SSCL-N-835

The Description of Magnetic Quadrupoles Using Optical Terminology and Techniques

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October 1993

*Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

1.0 INTRODUCTION

The purpose of this note is to lay the foundation for a discussion of magnetic lenses (quadrupoles) in terms of optical quantities such as focal lengths, focal points, and principal planes. In optics terminology a lens is an optical system bounded by two or more refracting surfaces having a common axis. If the lens has only two surfaces, it is called a simple lens. If the lens has more than two surfaces, it is called a compound lens. Shown in Figure 1(a) is a schematic diagram of a simple lens in an imaging configuration. Associated with any simple lens is a quantity know as the *focal length,* which we denote as *f*. This focal length is defined in terms of four other quantities, known as *first and second focal points* and *first and second principal planes.*

Shown in Figure 1(b) is a schematic representation of the first focal point and first principal plane of a simple (positive) lens. By definition, the first focal point of a lens, denoted as *F*, is the object point on the axis that is imaged to infinity by the lens. All rays exiting from the first focal point will propagate parallel to the axis after passing through the lens. The first principal plane, denoted as *H*, is an "imaginary" plane where the light rays exiting the first focal point intersect the backward projection of the rays exiting the lens. The distance from the first focal point to the first principal plane is equal to the focal length *f* of the lens.

Shown in Figure 1(c) is a schematic representation of the second focal point and second principal plane of a simple (positive) lens. By definition, the second focal point of a lens, denoted as F', is the image point on the axis of an object located at (negative) infinity. All rays propagating parallel to the axis before the lens will converge at the second focal point after passing through the lens. The second principal plane, denoted as H', is an "imaginary" plane located at the intersection of the extension of the rays propagating into the lens with those exiting the lens. The focal length *f* is also equal to the distance between the second focal point and the second principal plane. In Figure 2 we show these points and planes for a negative lens. Note that for a negative lens the focal length *f* is a negative quantity.

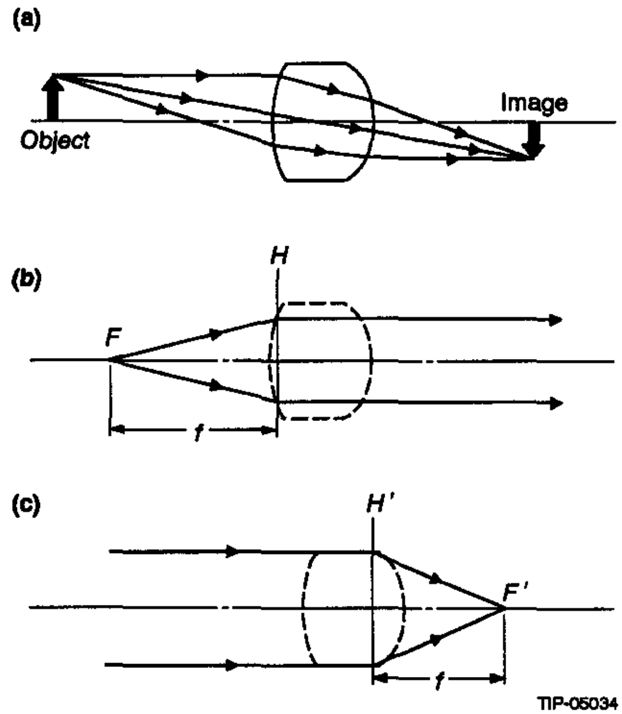

Figure 1. Schematic representation of (a) a positive lens in imaging configuration, (b) first focal point and first principal plane, (c) second focal point and second principal plane.

