \end{pmatrix}\,,\ e_\mu\bar{e}_\mu = -1\,, \tag{12}$$

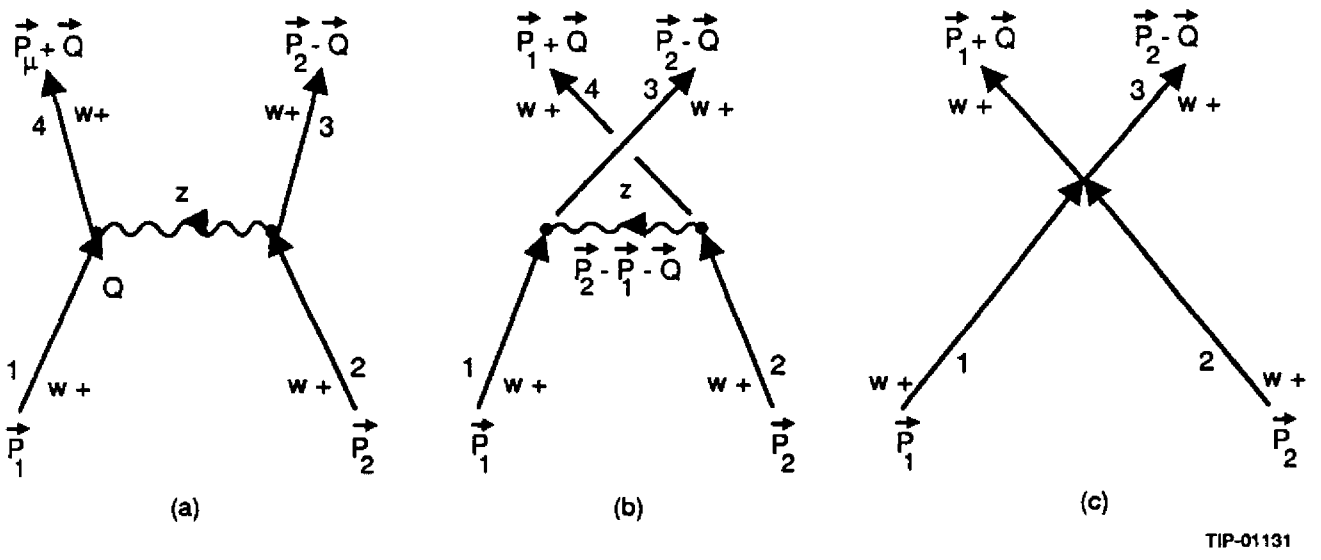

Figure 2. Elastic Amplitude of W^+W^+ Scattering

and it is easy to see, that (12) is transverse. To calculate the amplitude,it requires the three- and four-point vertices (see, for example[15]). It is prepared here for the case of W^+W^+ scattering, thus the isotopical structure is quite definite and the fact that external lines are transverse can be used, because the polarization vectors (12) and propagator (2) in the Landau gauge are transverse.

$$
\begin{aligned}
\cdot_{\mu\rho\lambda}(p_1, -p_1 - Q, Q) &= im\,\epsilon_{\mu\rho\lambda} + 2g_{\mu\rho}p_{1\lambda} - 2g_{\rho\lambda}Q_\mu + 2g_{\lambda\mu}Q_\rho \\
\Gamma_{\mu\rho\nu\sigma}(p_1, p_2, p_3, p_4) &= (g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\nu}g_{\rho\sigma})\ ,
\end{aligned}
\tag{13}
$$

where the additional ϵ term in the three-point vertex is due to the CS term in the Lagrangian. Compute the amplitude in the rest frame and choose the momenta of the in-going particles in the x directions. Due to this, the outgoing particles will have the momenta which are determined by the scattering angle θ. The polarization vectors are

$$
e_1 = \frac{1}{\sqrt{2}\,Mp}\begin{pmatrix} p^2 \\ Ep \\ -imp \end{pmatrix}, e_2 = \frac{1}{\sqrt{2}\,Mp}\begin{pmatrix} p^2 \\ -Ep \\ imp \end{pmatrix},
$$

$$
\bar{e}_3 = \frac{1}{\sqrt{2}\,Mp}\begin{pmatrix} p^2 \\ -Ep\cos\theta + imp\sin\theta \\ -Ep\sin\theta - imp\cos\theta \end{pmatrix},
$$

$$
\bar{e}_4 = \frac{1}{\sqrt{2}\,Mp}\begin{pmatrix} p^2 \\ -Ep\cos\theta - imp\sin\theta \\ -Ep\sin\theta + imp\cos\theta \end{pmatrix}, \tag{14}
$$

and now to calculate the amplitude (11) use the propagator (2), vertices (13) and polarization vectors (14). The total amplitude A is the sum of the three terms $A^a + A^b + A^c$, which are determined by the corresponding diagrams at Figure 2. This amplitude is sufficiently complicated and can be found in the appendix. Here only the limit of the small momentum is presented. As in the

case of the $U(1)$ theory there is the infrared singular term, due to the Aaharonov-Bohm interaction, manifesting itself as singularity of the scattering amplitude at angles $\theta = 0$ and $\theta = \pi$. To take the limit of small momenta, fix the scattering angle, and then tend momentum p go to zero, which also leads to the zero limit of momentum transfer Q. Finally, the amplitude (at small angles θ) is obtained:

$$A(\vec{p}, -\vec{p}; \vec{p}+\vec{Q}, -\vec{p}-\vec{Q}) = \frac{8\pi M}{k}\left(4 - \frac{i}{\theta}\right) + 0(p,q)$$

$$= \frac{8\pi M}{k}\left(4 - i\,\epsilon_{ij}\frac{p_i Q_j}{Q^2}\right) + 0(p,q)\,, \qquad (15)$$

and the sign of the real part corresponds to the attraction (it was checked during the calculation, using the fact that the P-even part of the propagator's contribution to the real part of the amplitude is a repulsive one, thus it is only required to know the relative sign of the total amplitude).

