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Comment on the Analysis of the Stability Domain for a Single Sextupole Map

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Abstract

We show that a simple and fast algorithm based on particle tracking can determine the border of the central island of a symplectic map modelling a single sextupole with desired accuracy on a personal computer, whereas a recently proposed "analytic" method requires the use of a Cray supercomputer.

The dynamic aperture can be defined as the maximum phase space area of a beam up to which the motion of the particles of the beam remain bounded. The standard method to determine this quantity is particle tracking. However, this method is purely numerical and does not give informations on the actual process of transport in phase space. Recently, a new method has been proposed [1], where the dynamic aperture is obtained by finding the minimal distance between the origin and the invariant manifolds of the unstable fixed point of the Hénon map [2],

$$x' = \cos(2\pi Q)x - \sin(2\pi Q)(y - x^2), \quad (1)$$

$$y' = \sin(2\pi Q)y + \cos(2\pi Q)(y - x^2), \quad (2)$$

where Q denotes the tune parameter. This map models particle dynamics through an individual sextupole. Recent experiments performed at Fermilab shows that the kind of effects predicted by the study of the Henon map can be reproduced in the real life [3-5].

To understand why this method works we must briefly review certain facts of dynamical system theory [6]. Any symplectic map, including the maps modelling particle dynamics in accelerators, have fixed points, that is equilibrium points where a particle would remain forever if left unperturbed. The Hénon map, for example, has two fixed points. The stable fixed point is the origin, and the unstable fixed point is known analytically: $X = (2\tan(\pi Q), 2\tan^2(\pi Q))$. The former one is at the center of the stability domain of the map, where one should place the beam. The latter one lies in the chaotic region of the phase space.

There is a set of points, the invariant manifold, associated to the unstable fixed point. The invariant manifold of X is defined as the set of all points going asymptotically to X as the number of turns of the map goes to $+\infty$ or $-\infty$. The $+\infty$ limit is the stable manifold of X, $M_s(X)$, and the $-\infty$ limit is the unstable manifold of X, $M_u(X)$. It is clear that invariant manifolds are fundamentally important in the understanding of transport processes in symplectic maps, because they are defined in terms of time asymptotic objects. Three examples are displayed in Fig. 1, 4 and 5, for different values of Q. They are obtained by means of the following procedure. The map is linearized at X, and the stable and unstable eigenvectors $\mathbf{e}_s$ and $\mathbf{e}_u$ of the corresponding linear map are found. Starting from M initial conditions distributed on $\mathbf{e}_u$ and $\mathbf{e}_s$, within a short distance ϵ from X, one iterates the

map N times, respectively forward and backward in time. Fig. 1, 4 and 5 have been made using $\epsilon = 0.001$, $M = 4000$ and $N = 15$. As it is possible to see from the pictures, the stable and unstable manifolds intersect at points called homoclinic points. Homoclinic points are points that converge to X in both directions of time. A Beside the fixed point X, a symplectic map like the Hénon map, has infinitely many periodic orbits. Each unstable periodic orbit has stable and unstable manifolds associated with it. Fig. 2 shows both invariant manifolds of the unstable fixed point and of the period five orbit, obtained using $M = 200$ and $N = 100$. The invariant manifolds of different periodic orbits in the chaotic region intersect each other at points called heteroclinic points. Heteroclinic points are points that in the infinitely remote past were very close to one periodic orbit and in the inifinitely far future will be very close to another periodic orbit. In Fig. 3, one sees how the invariant manifolds of X and of the period five orbit are interwoven in a beautifully complex fashion. Fig. 2 does not show any intersection of the two manifolds because we have not iterated for a sufficiently large number of turns, in order to show that the two manifolds are indeed two distinct objects, associated to different periodic orbits.

By heteroclinic intersections with successive periodic orbits, the invariant manifold of the fixed point X can reach regions as close to the stability domain of the map as one likes. Fig. 3 illustrates this concept. The stability domain of Hénon's map is the pentagonal region between the origin and the invariant manifold. Fig. 5 shows a similar construction for a different tune. Here the stability domain is the heart-shaped region close to the origin, completely surrounded by points of the invariant manifold of X. The conclusion is that one can obtain an exact definition of the dynamic aperture of a map by reconstructing the invariant manifolds associated to the unstable fixed point of the map in the chaotic region.

Our comment concerns the following observation. The border of the stability region of the map is, in terms of map iterations, infinitely far from the unstable fixed point both in the past and in the future. This is because the points in the stability region are forever confined there and will never go into the chaotic region. This means that the rate of filling of the invariant manifold becomes extremely slow when the manifold gets sufficiently close to the stability region.