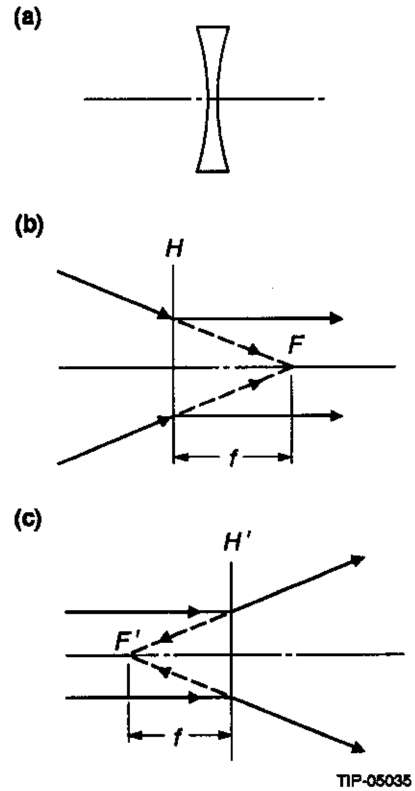

Figure 2. The negative lens: (a) schematic, (b) first focal point and first principal plane, (c) second focal point and second principal plane.

2.0 THE IMAGING CONDITION

In optics these parameters (focal length, focal points, *etc.*) are very useful because they can be used to describe the way that a lens forms an image. In particular, if s is the distance of an object from the first principal plane and s' is the distance of the image from the second principal plane, then these two distances are related by the simple lens equation, *i.e.*,

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \ , \tag{1}$$

where f is obviously the focal length of the lens. Such an imaging situation is shown schematically in Figure 3 for a positive lens.

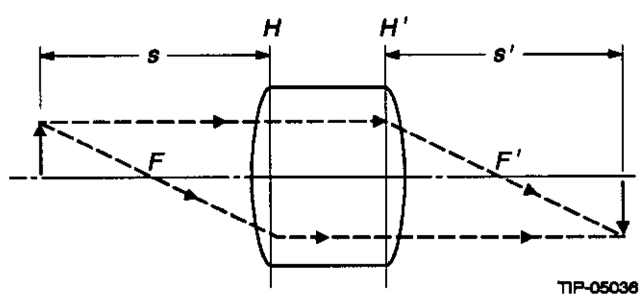

Figure 3. The imaging condition shown in terms of focal points, image distances, and principal planes.

These parameters are also useful to us because they provide a convenient way to combine a series of simple lenses into an equivalent single simple lens. This is accomplished by repeated application of the Lens Equation (1) to determine the location of the focal points. The focal length of the lens combination is given as

$$f = s_1'\left(\frac{-s_2'}{s_2}\right)\left(\frac{-s_3'}{s_3}\right) \cdots , \tag{2}$$

where s_i and $s_i'(i = 1, \ldots)$ are the object and image distances obtained during the process of determining the focal points. These comments are best illustrated with an example.

Consider the lens system shown schematically in Figure 4(a), in which two positive lenses are separated by a nominal distance of 10 cm. The first lens has a focal length $f_1 = 20$ cm, and the second lens a focal length $f_2 = 10$ cm. The principal planes of the first lens are separated by a distance of 2 cm, and of the second lens by 4 cm, as illustrated in Figure 4(b). (Note that for convenience we have attached an "absolute" scale in which distance is measured from the first principal plane of lens 1.) We begin by considering an object located at (negative) infinity, *i.e.*, $s_1 = -\infty$. Using Eq. (1) we see that this object is imaged at the focal length of the first lens, which is equal to a distance 20 cm past the second principal plane. Thus, $s_1' = f_1 = 20$. This then becomes the object for the second lens as measured from the first principal plane of the second lens. Inasmuch as this object lies to the right of H we have $s_2 = -12$. Again using Eq. (1) with $f = 10$, we obtain $s_2' = 5.45$. Thus, we see that the image is formed a distance of 5.45 cm past the second principal plane of lens 2. In terms of our absolute scale this is a distance equal to 19.45 cm. The focal length of the system is obtained using Eq. (2), with $s_1' = 20$, $s_2 = -12$, and $s_2' = 5.45$, *i.e.*,

