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Semi-Parameterization of Dragt-Finn Factorization Map and its Application to Irwin Kick Factorization

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Abstract

This note is based on the work we did in August, 1989 for parameterizing Irwin factorization kick map. A semi-parameterization method was employed in which the linear part of the map is three-dimensional (six-dimensional in phase space) while the nonlinear part of the map is two-and-one-half dimensional. The one-half dimension can be eliminated once the parameter (off-momentum) is updated at the beginning of each turn of the kick map.

1.0 INTRODUCTION

In 1989, Irwin proposed a method of converting a Dragt-Finn factorization[1] of a Taylor map into kicks.[2] The minimized highest-order rotation bases are used as the bases for all orders, while Lagrange multipliers are employed to (1) eliminate the extra independent bases in order to obtain a unique solution, and (2) suppress the higher-order (order higher than the order of the original Taylor map) artificial terms. If the artificial terms can be suppressed enough and be negligible, then this method may be promising for fast long-term tracking. Therefore, an investigation was started right after Irwin's proposal to verify its applicability for the Superconducting Super Collider (SSC) long-term tracking. In order to enhance tracking speed, a semi-parameterization method was employed to parameterize the nonlinear part of the Dragt-Finn factorization map and, therefore, reduce from three dimensions to two dimensions for the kicks proposed by Irwin. To accomplish this, the one-turn Taylor map should not involve any RF cavity so that the off-momentum (represented by the energy deviation $\delta = \Delta E/E$) is kept invariant and is treated as a parameter. To update the off-momentum, simply track the particles over the RF cavities separately.

Let a closed-orbit Taylor map (which can be extracted using Zmap[3] for the SSC) be extracted up to the Ω order for a beam line from after an RF cavity to before the next RF cavity. The map can be expressed as follows (we adapt the same notational convention as in ZLIB[4] manual):

$$m : \vec{\mathbf{z}} = \vec{\mathbf{U}}(\vec{\mathbf{z}}) = M\vec{\mathbf{z}} + \sum_{k=2}^{\Omega} \vec{\mathbf{U}}^{k}(\vec{\mathbf{z}}-),$$

where the transpose of the coordinates are

$$\begin{aligned} \vec{\mathbf{z}}^{T} &= [z_1, z_2, \ldots, z_n] = [\vec{\mathbf{x}}^{T}, \delta, P_\delta], \\ \vec{\mathbf{x}}^{T} &= [x, P_x, y, P_y], \end{aligned}$$

and the transpose of the VTPS (vector truncated power series) is

$$\vec{\mathbf{U}}^{T}(\vec{\mathbf{z}}) = [U_1(\vec{\mathbf{z}}),\ U_2(\vec{\mathbf{z}}),\ U_3(\vec{\mathbf{z}}),\ U_4(\vec{\mathbf{z}}),\ U_5(\vec{\mathbf{z}}),\ U_6(\vec{\mathbf{z}})].$$

Note that P_x, P_y, and P_δ are the conjugate momenta of x, y, and δ, respectively. $U_5(\vec{\mathbf{z}}) = \delta$, that is, the off-momentum, δ, is an invariant since no RF cavity is involved for acceleration. The nonlinear part of the map is dependent only on $\vec{\mathbf{x}}$ ($\vec{\mathbf{x}} \equiv [x, P_x, y, P_y]$) and δ, not on P_δ, as we have explicitly expressed $\vec{\mathbf{U}}^{k}(\vec{\mathbf{z}}-)$ for $k > 1$ as nonlinear VTPS of $\vec{\mathbf{z}}- = [x, P_x, y, P_y, \delta]$. Furthermore, each of the constant terms in U_i ($i = 1, 2, \ldots, 6$) is 0.

2.0 BLOCK DIAGONALIZATION OF M

Before the nonlinear Dragt-Finn factorization can be performed, block diagonalization and similarity transformation are necessary such that the transformed map is semi-normalized in the linear part. In order to parameterize the nonlinear Dragt-Finn generators, the transformed nonlinear part of the map should be independent of P_δ, and the transformed $U_5^k(\vec{\mathbf{z}})$ should be 0 for $k = 2, \Omega$. In this section, we describe the method we used to accomplish this.