The practical consequence of this fact is that, in order to achieve sufficient accuracy in the definition of the central region limited by the invariant

manifold, one needs much more particles and iterations (typically $M = 10^5$ and $N = 10^3$), making this method extremely expensive. We must also consider that calculations of the dynamic aperture for maps modelling accelerator dynamics accounting for higher order non-linear terms and both transversal and longitudinal oscillations, are bound to be even more time-consuming, because of the higher complexity of the map. ΘΠ We show here that standard particle tracking, with minor modifications, can yield the same degree of accuracy with affordable computational requirements. The idea is very simple. Instead of reaching the stability region from the unstable fixed point deep in the chaotic region, by means of the lengthy construction of the associated invariant manifold, we find directly the border between chaotic and regular region of phase space by a binary search algorithm, exactly as we were looking for the zero of a function.

Basically, we identify the orbit at the border between the central stable region and the surrounding chaotic region, by tracking a small number of particle (typically 10), along the line joining the origin with the unstable fixed point. The Lyapunov exponent or the escape time are calculated for each particle. A new smaller interval is defined by the last stable particle and the first unstable one. This procedure is repeated until the desired accuracy is obtained. With this method we obtained the border between regular and chaotic region plotted in Fig. 6, at the same value of the tune as in Fig. 5, with 8 digits accuracy in the determination of the initial conditions, running a simple *Mathematica* notebook on a NeXT station, for 155 sec. The method requiring the calculation of the invariant manifolds to the same accuracy requires on the same machine about 4.17 days. With a C or FORTRAN program, the requested time is decreased to a few hours, which is still too much, since the calculation must be repeated for a large number of values of the parameter, which would force us to use a supercomputer.

Fig. 7 shows in the square $[-1,1]\times[-1,1]$ the orbits separating the central regular region from the chaotic region, obtained with our method, at several values of the tune parameter Q. The cases $Q = 0.26$ and $Q = 0.35$ are close, respectively, to the $1/4$ and $1/3$ resonances. Fig. 8 shows the dynamic aperture obtained by our fast tracking algorithm, plotted against Q. The aperture is drastically reduced in the vicinity of the major resonances, as one would expect.

We are presently implementing the same procedure in the case of a 12th order six-dimensional symplectic map modelling beam dynamics in the SSC.

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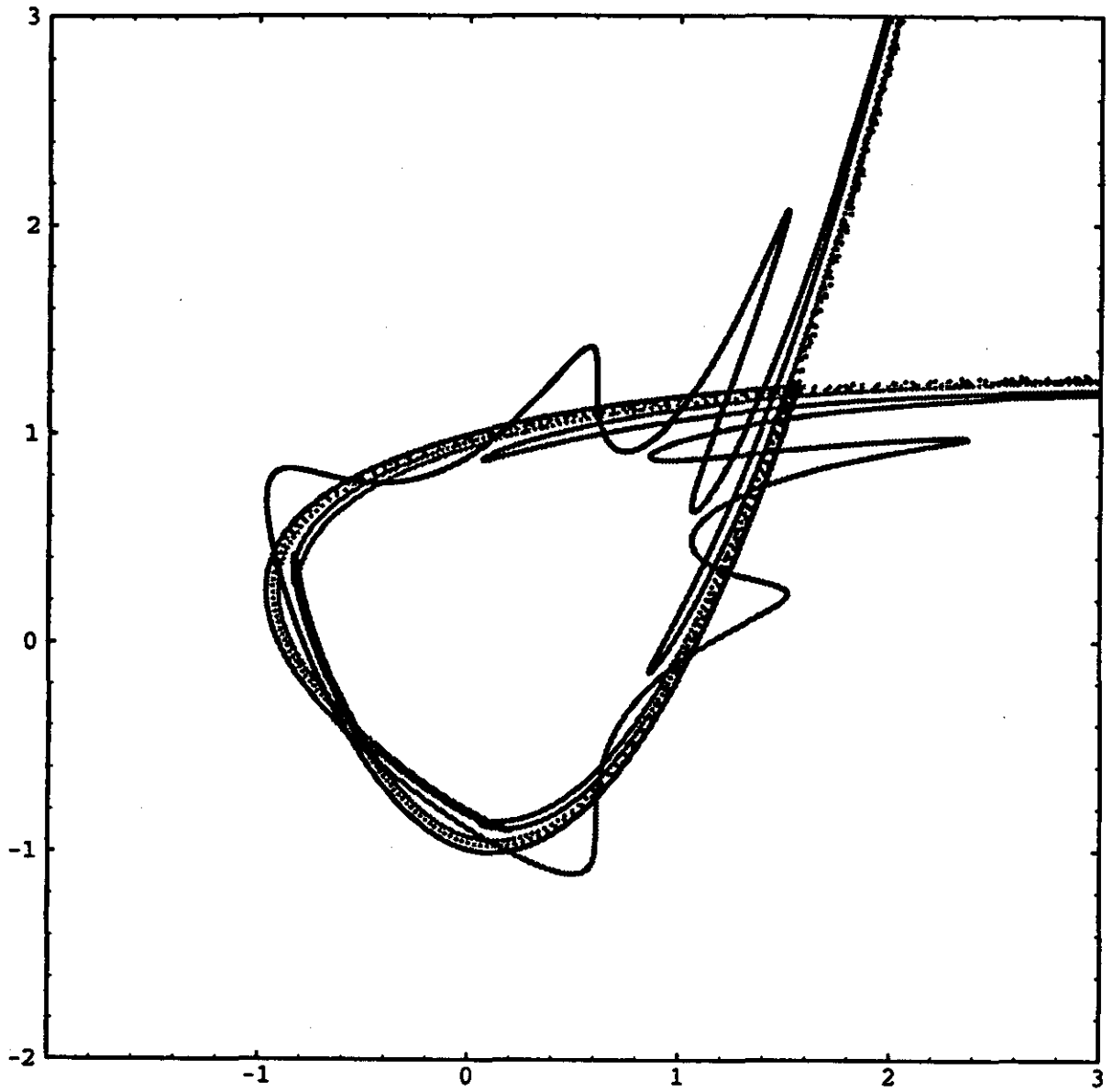