It is easy to see, that the same amplitude is obtained in the case of the W^-W^- scattering, and in the cases of the W^+W^- and W^+ or W^-Z scatterings, there is repulsion and attraction respectively [12]. So the attraction in the channels with the iso-spin projection equals to $+2, +1, -1, -2$ and the repulsion in the zero iso-spin channel is obtained.

4. THE EFFECTIVE QUANTUM HAMILTONIAN AND POSSIBLE VACUUM INSTABILITY IN THE ABELIAN TMGT

Now the obtained elastic amplitudes can be used to construct the effective quantum Hamiltonians for the charged degrees of freedom. Starting from the $U(1)$ case. What is the sense of the obtained amplitude (5)? It describes the fermion interactions at excluded gauge degree of freedom. This means in particular, that if this amplitude is iterated, the arbitrary fermionic amplitudes cannot be obtained. only those which don't contain the crossed photon lines (see Figures 3, 4(a) can be obtained, and (b) cannot). This is evident. because the

retarding effects are neglected. This approximation is quite good for the non-relativistic fermions and consider only the fermionic degree of freedom without making a fault. The amplitude (5) describes the interaction part of the Hamiltonian and the fermionic quantum Hamiltonian can be written, which correctly describes the quantum states where only nonrelativistic levels of energy are filled. What can be done with the imaginary part of the amplitude which has a singularity at small q? The answer is simple, the Hamiltonian is the hermitian operator, the initial four-fermion operator the complex conjugate one must be added to:

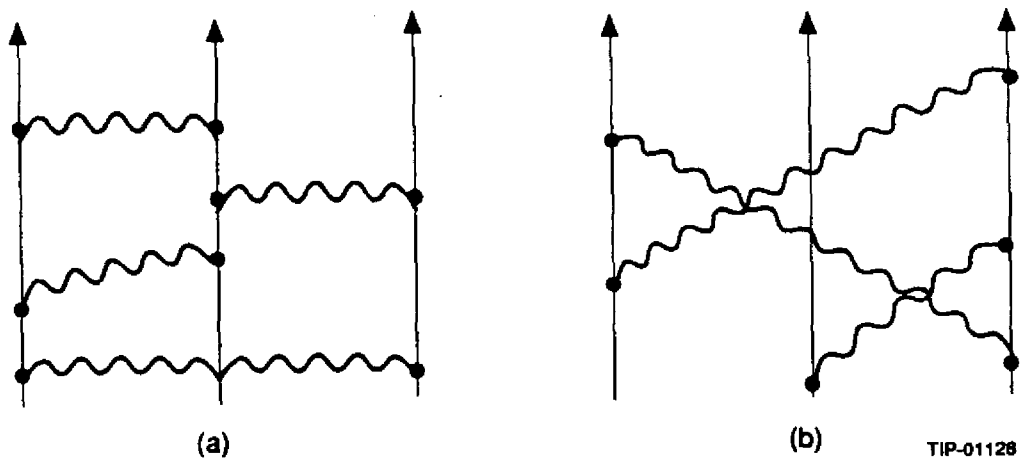

Figure 3. Fermionic Amplitudes

$$\frac{1}{2}\sum_{\vec{p_1},\vec{p_2},\vec{q}}\Big[A(\vec{p_1}-\vec{p_2},\vec{q})a^{+}_{\vec{p_1}+\vec{q}}\,a^{+}_{\vec{p_2}-\vec{q}}\,a_{\vec{p_1}}\,a_{\vec{p_2}}$$
$$+A^{*}(\vec{p_1}-\vec{p_2},\vec{q})a^{+}_{\vec{p_1}}\,a^{+}_{\vec{p_2}}\,a_{\vec{p_1}+\vec{q}}\,a_{\vec{p_2}-\vec{q}}\Big]$$
$$=\frac{1}{2}\sum_{\vec{p_1}\vec{p_2}\vec{q}}\Big[A(\vec{p_1}-\vec{p_2},\vec{q})a^{+}_{\vec{p_1}+\vec{q}}\,a^{+}_{\vec{p_2}-\vec{q}}\,a_{\vec{p_1}}\,a_{\vec{p_2}}$$
$$+A^{*}(\vec{p_1}-\vec{p_2}-2\vec{q},\vec{q})a^{+}_{\vec{p_1}-\vec{q}}\,a^{+}_{\vec{p_2}+\vec{q}}\,a_{\vec{p_1}}\,a_{\vec{p_2}}\Big]$$

$$\simeq \sum_{\vec{p_1}\vec{p_2}\vec{q}} \Big\{ \mathrm{Re}\ A(\vec{p_1}-\vec{p_2},\vec{q}) \left[a^+_{\vec{p_1}+\vec{q}}\, a^+_{\vec{p_2}-\vec{q}} + a^+_{\vec{p_1}-\vec{q}}\, a^+_{\vec{p_2}+\vec{q}} \right]$$

$$\times a_{\vec{p_1}} a_{\vec{p_2}} + i\ \mathrm{Im}\ A(\vec{p_1}-\vec{p_2},\vec{q}) \left[a^+_{\vec{p_1}+\vec{q}}\, a^+_{\vec{p_2}-\vec{q}} - a^+_{\vec{p_1}-\vec{q}} a^+_{\vec{p_2}+\vec{q}} \right]$$

$$\times a_{\vec{p_1}} a_{\vec{p_2}} \Big\}\,, \tag{16}$$

where the fact is used, that in the approximation, the amplitude is unchanged when $p_1 - p_2$ by q (the real part is constant and the imaginary part is not changed due to the vector product of $p_1 - p_2$ and q) is shifted. The matrix elements of the four-fermion operator standing behind the singular imaginary part will contain the additional power of q and there will be no singularity in the matrix elements. Substituting the standard plane wave representation, for the fermionic creating and destroying operators, the following Hamiltonian is obtained:

$$\hat{H} = \sum_{i=1}^{N_f} \sum_{\vec{p}} \epsilon_i(\vec{p}) a^{+i}_{\vec{p}} a^i_{\vec{p}}$$

$$- \sum_{i,j=1}^{N_f} \sum_{\vec{p_1},\vec{p_2},\vec{q}} \frac{\gamma}{4V} \frac{1}{\vec{q}^2 + M^2} \Bigg[\frac{1}{2} \left(\frac{M}{m_{ij}} - 1 \right) \Big(a^{+i}_{\vec{p_1}+\vec{q}}\, a^{+j}_{\vec{p_2}+\vec{q}}$$

$$+ a^{+i}_{\vec{p_1}-\vec{q}}\, a^{+j}_{\vec{p_2}+\vec{q}} \Big) a^i_{\vec{p_1}} a^j_{\vec{p_2}} - i \frac{M}{m_{ij}} \epsilon_{k\ell} \frac{(p_1 - p_2)_k q_\ell}{\vec{q}^2}$$

$$\times \frac{1}{2} (a^{+i}_{\vec{p_1}+\vec{q}}\, a^{+j}_{\vec{p_2}-\vec{q}} - a_{\vec{p_1}} - \vec{q}^{+i} a^{+j}_{\vec{p_2}-\vec{q}}) a^i_{\vec{p_1}} a^j_{\vec{p_2}} \Bigg]\ ;$$

$$\gamma = \frac{4\pi M}{k}\,, \tag{17}$$

where V is the area of the system (2-D volume) and

$$\epsilon_i(\vec{p}) = \sqrt{\vec{p}^2 + m_i^2}$$

$$\frac{2}{m_{ij}} = \frac{1}{m_i} + \frac{1}{m_j}\,. \tag{18}$$

Later a consideration will be made in the case of equal masses, for the sake of simplicity. The main question now is about the ground state of this Hamiltonian. The standard vacuum has, by definition, zero energy and is the eigenfunction of (17). However, if the probe wave function can be constructed with average energy less than zero, this means, that the standard vacuum is unstable. Choose this function in the form

$$|\phi\rangle_f = \prod_{i=1}^{N_f} |\phi_i\rangle_f$$

$$|\phi_i\rangle_f = \prod_{\vec{p}} (u_i(\vec{p}) + f_i(\vec{p}) a^{i+}_{\vec{p}})|\phi_i\rangle \ , \tag{19}$$

where the correlation between the different flavors is neglected. The normalization conditions are

$$\langle\phi_i|\phi_i\rangle = \prod_{\vec{p}} (|u_i(\vec{p})|^2 + |f_i(\vec{p})|^2) \ ;$$

$$|u_i(\vec{p})|^2 + |f_i(\vec{p})|^2 = 1 \ . \tag{20}$$

The functions $f_i(p)$ describe the density of the fermions with momentum p

$$\begin{aligned} N_i(\vec{p}) &= \langle\phi_i|a^{i+}_{\vec{p}} a^i_{\vec{p}}|\phi_i\rangle = |f_i(\vec{p})|^2 \\ N_i &= \sum_{\vec{p}} N_i(\vec{p}) = \sum_{\vec{p}} |f_i(\vec{p})|^2 \ , \end{aligned} \tag{21}$$

and N_i are the total average numbers of the fermions. To calculate the average energy, the matrix element must be known

$$\langle\phi_i|a^{i+}_{\vec{p_1}} a^i_{\vec{p_2}}|\phi_i\rangle = \delta_{\vec{p_1}\vec{p_2}}|f_i(\vec{p_1})|^2 + \prod_{\vec{p}=\vec{p_1}}^{\vec{p_2}} (|u_i(\vec{p})|^2 - |f_i(\vec{p})|^2)$$

$$\times\, u_i^*(\vec{p_2}) f_i(\vec{p_2}) u_i(\vec{p_1}) f_i^*(\vec{p_1})$$

$$= \delta_{\vec{p_1}\vec{p_2}} |f_i(\vec{p_1})|^2 + \Delta_i(\vec{p_1}\vec{p_2}) \ , \tag{22}$$

where $\Delta_i(\vec{p_1}\vec{p_2})$ is the quickly oscillating function and can be chosen arbitrarily small, if the functions f and u with the small overlapping regions are considered. Consider now, the average energy of this probe wave function:

$$\begin{aligned}
\langle \phi_f | \hat{H} | \phi_f \rangle = & \sum_{i=1}^{N_f} \sum_{\vec{p}} \epsilon_i(\vec{p}) |f_i(\vec{p})|^2 \\
& - \frac{\pi}{kMV} \sum_{\substack{i,j \\ i>j}}^{N_f} \sum_{\vec{p_1},\vec{p_2},\vec{q}} \Bigg\{ \left(\frac{M}{m} - 1 \right) \Bigg[|f_i(\vec{p_1})|^2 \, |f_j(\vec{p_2})|^2 \, \delta_{\vec{q},0} \\
& + \frac{1}{2} \left(\Delta_i(\bar{p_1} + \bar{q}, \bar{p_1}) \Delta_j(\bar{p_2} - \bar{q}, \bar{p_2}) + (\vec{q} - \vec{q}) \right) \Bigg] \\
& - \frac{iM}{2m} \frac{[(\bar{p_1} - \bar{p_2})\bar{q}]}{\bar{q}^2} [\Delta_i(\bar{p_1} + \bar{q}, \bar{p_1}) \Delta_j(\bar{p_2} - \bar{q}, \bar{p_2}) \\
& - \Delta_i(\bar{p_1} - \bar{q}, \bar{p_1}) \Delta_j(\bar{p_2} + \bar{q}, \bar{p_2})] \Bigg\} \ .
\end{aligned} \tag{23}$$

Note, that due to the Pauli principle there is no contribution from the scattering of fermions with equal flavors, because the local interaction (in real part of amplitude) is restricted. As shown in [11], there is P-wave scattering for one flavor. producing the BCS P-wave superconducting ground state at Fermi surface.

