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Estimate of Effects of Pileup on the Mean Energy and Spread of Energy in a Tower

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Consider the following simplified diagram (Figure 1) of the signal processing for a single calorimeter tower.

Figure 1. Signal Processing for a Single Calorimeter Tower.

This note estimates the average signal at point A (before digitization and before any thresholds are applied) due to event pileup. The average signal is denoted by $\bar{E}_{\text{tower}}$, and the variation as ΔE_{tower}. These depend on the following variables:

1. Average number of events per beam crossing $\equiv \bar{n}_{\text{ev}}$.
2. Average number of particles produced per unit $n \equiv \bar{n}_p$ (it is assumed that the particles are uniformly distributed in n and ϕ).
3. Average energy per particle $\bar{E}$.
4. RMS spread in particle energies ΔE.
5. The shaping function $h(t)$. This includes both the integrator and the shaper. It is defined so that if energy E is deposited as a delta function in time at $t = 0$, then the signal at A is $E\,h(t)$.

The signal at A will depend on the energy deposited in the tower on the previous beam crossings waited by the function $h(t)$. Let t_k ($k = 1, 2, \ldots$) be the times of the beam crossing. Let n^k_{ev} be the actual number of pp interactions in the k^{th} crossing. Let n^{jk} be the number of particles that deposit energy in the tower from the j^{th} event ($j \leq n^k_{\text{ev}}$) of the k^{th} beam crossing. Finally, let E_{ijk} be the energy deposited by the i^{th} particle ($i \leq n^{jk}$) of the j^{th} event of the k^{th} beam crossing. Then, the signal at A is

$$E_{\text{tower}} = \sum_k \sum_{j=0}^{n^k_{\text{ev}}} \sum_{i=0}^{n^{jk}} E_{ijk}\, h(t_k) \tag{1}$$

Now, we have to average over (1) particle energies, (2) number of particles per tower per event, and (3) number of events in each beam crossing. Denote these averages as $\langle\,\rangle_E$, $\langle\,\rangle_n$, and $\langle\,\rangle_{n_{\rm ev}}$, and do them in the following order:

$$\bar{E}_{\rm tower} = \left\langle\left\langle\left\langle \sum_k \sum_{j=0}^{n_{\rm ev}^k} \sum_{i=0}^{n^{jk}} E_{ijk}\, h(t_k) \right\rangle_E\right\rangle_n\right\rangle_{n_{\rm ev}}$$

$$= \sum_k \left\langle \sum_{j=0}^{n_{\rm ev}^k} \left\langle \sum_{i=0}^{n^{jk}} \langle E_{ijk}\rangle_E \right\rangle_n \right\rangle_{n_{\rm ev}} h(t_k).$$

All the E_{ijk}'s are statistically independent and are sampled from the same distribution, which is assumed to have mean $\bar{E} \Longrightarrow \langle E_{ijk}\rangle_E = \bar{E}$.

$$\Longrightarrow \bar{E}_{\rm tower} = \bar{E} \sum_k \left\langle \sum_{j=0}^{n_{\rm ev}^k} \left\langle \sum_{i=0}^{n^{jk}} 1 \right\rangle_n \right\rangle_{n_{\rm ev}} h(t_k)$$

$$= \bar{E} \sum_k \left\langle \sum_{j=0}^{n_{\rm ev}^k} \left\langle n^{jk} \right\rangle_n \right\rangle_{n_{\rm ev}} h(t_k).$$

All the n^{jk}'s are statistically independent and Poisson-distributed with mean $\bar{n} = \frac{\bar{n}_p}{2\pi}\Delta n \Delta\phi$, where $\Delta n \Delta\phi$ is the size of the tower.

$$\Longrightarrow \left\langle n^{jk} \right\rangle_n = \bar{n} = \frac{\bar{n}_p \Delta n \Delta\phi}{2\pi}$$

$$\Longrightarrow \bar{E}_{\rm tower} = \bar{E}\, \frac{\bar{n}_p \Delta n \Delta\phi}{2\pi} \sum_k \left\langle \sum_{j=0}^{n_{\rm ev}^k} 1 \right\rangle_{n_{\rm ev}} h(t_k)$$

$$= \bar{E}\, \frac{\bar{n}_p \Delta n \Delta\phi}{2\pi} \sum_k \left\langle n_{\rm ev}^k \right\rangle_{n_{\rm ev}} h(t_k)$$