Fig. 1: Stable and unstable manifolds of the unstable fixed point of the Hénon map, at the tune Q = 0.2114.

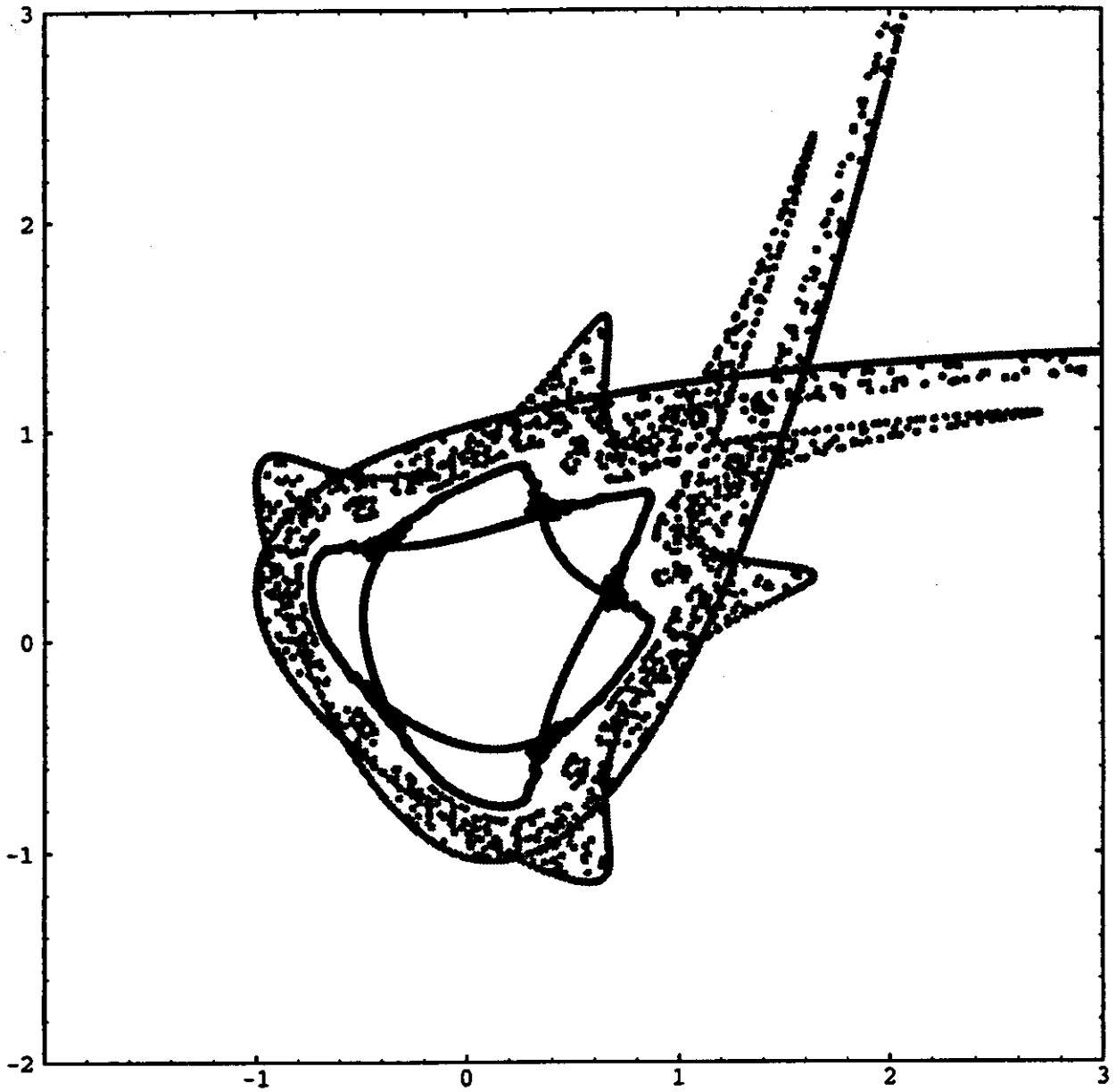

Fig. 2: Stable and unstable manifolds of the unstable fixed point of the Hénon map, at the tune Q = 0.215.