Now, to define the f_i functions. The simplest way is to define them as shown in Figure 4, is to obtain the Fermi-sphere like distribution with Fermi momentum p_F. It is obvious there is no infrared singularities in the imaginary part contribution and choosing the overlapping regions as narrow as desired (thus getting the usual Fermi surface), the imaginary part contribution as well as overlapping contribution in real part can be neglected. Choosing the Fermi momenta equal for all flavors, the variational energy depending on one variational parameter is obtained—the Fermi momentum p_F

$$\langle \phi_f | \hat{H} | \phi_f \rangle = N_f V \int \frac{d^2\vec{p}}{(2\pi)^2} \sqrt{\vec{p}^2 + m^2} \, \theta(p_F - |\vec{p}|)$$

$$-\frac{\pi V}{kM}\frac{N_f(N_f-1)}{2}\int\int\frac{d^2\vec{p_1}}{(2\pi)^2}\frac{d^2\vec{p_2}}{(2\pi)^2}\left(\frac{M}{m}-1\right)$$

$$\times\,\theta(p_F-|\vec{p_1}|)\theta(p_F-|\vec{p_2}|)\,. \tag{24}$$

Using the fact that the integral over the Fermi sphere equals to the fermionic density

$$n=\int\frac{d^2p}{(2\pi)^2}\,\theta(p_F-|\vec{p}|)=\frac{p_F^2}{4\pi} \tag{25}$$

and dividing energy (24) by the volume of the system V, the energy density as a function of the density of the fermions is obtained (see Figure 5):

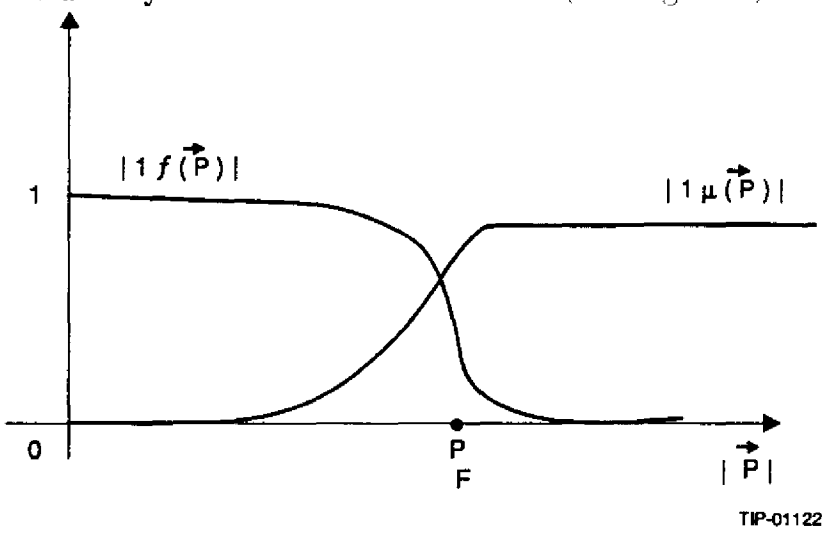

Figure 4. f_i Functions

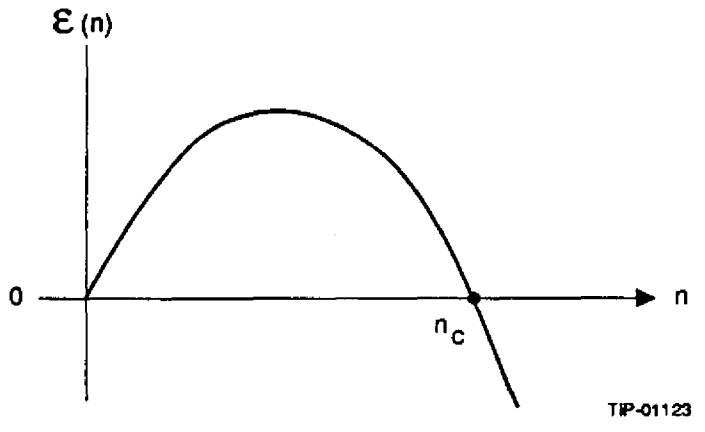

Figure 5. Fermion Density

$$\epsilon(n) = N_f m n - \frac{N_f(N_f-1)}{2}\,\frac{\pi}{kM}\left(\frac{M}{m}-1\right)n^2\,. \tag{26}$$

At critical density, n_c, the energy density becomes zero

$$\begin{aligned}\epsilon(n_c) &= 0\\ n_c &= \frac{2}{\pi(N_f-1)}\,\frac{kmM}{\frac{M}{m}-1}\,,\end{aligned} \tag{27}$$

and for larger densities, it becomes negative,which means that the standard vacuum is metastable, i.e. it is stable classically, but unstable quantum mechanically. To be quite rigorous, the scenario is correct only for nonrelativistic electrons and it is necessary for critical Fermi momentum to be smaller than fermion mass m. The corresponding inequality is:

$$\frac{N_f-1}{k} > \frac{\frac{M}{m}}{\frac{M}{m}-1}\,. \tag{28}$$

For large k and sufficiently light fermions, it follows from this inequality, that the number of flavors must be larger than k. It may be dangerous, because the perturbation expansion parameter will now be N_f/k and the radiative corrections to the amplitude may become large. However, there are some situations when these corrections are irrelevant. Consider the one loop corrections to the amplitude, in which the Feynman diagrams are shown at the Figure 6(a), (b), and (c). The only dangerous diagram is the last, because the fermion loop is needed to get the N_f factor. This renormalization of the photon propagator leads to the two things: the renormalization of γ, i.e. the photon mass M, which is insignificant, and the renormalization of k which can change the complete picture. However, consider the theory where there are an equal (or approximately equal) number of fermions with positive and negative signs of mass, the renormalization of k will be zero (or small) and all other corrections will be in the order of $1/k$. Now consider the higher order diagrams with fermionic loops, for example the diagram