Applying the same arguments, $\langle n_{\rm ev}^k\rangle_{n_{\rm ev}} = \bar{n}_{\rm ev}$

$$\Longrightarrow \bar{E}_{\rm tower} = \frac{\bar{E}\, \bar{n}_p \Delta n \Delta\phi\, \bar{n}_{\rm ev}}{2\pi} \sum_k h(t_k).$$

Let Δt be the time between beam crossings.

$$\sum_k h(t_k) = \frac{1}{\Delta t} \sum_k h(t_k)\,\Delta t \simeq \frac{1}{\Delta t} \int_0^\infty h(t)dt.$$

[Note $h(t) = 0$ for $t < 0$ since it is not possible to build an anticipator box.]

Thus,

$$\bar{E}_{\text{tower}} = \frac{\bar{E}\,\bar{n}_p \Delta n \Delta\phi\,\bar{n}_{\text{ev}}}{2\pi\Delta t} \int_0^\infty h(t)dt.$$

This is quite sensible in that $\bar{E}_{\text{tower}}$ is linear in $\bar{E}$, $\bar{n}_p$, and $\bar{n}_{\text{ev}}$.

Now we need to estimate ΔE_{tower}. Since we have $\bar{E}_{\text{tower}}$, we need $\overline{E^2_{\text{tower}}}$. From Eq. (1),

$$E^2_{\text{tower}} = \left[\sum_k \sum_{j=0}^{n^k_{\text{ev}}} \sum_{i=0}^{n^{jk}} E_{ijk}\,h(t_k)\right]^2$$
$$= \sum_{k_1} \sum_{k_2} \sum_{j_1=0}^{n^{k_1}_{\text{ev}}} \sum_{j_2=0}^{n^{k_2}_{\text{ev}}} \sum_{i_1=0}^{n^{j_1 k_1}} \sum_{i_2=0}^{n^{j_2 k_2}} E_{i_1 j_1 k_1}\,E_{i_2 j_2 k_2}\,h(t_{k_1})h(t_{k_2})$$

$$\overline{E^2_{\text{tower}}} = \left\langle\left\langle\left\langle \sum_{k_1} \sum_{k_2} \sum_{j_1=0}^{n^{k_1}_{\text{ev}}} \sum_{j_2=0}^{n^{k_2}_{\text{ev}}} \sum_{i_1=0}^{n^{j_1 k_1}} \sum_{i_2=0}^{n^{j_2 k_2}} E_{i_1 j_1 k_1}\,E_{i_2 j_2 k_2} \otimes h(t_{k_1})\,h(t_{k_2}) \right\rangle_E\right\rangle_n\right\rangle_{n_{\text{ev}}}$$
$$= \sum_{k_1} \sum_{k_2} \left\langle \sum_{j_1=0}^{n^{k_1}_{\text{ev}}} \sum_{j_2=0}^{n^{k_2}_{\text{ev}}} \left\langle \sum_{i_1=0}^{n^{j_1 k_1}} \sum_{i_2=0}^{n^{j_2 k_2}} \left\langle E_{i_1 j_1 k_1}\,E_{i_2 j_2 k_2}\right\rangle_E \right\rangle_n \right\rangle_{n_{\text{ev}}} \otimes h(t_{k_1})\,h(t_{k_2})$$

If $i_1 \neq i_2$ *or* $j_1 \neq j_2$ *or* $k_1 \neq k_2$, then $E_{i_1 j_1 k_1}$ and $E_{i_2 j_2 k_2}$ are statistically independent variables $\Longrightarrow \left\langle E_{i_1 j_1 k_1}\,E_{i_2 j_2 k_2}\right\rangle_E = \left\langle E_{i_1 j_1 k_1}\right\rangle_E \left\langle E_{i_2 j_2 k_2}\right\rangle_E = \bar{E}^2$.