$$f = 20\left(\frac{5.45}{12}\right) = 9.09 .$$

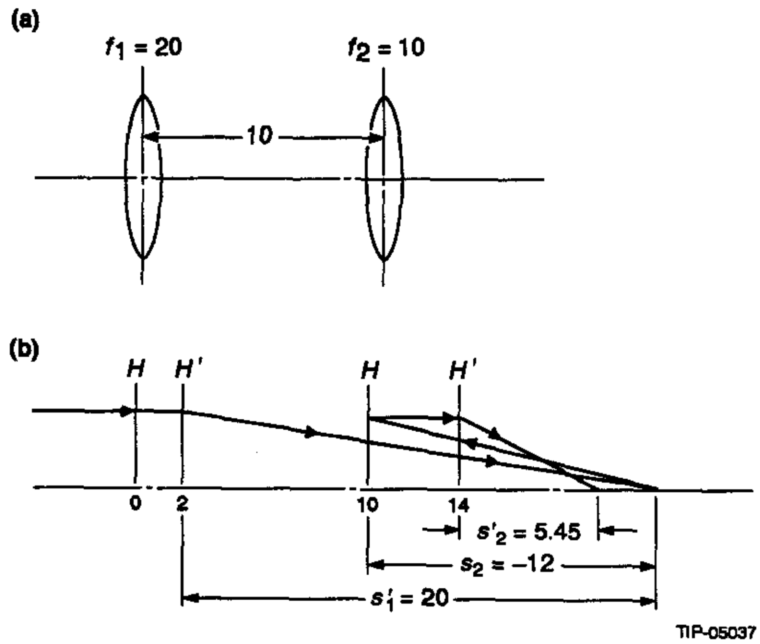

Figure 4. (a) Lens system containing two simple lenses. (b) The principal planes of the lenses and the imaging process used to find the equivalent second principal plane and focal length of a single lens system.

The second principal plane of the lens system lies a focal length to the left of the second focus. Thus, we see that H' is located at 10.36 cm. The first focal point F is obtained by propagating a parallel beam backward (from infinity) through the system to determine the location of the point of origin of the rays. From a practical point of view this operation is best accomplished by inverting the original system and repeating the process by which we determined the second focal point. This inverted system is shown in Figure 5. (Note that in the inverted system H becomes H'.) Using similar logic as before, we determine that $s_1' = 10$, $s_2 = -2$, $s_2' = 1.18$, and again $f = 9.09$. From these values (see Figure 5(b)) we determine that in the inverted system this principal plane lies at 6.73 cm. When this is transposed back to the original system we find that the first principal plane lies at 7.27 cm. The equivalent single simple lens system is shown in Figure 6.

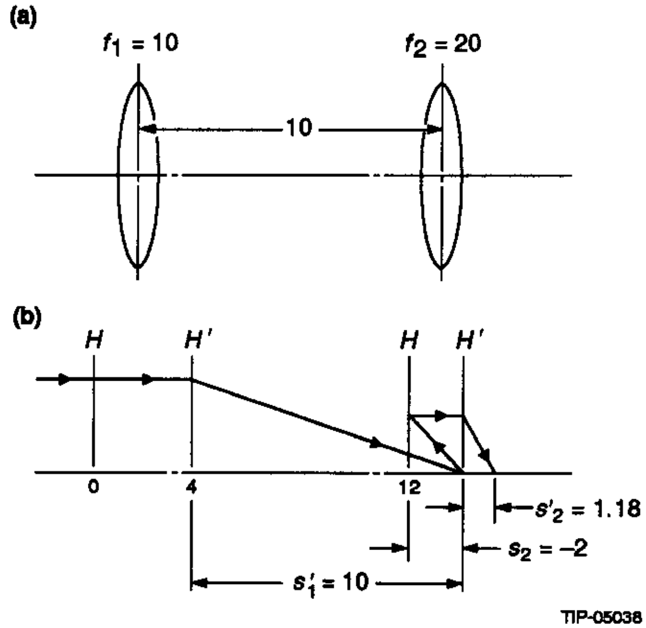

Figure 5. Inverted optical system (a) in which the imaging process (b) to determine the first focal point and principal plane are schematically illustrated.