shown in Figure 6(d). Estimating this information, the magnitude by the order of N_f/k^2 is obtained, which is larger in comparison with the magnitudes of the one-loop diagrams, but is small when comparing the value of the tree amplitude:

$$A = A_0 \left(1 + \frac{1}{k} f_1 \left(\frac{N}{k}\right) + \frac{1}{k^2} f_2 \left(\frac{N}{k}\right) + \cdots\right) . \tag{29}$$

Then, if the parameters are:

$$1 \ll k \ll N_f \ll k^2 , \tag{30}$$

then the perturbative region as well as the critical Fermi-momentum is small, and the picture of the vacuum instability is self-consistent.

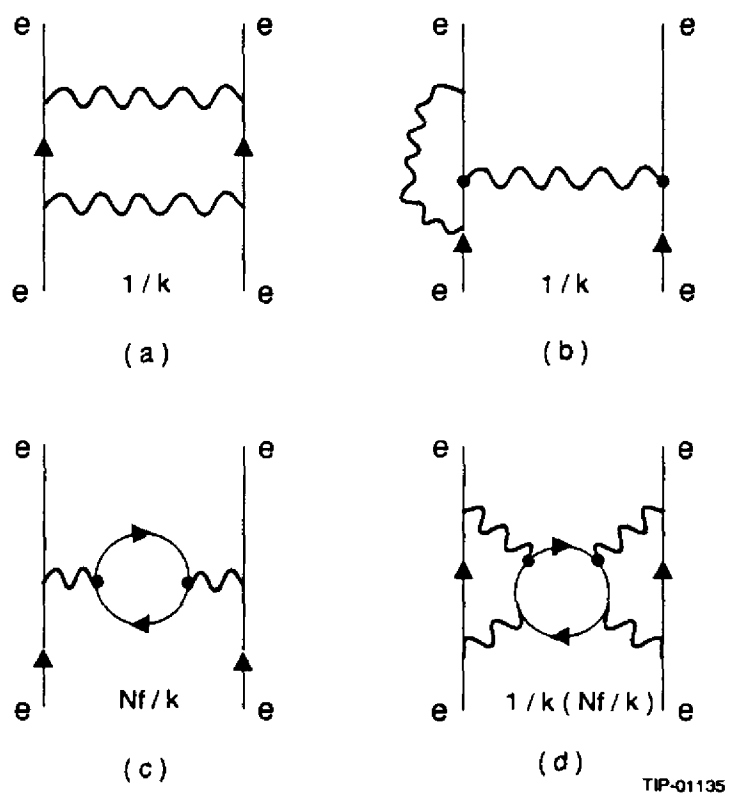

Figure 6. High Order Fermionic Loop Diagram(s)

5. THE EFFECTIVE QUANTUM HAMILTONIAN AND POSSIBLE VACUUM INSTABILITY IN THE NON-ABELIAN $SU(2)$ TMGT

Now, use the obtained elastic amplitude for W bosons to construct the effective quantum Hamiltonian for the charged degrees of freedom in the non-Abelian $SU(2)$ TMGT. However, as in the Abelian case, it is impossible to get the total Hamiltonian, even in the nonrelativistic approximation. The problem is that due to the four-point vertices in the non-Abelian theory, there are new many-particles amplitudes (see Figure 7). The calculation of this amplitude is rather complicated and thus we did not accomplish it. Therefore, the reduced toy Hamiltonian was constructed by using only the four-particle amplitude (15). Making this in a complete analogue, with the Abelian case, the Hamiltonian is obtained:

$$
\begin{aligned}
H_w = M \sum_{\vec{p}} a^{+}_{\vec{p}} a_{\vec{p}} \\
- \frac{2\pi}{kMV} \sum_{\vec{p_1},\vec{p_2},\vec{q}} \Big\{ 2 \left(a^{+}_{\vec{p_1}+\vec{q}}\, a^{+}_{\vec{p_2}-\vec{q}} + (\vec{q} \to -\vec{q}) \right) a_{\vec{p_1}} a_{\vec{p_2}} \\
- \frac{i[(\vec{p_1} - \vec{p_2})\vec{q}]}{\vec{q}^{\,2}} \left(a^{+}_{\vec{p_1}+\vec{q_1}}\, a^{+}_{\vec{p_2}-\vec{q}} - (\vec{q} \to -\vec{q}) \right) a_{\vec{p_1}} a_{\vec{p_2}} \Big\} ,
\end{aligned}
\tag{31}
$$

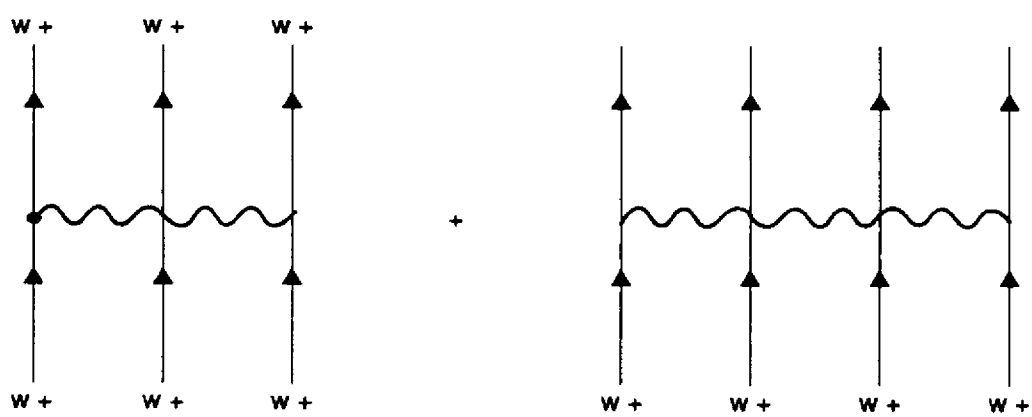

Figure 7. New Particle Amplitudes

where all the energies of the particles equal to mass M due to the nonrelativistic approximation. Now choose the probe wave function in the simplest form:

$$|\phi\rangle_w = \prod_{\vec{p}} \frac{1}{\sqrt{N(\vec{p})!}} (a^+_{\vec{p}})^{N(\vec{p})}|0\rangle \ , \tag{32}$$

where the variational parameters are the filling numbers

$$\langle_w\phi|a^+_{\vec{p}}a_{\vec{p}}|\phi\rangle_w = N(\vec{p}) \ . \tag{33}$$

This wave function has the definite number of particles and when averaging the Hamiltonian (31) over the (32), the only nonzero contribution will give the terms with Q equal to zero, $p_1 - p_2$, or $p_2 - p_1$. The last two are equal to the first one, as a consequence of the Bose-statistics. The contribution of the imaginary part of the amplitude to the average energy equals zero by the same reason, as in the Abelian case, and the real part after the averaging over the (32), gives the average energy:

$$\langle\phi|H_w|\phi\rangle = M\sum_{\vec{p}} N(\vec{p}) - \frac{8\pi}{kMV}\sum_{\vec{p_1},\vec{p_2}} (N(\vec{p_1})N(\vec{p_2})$$

$$+ \delta_{\vec{p_1},\vec{p_2}} N(\vec{p_1})[N(\vec{p_1}) - 1]) \ . \tag{34}$$

To obtain this energy the next matrix element was used

$$\langle\phi|a^+_{\vec{p_2}}a^+_{\vec{p_1}}a_{\vec{p_2}}a_{\vec{p_1}}|\phi\rangle = N(\vec{p_2})N(\vec{p_1}) + \delta_{\vec{p_1}\vec{p_2}}N(\vec{p_1})[N(\vec{p_1}) - 1] \ , \tag{35}$$

where the first term is only for different momenta. Now consider two different types of the probe functions. The first one is the Bose condensate at some momentum p_0:

$$\begin{aligned} N(\vec{p}) &= N \qquad \vec{p} = \vec{p}_0 \\ N(\vec{p}) &= 0 \qquad \vec{p} \neq \vec{p}_0 \end{aligned} \tag{36}$$

and the energy density is

$$n = \frac{N}{V} \ ,$$

$$\epsilon = Mn - \frac{8\pi}{kM}\, n \left(n - \frac{1}{V} \right) \ . \tag{37}$$

The critical density n_c in the thermodynamic limit equals

$$n_c \simeq \frac{1}{8\pi} k\, M^2 \ . \tag{38}$$

Consider instead, of (36) some constant distribution

$$\begin{aligned} N(\vec{p}) &= \frac{4\pi N}{V P_0^2} \qquad & |\vec{p}| < P_0 \\ N(\vec{p}) &= 0 \qquad & |\vec{p}| > P_0 \end{aligned} \tag{39}$$

the energy density will be

$$\epsilon = Mn \left(1 + \frac{8\pi}{kM^2 V} \right) - \frac{8\pi}{kM}\, n^2 \left(1 + \frac{4\pi}{V p_0^2} \right) \ , \tag{40}$$

and in the thermodynamic limit, the critical density will be the same as in the first case. Unfortunately, the contribution to the energy from the diagrams shown in Figure 7, at these densities, become the same order of magnitude and the vacuum instability cannot be proven, but consider it as the possibility, if the higher order corrections do not destroy it. At the end, the case of the general group G instead of $SU(2)$ was discussed. Choose some Cartan subgroup H examples and consider the interaction of the "charged degrees of freedom," belonging to the coset G/H. The gluons of the H subgroup cannot interact themselves, but interact with coset gluons. Thus the same picture as in $SU(2)$ is obtained, but with more complicated laws of attractions–there are some charges now, the number equals to the group rank $r = \dim H$. However if this rank becomes large enough, the strong coupling regime is obtained, because the perturbative expansion parameter is r/k. In this region the radiative corrections are extremely significant and to get the answer it is necessary to sum up all the leading corrections, (neglecting terms of order r/k^2).

6. DISCUSSION AND CONCLUSION

The vacuum instability (only hypothetical for gluons) is obtained, but what is the nature of the ground state? If the density is larger, then what criticality will stabilize the vacuum? To answer this question it is necessary to remember, that attraction is obtained considering the vector particles ("photon" or "gluon") exchange between the charges. However, when there is some charge density, the propagator of the vector particle is changed and the attraction amplitude will also be changed. In the self-consistent approximation, the energy density can have a minimum at some n^* (see Figure 8). Consider, for example, the simplest suggestion, that this charge condensate is described by some effective Higgs field. The propagator equals [15]

$$G_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + iM\epsilon_{\mu\nu\lambda}\frac{p_\lambda}{p^2 - m_s^2} \right) \frac{p^2 - m_s^2}{(p^2 - m_+^2)(p^2 - m_-^2)}$$

$$m_\pm^2 = m_s^2 + \frac{M^2}{2} \pm \frac{1}{2}\sqrt{M^2(M^2 + 4m_s^2)} \; ;$$

$$m_s^2 \ll M^2 \quad , \quad m_+^2 = M^2 \quad , \quad m_-^2 \simeq \frac{m_s^4}{M^2} \; , \tag{41}$$

where m_s is the Higgs mass of the photon, m_s^2 is by order of the density n. The elastic amplitude for the nonrelativistic fermions equal to

$$A \sim \frac{\gamma}{M^2} \frac{\vec{Q}^2 + m_s^2}{\vec{Q}^2 + \frac{m_s^4}{M^2}} \left(1 - \frac{M}{m} \frac{\vec{Q}^2}{\vec{Q}^2 + m_s^2} \right) \; , \tag{42}$$

and that at zero momentum transfer, there is repulsion, but starting from

$$\vec{Q}^2 > m_s^2 \cdot \frac{m}{M} \sim \frac{m_s^2}{k} \sim \frac{n}{k} \; . \tag{43}$$