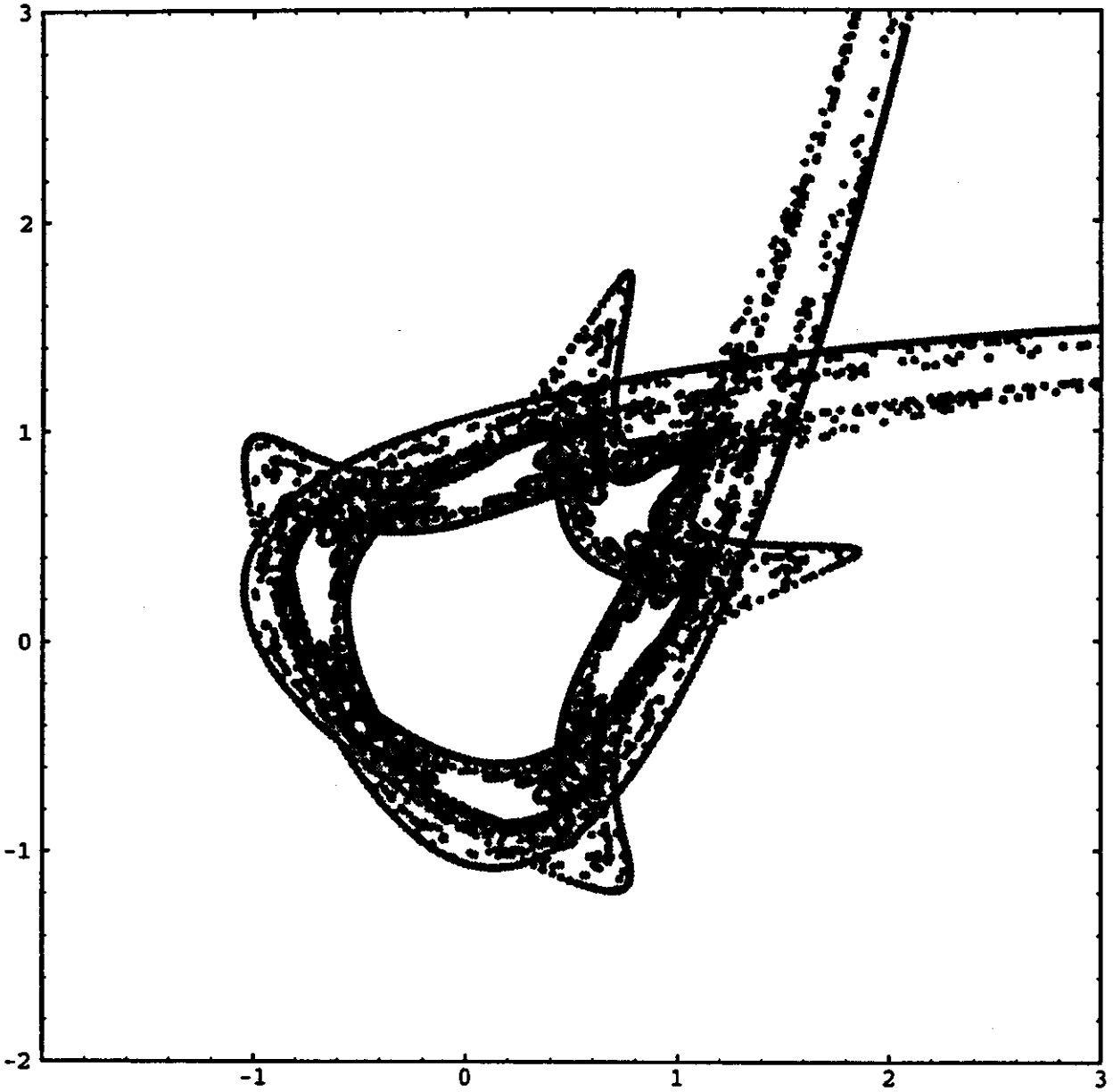

Fig. 3: Stable and unstable manifolds of the unstable fixed point and of the unstable period five orbit of the Hénon map, at the tune Q = 0.215.