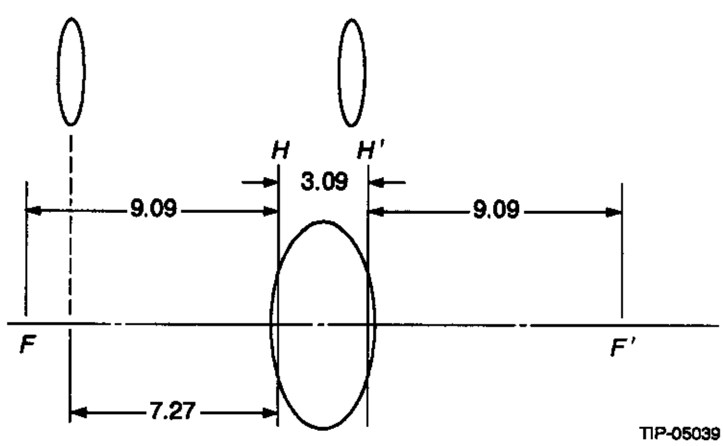

Figure 6. Single lens system equivalent to dual lens system of Figure 4.

3.0 THE QUADRUPOLE AS A SIMPLE LENS

In this section we extend the previous results to describe quadrupole magnets in terms of optical parameters such as focal lengths, principal planes, and focal points. Shown in Figure 7 is a schematic representation of a positive (magnetic) lens of length L. A charged particle beam is assumed to be propagating from left to right. In this figure the state of the particle entering the lens is given as $[x_i, y_i]$ and that exiting the lens is given as $[x_o, y_o]$.

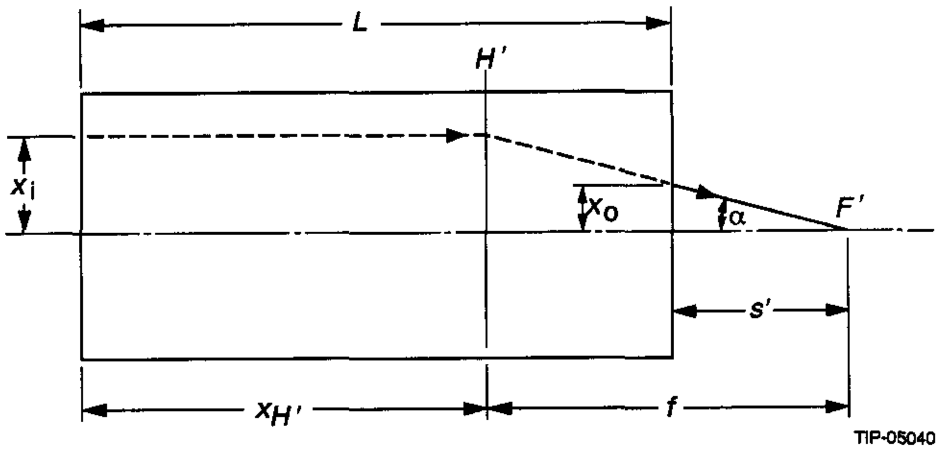

Figure 7. Schematic representation of a focusing quadrupole lens showing the second focal point F' and second principal plane H'.

Building on the optical concepts presented in Section 2.0, we define the second focal point F' of a quadrupole lens as the location on the axis where a parallel particle beam entering the lens will focus upon exiting the magnet. To define a parallel particle beam we simply set $y_i = 0$. We next denote the distance from the output of the lens to the point where the beam crosses the axis (*i.e.*, $x = 0$) as s'. It is a relatively simple matter to determine this distance in terms of the lens parameters (K and L). To accomplish this we first track the parallel particle beam through the magnetic quadrupole using the transfer matrix for a thick lens, *i.e.*,

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{bmatrix} \begin{bmatrix} x_i \\ 0 \end{bmatrix} = \begin{bmatrix} x_i\cos\sqrt{K}L \\ -x_i\sqrt{K}\sin\sqrt{K}L \end{bmatrix} .$$

We next use the transfer matrix for a drift of length s' and require that the state of a particle at the end of the drift be equal to $[0, y_f]$. This results in

$$\begin{bmatrix} 0 \\ y_f \end{bmatrix} = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_i\cos\sqrt{K}L \\ -x_i\sqrt{K}\sin\sqrt{K}L \end{bmatrix} = \begin{bmatrix} x_i\cos\sqrt{K}L - x_i s'\sqrt{K}\sin\sqrt{K}L \\ -x_i s'\sqrt{K}\sin\sqrt{K}L \end{bmatrix} .$$