If $i_1 = i_2$ *and* $j_1 = j_2$ *and* $k_1 = k_2$ then

$$\left\langle E_{i_1 j_1 k_1}\,E_{i_2 j_2 k_2}\right\rangle_E = \left\langle E^2_{i_1 j_1 k_1}\right\rangle = \overline{E^2}.$$

Thus, $\langle E_{i_1 j_1 k_1}\, E_{i_2 j_2 k_2}\rangle = \bar{E}^2 + \delta_{i_1 i_2}\delta_{j_1 j_2}\delta_{k_1 k_2}(\overline{E^2} - \bar{E}^2) = \bar{E}^2 + \delta_{i_1 i_2}\delta_{j_1 j_2}\delta_{k_1 k_2}\,\Delta E^2$

$$\overline{E^2_{\text{tower}}} = \sum_{k_1}\sum_{k_2}\left\langle \sum_{j_1=0}^{n_{\text{ev}}^{k_1}}\sum_{j_2=0}^{n_{\text{ev}}^{k_2}}\left\langle \sum_{i_1=0}^{n^{j_1 k_1}}\sum_{i_2=0}^{n^{j_2 k_2}}\left[\bar{E}^2 + \delta_{i_1 i_2}\delta_{j_1 j_2}\delta_{k_1 k_2}\,\Delta E^2\right]\right\rangle_n\right\rangle_{n_{\text{ev}}} \otimes h(t_{k_1})h(t_{k_2})$$

$$= \bar{E}^2 \sum_{k_1}\sum_{k_2}\left\langle \sum_{j_1=0}^{n_{\text{ev}}^{k_1}}\sum_{j_2=0}^{n_{\text{ev}}^{k_2}}\left\langle n^{j_1 k_1} n^{j_2 k_2}\right\rangle_n\right\rangle_{n_{\text{ev}}} h(t_{k_1})h(t_{k_2})$$

$$+ \Delta E^2 \sum_k \left\langle \sum_{j=0}^{n_{\text{ev}}^{k}} \left\langle n^{jk}\right\rangle_n\right\rangle_{n_{\text{ev}}} [h(t_k)]^2.$$

Using the same argument, $\langle n^{j_1 k_1}\, n^{j_2 k_2}\rangle_n = \bar{n}^2 + \delta_{j_1 j_2}\delta_{k_1 k_2}(\overline{n^2} - \bar{n}^2) = \bar{n}^2 + \delta_{j_1 j_2}\delta_{k_1 k_2}\Delta n^2$

$$\Longrightarrow \overline{E^2_{\text{tower}}} = \bar{E}^2 \sum_{k_1}\sum_{k_2}\left\langle \sum_{j_1=0}^{n_{\text{ev}}^{k_1}}\sum_{j_2=0}^{n_{\text{ev}}^{k_2}}\left[\bar{n}^2 + \delta_{j_1 j_2}\delta_{k_1 k_2}\Delta n^2\right]\right\rangle_{n_{\text{ev}}} h(t_{k_1})h(t_{k_2})$$

$$+ \Delta E^2 \bar{n}\,\bar{n}_{\text{ev}} \sum_k [h(t_k)]^2$$

$$= \bar{E}^2\,\bar{n}^2 \sum_{k_1}\sum_{k_2}\left\langle n_{\text{ev}}^{k_1}\, n_{\text{ev}}^{k_2}\right\rangle_{n_{\text{ev}}} h(t_{k_1})h(t_{k_2}) + \bar{E}^2 \Delta n^2 \sum_k \left\langle n_{\text{ev}}^{k}\right\rangle_{n_{\text{ev}}} [h(t_k)]^2$$

$$+ \Delta E^2 \bar{n}\,\bar{n}_{\text{ev}} \sum_k [h(t_k)]^2$$

$$= \bar{E}^2\,\bar{n}^2 \sum_{k_1}\sum_{k_2}\left[\bar{n}_{\text{ev}}^2 + \delta_{k_1 k_2}\Delta n_{\text{ev}}^2\right] h(t_{k_1})h(t_{k_2})$$

$$+ \left(\bar{E}^2 \Delta n^2\,\bar{n}_{\text{ev}} + \Delta E^2 \bar{n}\,\bar{n}_{\text{ev}}\right) \sum_k [h(t_k)]^2$$