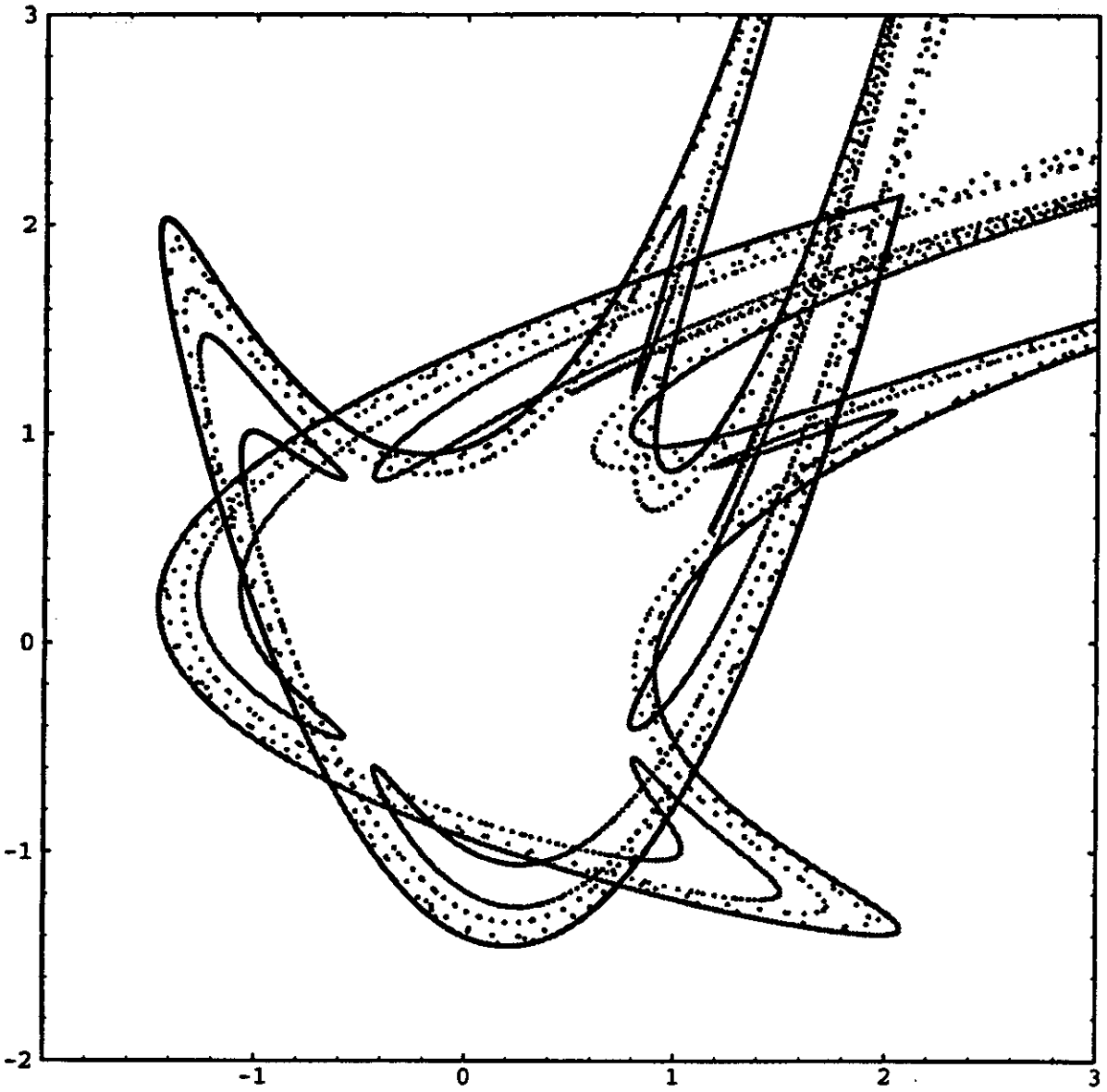

Fig. 4: Stable and unstable manifolds of the unstable fixed point of the Hénon map, at the tune Q = 0.255.