From the first of the above two implied equations we find

$$s' = \frac{1}{\sqrt{K}} \frac{\cos\sqrt{K}L}{\sin\sqrt{K}L} . \tag{3}$$

Again extending the optical analogy of Section 2.0, we define the *principal plane* of a quadrupole as a plane (perpendicular to the optical axis) located at the intersection of the forward projection of the entry particle ray and the backward projection of the output particle ray. This plane is shown in Figure 7 and is denoted as H'. The distance from this principal plane to the second focal point F', or "focus" of the beam, is defined as the focal length and is denoted as f. Using basic trigonometric analysis we are able to easily solve for this distance in terms of the parameters of the magnetic quadrupole, *i.e.*,

$$\tan\alpha = \frac{x_o}{s'} = \frac{x_i}{f} \ , \qquad \text{or}$$

$$f = \frac{x_i\, s'}{x_o} = x_i\left[\frac{\cos\sqrt{K}L}{\sqrt{K}\sin\sqrt{K}L}\right]\left[\frac{1}{x_i\cos\sqrt{K}L}\right] , \qquad \text{or}$$

$$f = \frac{1}{\sqrt{K}\sin\sqrt{K}L} \ . \tag{4}$$

Note that in the above equation for small values of L (*i.e.*, a thin magnetic lens), we approximate the sine of an argument by the argument itself (*i.e.*, $\sin\alpha \approx \alpha$) and obtain

$$f \approx \frac{1}{KL} \ ,$$

which is the well-known expression for the focal length of a thin magnetic lens.

Combining Eqs. (3) and (4) we obtain a clean expression for the distance s, *i.e.*,

$$s' = f\cos\sqrt{K}L \ . \tag{5}$$

The location of the second principal plane of a focusing quadrupole (measured from the entrance of the magnet) is given as

$$x_{H'} = L - (f - s') = L - f\left(1 - \cos\sqrt{K}L\right) \qquad \text{or}$$

$$x_{H'} = L - \frac{\left(1 - \cos\sqrt{K}L\right)}{\sqrt{K}\sin\sqrt{K}L} \ . \tag{6}$$

To determine the first focal point F and principal plane H we simply invert the system and propagate the beam backward through the magnet. However, due to the complete symmetry of the system we will obtain the exact same parameters as just described. In other words, the first principal plane will be located a distance x_H' from the exit of the magnet. Consequently, when expressed in terms of the distance from the entrance of the magnet, the location of the first principal plane is given as

$$x_H = \frac{\left(1 - \cos\sqrt{K}L\right)}{\sqrt{K}\sin\sqrt{K}L} \ . \tag{7}$$

We now examine the situation for a defocusing quadrupole, which is shown schematically in Figure 8. In this figure we show a parallel particle ray entering the quad, whose state is denoted again as $[x_i, 0]$. The effect of this negative lens is to deflect, or "defocus," the beam away from the axis. The state of the particle ray as it exits this element is denoted as $[x_o, y_o]$. The relationship between the input and output states is obtained by using the transfer matrix for a defocusing lens, *i.e.*,

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ -\sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{bmatrix} \begin{bmatrix} x_i \\ 0 \end{bmatrix} = \begin{bmatrix} x_i\cosh\sqrt{K}L \\ -x_i\sqrt{K}\sinh\sqrt{K}L \end{bmatrix} .$$

Equating components, we see

$$x_o = x_i\cosh\sqrt{K}L \qquad \text{and} \qquad y_o = x_i\sqrt{K}\sinh\sqrt{K}L ,$$

where y_o is the slope of the output ray. However, from the figure we also see that

$$y_o = \frac{x_o}{d} ,$$

which we will take advantage of shortly. For a negative lens, the second focal point F' is obtained by projecting the exit particle ray backward until it intersects the "optical" axis. Again (as we did for the positive lens case) the second principal plane H' is located by the intersection of (the extension of) the input particle ray and (the backward projection of) the output particle ray. This plane is denoted as H' in the figure. From the geometry presented in Figure 8 we see that