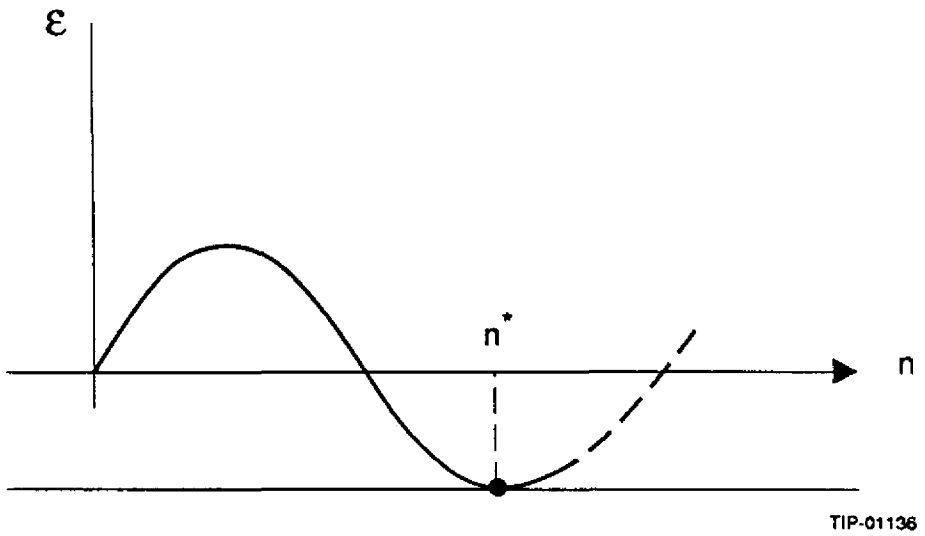

Figure 8. Energy Density

the attraction, it appears again (see Figure 9). If density becomes extremely large, then the attraction region disappears. Then the energy density will be bounded below. However, because there is no attraction at zero momentum, use the probe function like (20) with the large overlapping regions. The detailed analysis of this situation will be published separately. This Higgs model is only a rough approximation, of course. The detailed picture needs the calculation of the polarization operator in the given fermionic background, using the average of the two product currents over the probe wave function:

$$\prod_{\mu\nu}(q) = \int d^3x\, e^{-iqx} \langle \phi | T \tau_\mu(x) \tau_\nu(0) | \phi \rangle \ . \tag{44}$$

The same stabilization mechanism can work, and also in the non-Abelian case, if the instability will occur.

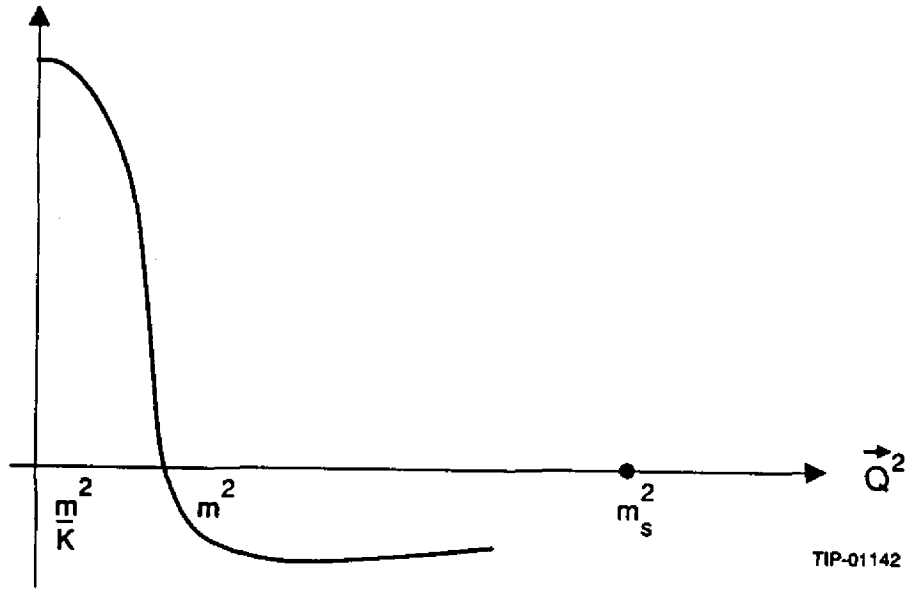

Figure 9. Zero Momentum Transfer Repulsion

Now observe the possible application of vacuum instability. Besides the possible new phases in the planar condensed matter systems, there are two possible applications in the field theory. One is connected with the Open Topological Membrane (OTM) theory, which describes strings as low-energy excitations of the open membranes [9]. The string propagation on some compact space is described by the low-energy limit of some TMGT (may be a direct sum of such models with different groups). If the standard vacuum in this model will be unstable, then the correct low-energy limit will have nothing in common with the Chern-Simons theory, and the string interpretation will be loosened completely. So, vacuum instability can be a serious constraint on possible types of compactification. The second possible application is the high-temperature limit of the usual four-dimensional gauge theories, after the effective 3-dimensional gauge theory is obtained. Without the Higgs mass there is singular infrared behavior and the standard loop expansion does not work [16, 17]. The CS term allows the use of the standard expansion, but the question arises, is it possible to generate it spontaneously. It appears, for example, there is a nonzero chemical potential for fermions in the four-dimensional theory. If creating a new vacuum with the CS term will be energetically favorable, then it is possible to generate some nonzero

chemical potential to create the CS term and after that to get the new vacuum with the smaller energy. If this speculative idea will work, it provides some new mechanisms of the creation of particles at high temperatures. These and other possible applications will be considered in further publications.