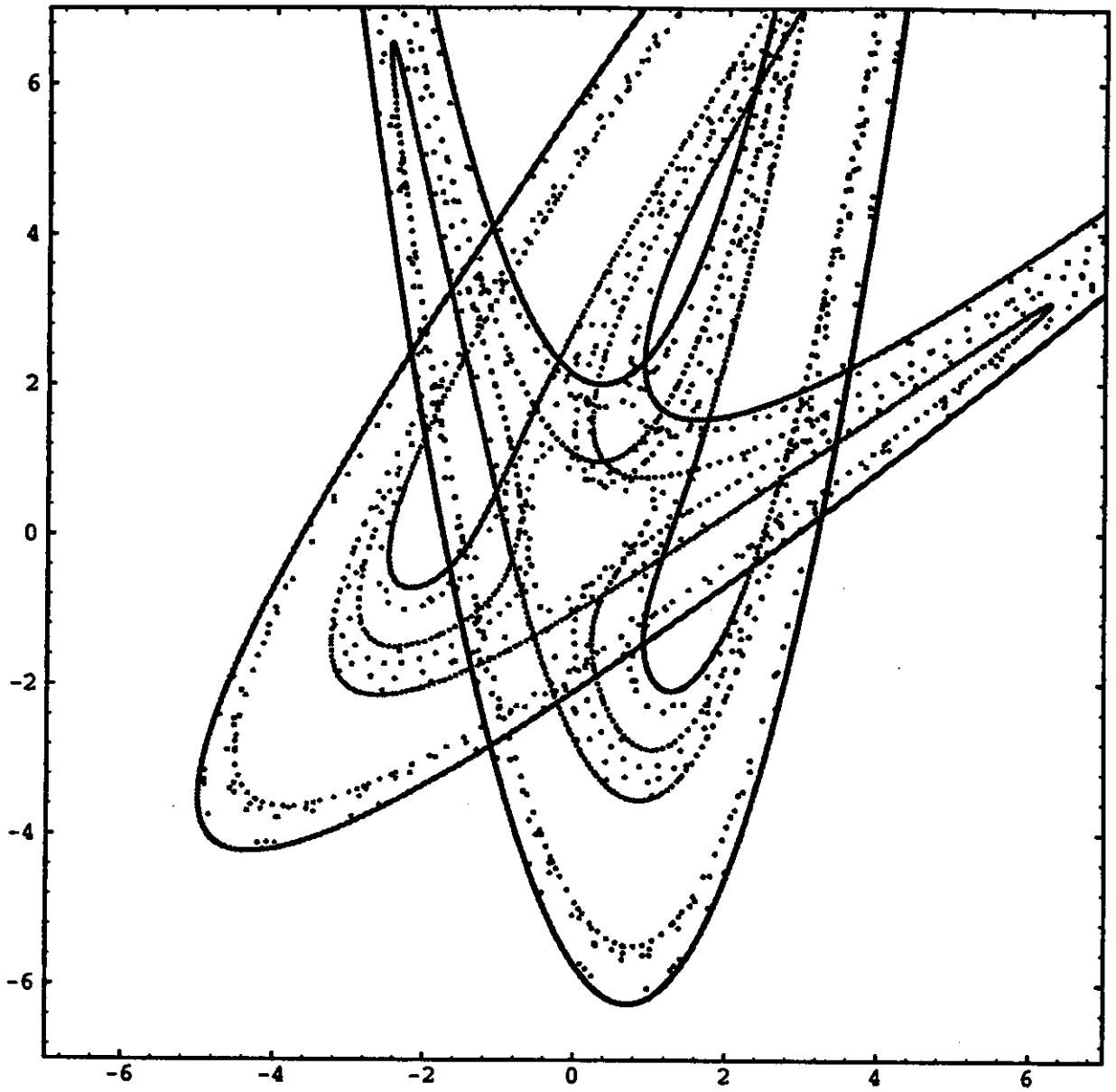

Fig. 5: Stable and unstable manifolds of the unstable fixed point of the Hénon map, at the tune Q = 0.38.