$$\frac{x_i}{f} = \frac{x_o}{d} = y_o = x_i\sqrt{K}\sinh\sqrt{K}L , \tag{8}$$

or

$$f = \frac{1}{\sqrt{K}\sinh\sqrt{K}L} . \tag{9}$$

A slight rearrangement of Eq. (8) gives

$$d = \frac{x_o}{x_i}f = \frac{x_i\cosh\sqrt{K}L}{x_i}f = f\cosh\sqrt{K}L .$$

Finally, we have (see Figure 8)

$$s' = d - L \qquad \text{or}$$

$$s' = f\cosh\sqrt{K}L - L .$$

The location of the second principal plane when measured from the entrance of the magnet is given as

$$x_H{}' = f - s' = f - f\cosh\sqrt{K}L + L ,$$

$$x_H{}' = L + \frac{\left(1 - \cosh\sqrt{K}L\right)}{\sqrt{K}\sinh\sqrt{K}L} . \tag{10}$$

Again, using arguments of symmetry we determine that the distance from the entrance of the magnet to the location of the first principal plane H is given as

$$x_H = \frac{\cosh\sqrt{K}L - 1}{\sqrt{K}\sinh\sqrt{K}L} . \tag{11}$$

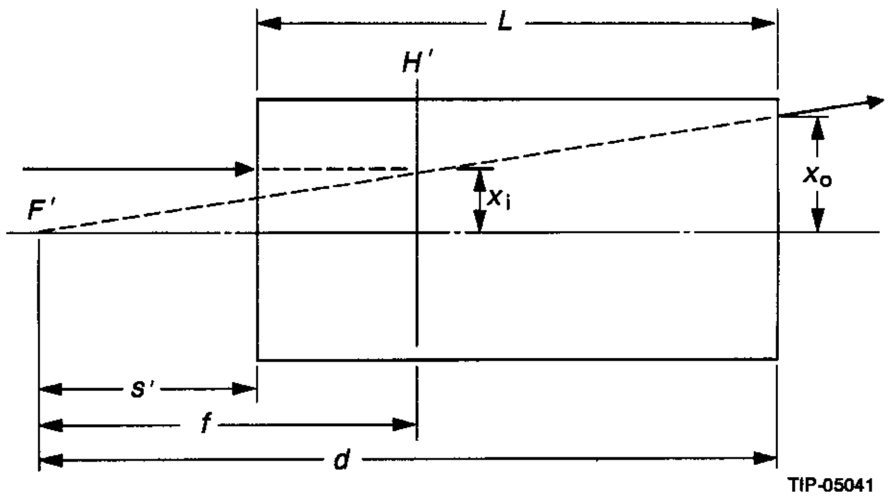

Figure 8. Schematic representation of a defocusing quadrupole lens showing the second focal point *F'* and the second principal plane *H'*.

3.1 Example 1

As an example we consider a Superconducting Super Collider (SSC) arc sector FODO quadrupole with a magnetic length of 5.2 m and a magnetic gradient of 205.6 T/m, which corresponds to a value* of $K = 3.0819 \times 10^{-3}$ m^{-2}. When this magnet is used in the positive mode, Eq. (4) tells us that the focal length is equal to 63.2742 m. The first and second principal planes are located 2.618 21 m and 2.581 79 m, respectively, from the entrance of the component. When used in the negative mode we obtain a focal length of 61.5407 m, with the first and second principal planes located at 2.582 09 m and 2.617 91 m, respectively. In Figure 9 we show this magnet in the positive mode. We have placed this magnet between two drifts of lengths 117.38 m and 131.23 m, respectively. These lengths were chosen such that the imaging relationship of Eq. (1) is satisfied (*i.e.*, $s = 120$ and $s' = 133.85$). Several particles at two different transverse locations from the axis were then propagated (tracked) through this system. Note that the particles do indeed obey the imaging condition.