In conclusion, it is necessary to emphasize that the problem of the ground state in the TMGT is very important and require further investigations. Y. Kogan would like to thank Prof. F. Gilman and all members of the Physics Research Division of the SSC Laboratory for the hospitality at SSC, where this work was finished.

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APPENDIX 1

Here is presented the explicit expressions for the $W^+W^+ \to W^+W^+$ amplitude A.

The diagram [Figure 2(a)] contribution is:

$$
\begin{aligned}
A^a = & \frac{\gamma}{(Q^2 - M^2)} \, e_{1\mu} e_{2\nu} \bar{e}_{3\sigma} \bar{e}_{4\rho} \\
& \times \Big\{ -9 g_{\mu\rho} g_{\sigma\nu} \left[(p_1 p_2) + \frac{Q^2}{4} + \frac{iM}{Q^2} (p_1 p_2 Q) \right] \\
& + 25 Q_\sigma Q_\rho \left[\frac{1}{2} g_{\mu\nu} - \frac{Q_\mu}{Q^2} (p_1 + Q)_\nu \right] \\
& + 15 g_{\mu\rho} Q_\sigma \left[\frac{1}{2} p_{1\nu} + \frac{1}{2} Q_\nu + \frac{(p_1 p_2)}{Q^2} Q_\nu \right] \\
& - 15 g_{\sigma\nu} Q_\rho \left[\frac{1}{2} p_{2\mu} - \frac{1}{2} Q_\mu - \frac{(p_1 p_2)}{Q^2} Q_\mu \right] \Big\} \, ,
\end{aligned}
\tag{A1}
$$

where

$$
Q^2 = Q_0^2 - \vec{Q}^{\,2} = -\vec{Q}^{\,2} \; ; \; (p_1 p_2 Q) = \epsilon_{\mu\nu\lambda} p_{1\mu} p_{2\nu} Q_\lambda \, ,
$$

and $e_{i\mu}$ are as in (14).

The diagram [Figure 2(b)] contribution is

$$
\begin{aligned}
A^b = & \frac{\gamma}{\tilde{Q}^{\,2} - M^2} \, e_{1\mu} e_{2\nu} \bar{e}_{3\sigma} \bar{e}_{4\rho} \\
& \times \Big\{ -9 g_{\mu\sigma} g_{\nu\rho} \left[(p_1 p_2) + \frac{\tilde{Q}^{\,2}}{4} + \frac{iM (p_1 p_2 \tilde{Q}}{\tilde{Q}^{\,2}} \right] \\
& + 25 \tilde{Q}_\rho \tilde{Q}_\sigma \left[\frac{1}{2} g_{\mu\nu} - \frac{\tilde{Q}_\mu}{\tilde{Q}^{\,2}} (p_1 + \tilde{Q})_\nu \right]
\end{aligned}
$$

$$+ 15 g_{\mu\sigma}\tilde{Q}_\rho \left[\frac{1}{2}(p_1 + \tilde{Q})_\nu + \frac{(p_1 p_2)}{\tilde{Q}^2}\tilde{Q}_\nu\right]$$

$$- 15 g_{\rho\nu}\tilde{Q}_\sigma \left[\frac{1}{2}(p_2 - \tilde{Q})_\mu - \frac{(p_1 p_2)}{\tilde{Q}^2}\tilde{Q}_\mu\right]\Bigg\} , \qquad (A2)$$

and the last diagram [Figure 2(c)] contribution is

$$A^c = e_{1\mu} e_{2\nu} \bar{e}_{3\sigma} \bar{e}_{4\rho} (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - 2 g_{\mu\nu} g_{\rho\sigma}) . \qquad (A3)$$

In the center of mass frame $\vec{p_1} = -\vec{p_2}$ the momenta transfers are

$$\vec{Q}^2 = 2\vec{p}^2(1 - \cos\theta)$$

$$\vec{\tilde{Q}}^2 = (\vec{Q} - 2\vec{p})^2 = 2\vec{p}^2(1 + \cos\theta) . \qquad (A4)$$

To calculate (A1), (A2), and (A3) use the next equations:

$$(e_1 e_2) = (\bar{e}_3 \bar{e}_4) = \frac{p^2}{M^2}$$

$$(e_1 \bar{e}_3) = (e_2 \bar{e}_4) = \frac{1}{2M^2}[p^2 + (E^2 + M^2)\cos\theta - 2iME\sin\theta]$$

$$(e_1 \bar{e}_4) = (e_2 \bar{e}_3) = \frac{1}{2m^2}[p^2 - (E^2 + M^2)\cos\theta + 2iME\sin\theta] . \qquad (A5)$$

$$(e_1 Q) = -(e_2 Q) = (e_3 Q) = -(e_4 Q) = \frac{p^2[E(1 - \cos\theta) + iM\sin\theta]}{\sqrt{2}\, Mp}$$

$$(e_1 p_1) = -(e_1 p_2) = (e_2 p_2) = -(e_2 p_1) = \sqrt{2}\, p\, \frac{E}{M}$$

$$(\bar{e}_4 p_2) = (\bar{e}_3 p_1) = \frac{p}{\sqrt{2}\, M}[E(1 + \cos\theta) - iM\sin\theta] . \qquad (A6)$$

Substituting (A4), (A5) and (A6) into (A1), (A2) and (A3) and taking the limit $\vec{p}^2 \to 0$, Eq. (15) is derived.