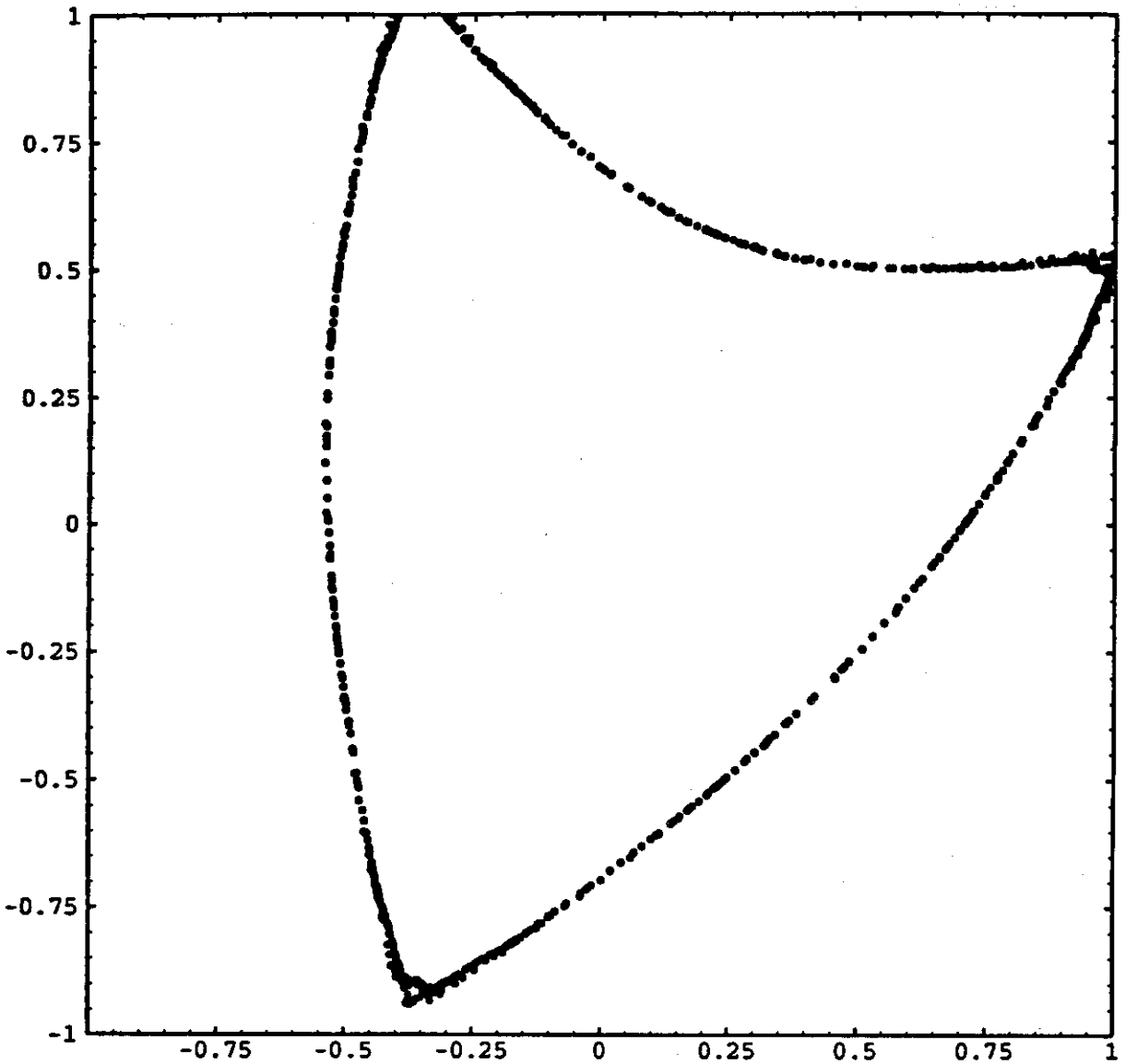

Fig. 6: Orbit at the border separating the stable and the chaotic regions of the phase space of Hénon's map, at the tune Q = 0.38.