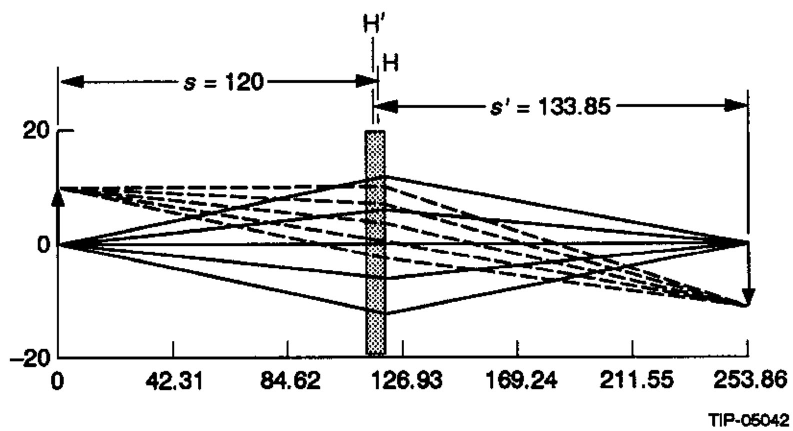

Figure 9. FODO quadrupole in focusing mode shown in an imaging configuration.

*This value is obtained (for 20 TeV) by multiplying the gradient value by 1.49898×10^{-5}.

3.2 Example 2

Let us now consider the SSC Baseline Interaction Region (IR) QVD/QVF quadrupoles, which have a magnetic length of 14.252 m and a magnetic gradient of 180.01 ($K = 2.6983 \times 10^{-3}$). When used in the focusing mode these quads have a focal length of 28.54 m, with the first and second principal planes located at 7.47 m and 6.78 m, respectively, from the entrance of the magnets. Now let us place two of these magnets in tandem separated by a distance d, as illustrated in Figure 10. Using techniques identical to those used for the optical system shown in Figure 4, we construct Table 1 for various separation distances. (Note that all distances for H and H' are given in meters and are measured from the entrance of the first quadrupole.)

TABLE 1. SEPARATION DISTANCES.

d	H	H'	f
5	24.01	9.50	22.53
10	31.25	7.23	26.16
15	41.25	2.25	31.16
20	55.99	–7.49	38.53
25	79.87	–26.36	50.47

Shown in Figure 11 is such a system with a separation distance of $d = 10$ m and set up for one-to-one imaging, with $s = s' = 2f$.

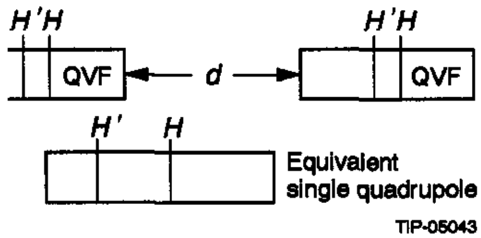

Figure 10. Schematic representation of two QVF quadrupoles separated by a distance *d*.

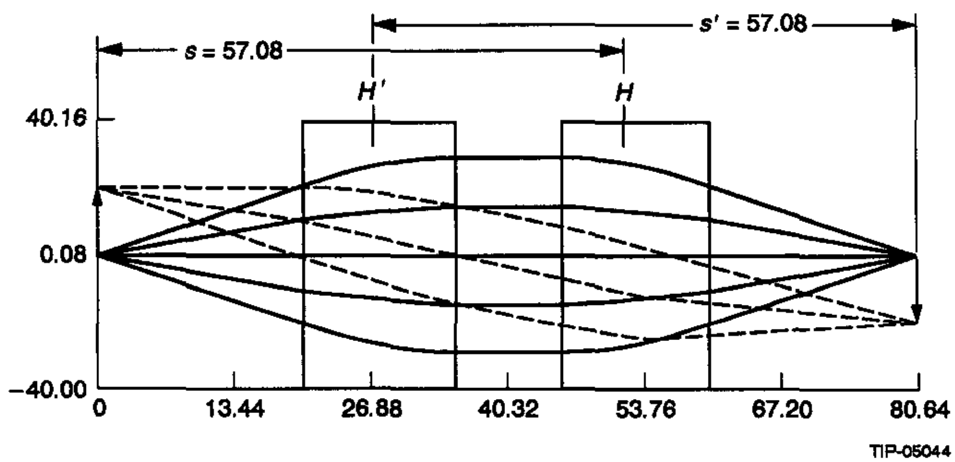

Figure 11. Two QVF quadrupoles separated by a distance of 10 m and used in a one-to-one imaging configuration.