$$= \bar{E}^2\,\bar{n}^2\,\bar{n}_{\text{ev}}^2 \left[\sum_k h(t_k)\right]^2 + \bar{E}^2\,\bar{n}^2 \Delta n_{\text{ev}}^2 \sum_k [h(t_k)]^2$$

$$+ \left(\bar{E}^2 \Delta n^2\,\bar{n}_{\text{ev}} + \Delta E^2 \bar{n}\,\bar{n}_{\text{ev}}\right) \sum_k [h(t_k)]^2$$

$$\sum_k h(t_k) \simeq \frac{1}{\Delta t} \int_0^\infty h(t) dt$$

$$\sum_k [h(t_k)]^2 = \frac{1}{\Delta t} \sum_k [h(t_k)]^2_{\Delta t} \simeq \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt$$

$$\Longrightarrow \overline{E^2_{\rm tower}} = \frac{\bar{E}^2 \, \bar{n}^2 \, \bar{n}^2_{\rm ev}}{(\Delta t)^2} \left[\int_0^\infty h(t) dt \right]^2$$

$$+ \left[\bar{E}^2 \left(\bar{n}^2 \, \Delta n^2_{\rm ev} + \Delta n^2 \, \bar{n}_{\rm ev} \right) + \Delta E^2 \, \bar{n} \, \bar{n}_{\rm ev} \right] \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt$$

$$\Delta E^2_{\rm tower} = \overline{E^2_{\rm tower}} - \overline{E_{\rm tower}}^2$$

$$= \left[\bar{E}^2 \left(\bar{n}^2 \, \Delta n^2_{\rm ev} + \Delta n^2 \, \bar{n}_{\rm ev} \right) + \Delta E^2 \, \bar{n} \, \bar{n}_{\rm ev} \right] \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt.$$

Since n and $n_{\rm ev}$ are Poisson-distributed,

$$\Delta n^2 = \bar{n} \quad \text{and} \quad \Delta n^2_{\rm ev} = \bar{n}_{\rm ev}$$

$$\Longrightarrow \Delta E^2_{\rm tower} = \bar{n}_{\rm ev} \, \bar{n} \left[\bar{E}^2 (1 + \bar{n}) + \Delta E^2 \right] \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt.$$

$$\Delta E^2_{\rm tower} = \bar{n}_{\rm ev} \, \frac{\bar{n}_p \Delta n \Delta \phi}{2\pi} \left[\bar{E}^2 \left(1 + \frac{\bar{n}_p \Delta \phi \Delta n}{2\pi} \right) + \Delta E^2 \right] \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt.$$

Thus, there are basically three contributions to $\Delta E_{\rm tower}$:

1. The $\bar{n}_{\rm ev} \, \bar{n} \, \Delta E^2$ term from fluctuations in the energy deposited per particle.
2. The $\bar{n}_{\rm ev} \, \bar{n} \, \bar{E}^2$ term from fluctuations in the number of particles per event.
3. The $\bar{n}_{\rm ev} \, \bar{n}^2 \, \bar{E}^2$ term from fluctuations in the number of events per crossing.

Note 1: for the size of towers in SDC ($\Delta \phi \Delta n = (.05)^2$), $\bar{n} = \dfrac{\bar{n}_p \Delta n \Delta \phi}{2\pi} \ll 1$

$$\Longrightarrow \Delta E^2_{\rm tower} \simeq \frac{\bar{n}_{\rm ev} \bar{n}_p \Delta n \Delta \phi}{2\pi} \left[\bar{E}^2 + \Delta E^2 \right] \frac{1}{\Delta t} \int_0^\infty [h(t)]^2 dt$$

Note 2: The shaping enters through the integrals $\int_0^\infty h(t)dt$ for $\bar{E}_{\text{tower}}$ and $\sqrt{\int_0^\infty [h(t)]^2 dt}$ for ΔE_{tower}. It is possible to make the first integral zero, but it is not possible to make the second one zero. It may be preferable to minimize $\int_0^\infty [h(t)]^2 dt$ [and hence ΔE_{tower}] at the expense of a non-zero $\bar{E}_{\text{tower}}$ and then subject off $\bar{E}_{\text{tower}}$ offline.