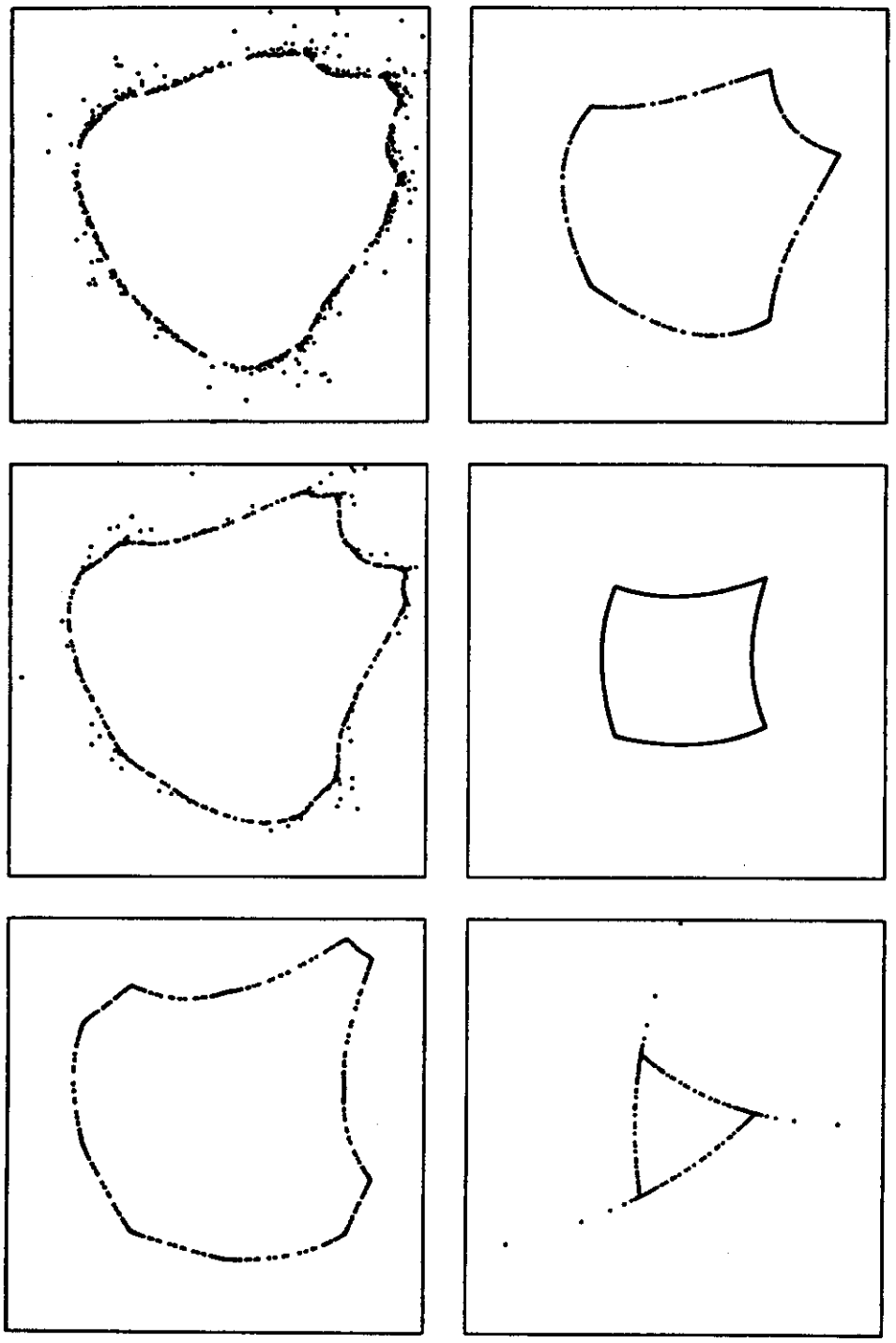

Fig. 7: Orbits separating the stable and the chaotic regions of the Hénon map, for Q = 0.205, 0.22, 0.235, 0.26, 0.285, 0.35, going from the left to the right and from the top to the bottom.

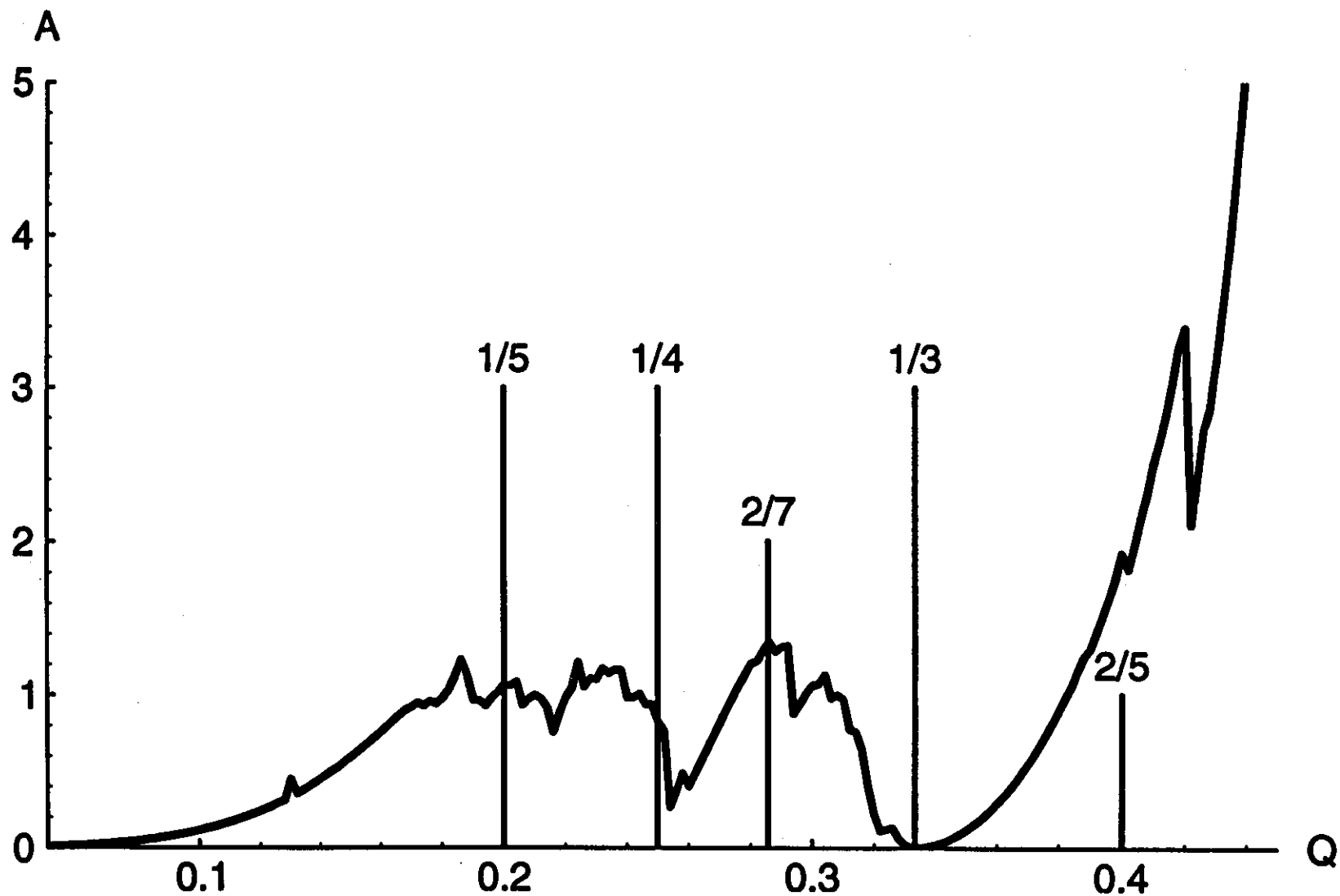

Fig. 8: The dynamic aperture of the Hénon map plotted against the tune Q.