The linear part of the Taylor map can be generally expressed as a symplectic matrix given by

$$M = \begin{pmatrix} N_{4\times 4} & \vec{\mathbf{m}} & \vec{\mathbf{0}} \\ \vec{\mathbf{0}}^T & 1 & 0 \\ \vec{\mathbf{n}}^T & \alpha & 1 \end{pmatrix},$$

with the constraint that M and N (N is a 4×4 matrix) are both symplectic, that is, $M^T SM = S = S_{6\times 6}$ and $N^T SN = S = S_{4\times 4}$, where $\vec{\mathbf{m}}$ and $\vec{\mathbf{n}}$ are vectors with 4 elements and

$$S_{6\times 6} = \begin{pmatrix} S_{2\times 2} & O_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & S_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} & S_{2\times 2} \end{pmatrix}, \qquad S_{4\times 4} = \begin{pmatrix} S_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & S_{2\times 2} \end{pmatrix}$$

with

$$S_{2\times 2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

First, we found the first-order parameter- (δ-) dependent closed orbit $\vec{\mathbf{x}}_c$ by imposing that

$$\vec{\mathbf{x}}_c = N\vec{\mathbf{x}}_c + \delta\vec{\mathbf{m}};$$

thus,

$$\vec{\mathbf{x}}_c = (I - N)^{-1}\delta\vec{\mathbf{m}} = \vec{\xi}\delta,$$

where

$$\vec{\xi} = (I - N)^{-1}\vec{\mathbf{m}}.$$

Then, we found a symplectic generation matrix M_1 to transform the coordinates $\vec{\mathbf{x}}$ into the linear-order-δ-dependent-closed-orbit coordinates, while keeping the parameter, δ, invariant. P_δ is inevitably transformed to have M_1 symplectic, that is, to have $M_1^T SM_1 = S$.

The generator M_1 is given by

$$M_1 = \begin{pmatrix} I_{4\times4} & \vec{\xi} & \vec{0} \\ \vec{0} & 1 & 0 \\ \vec{\xi}^T S & 0 & 1 \end{pmatrix},$$

and its inverse is given by

$$M_1^{-1} = \begin{pmatrix} I_{4\times4} & -\vec{\xi} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ -\vec{\xi}^T S & 0 & 1 \end{pmatrix},$$

where $S = S_{4\times4}$. The forms of the generator M_1 and its inverse M_1^{-1} show that through the similarity transformation, we would still have the two required properties (δ is an invariant and $\vec{\mathbf{x}}$ is independent of P_δ) for parameterizing the nonlinear Dragt-Finn generators. Performing the similarity transformation we have

$$m' : \vec{\mathbf{z}} = (\mathcal{M}_1\, m\, \mathcal{M}_1^{-1}) : \vec{\mathbf{z}} = (M_1^{-1} M M_1)\vec{\mathbf{z}} + \sum_{k=2}^{\Omega} M_1^{-1} \vec{\mathbf{U}}^k (M_1 \vec{\mathbf{z}}),$$

or simply,

$$m' : \vec{\mathbf{z}} = M' \vec{\mathbf{z}} + \sum_{k=2}^{\Omega} \vec{\mathbf{U}}'^k (\vec{\mathbf{z}}),$$

where

$$M' = M_1^{-1} M M_1 = \begin{pmatrix} N_{4\times4} & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & \alpha_c & 1 \end{pmatrix},$$

and $\alpha_c = \alpha + \vec{\mathbf{n}}^T S \vec{\xi}$ is the momentum compaction. Note that $\mathcal{M}_1$ and $\mathcal{M}_1^{-1}$ are the symplectic transformation generator representing the matrices M_1 and M_1^{-1}, respectively. However, for convenience we have followed the convention that $\mathcal{M}_1$ and $\mathcal{M}_1^{-1}$ operate on the global variables, while their matrix representations, M_1 and M_1^{-1}, operate on local variables. Please also note that $U_5^k(\vec{\mathbf{z}}) = 0$ and $\vec{\mathbf{U}}'^k(\vec{\mathbf{z}})$ are independent of P_δ for $k \geq 2$.

Now that the transverse linear map N is decoupled from the longitudinal space, the linear part of the map, M', can be semi-normalized by another similarity transformation as follows:

$$\begin{pmatrix} R_{4\times 4} & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & \alpha_c & 1 \end{pmatrix} = M_R = M_1'^{-1} M' M_1'$$

$$= \begin{pmatrix} N_1^{-1} & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} N_{4\times 4} & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & \alpha_c & 1 \end{pmatrix} \begin{pmatrix} N_1 & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} N_1^{-1} N_{4\times 4} N_1 & \vec{0} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{0}^T & \alpha_c & 1 \end{pmatrix},$$

where

$$R_{4\times 4} = N_1^{-1} N_{4\times 4} N_1$$

is the usual normalization for a four-dimensional symplectic matrix. The map is thus transformed as follows:

$${}_1m : \vec{\mathbf{z}} = (\mathcal{M}_1'\, m'\, \mathcal{M}_1'^{-1}) : \vec{\mathbf{z}} = (M_1'^{-1} M' M_1')\vec{\mathbf{z}} + \sum_{k=2}^{\Omega} M_1'^{-1} \vec{\mathbf{U}}^k (M_1' \vec{\mathbf{z}}),$$

or simply,

$${}_1m : \vec{\mathbf{z}} = M_R \vec{\mathbf{z}} + \sum_{k=2}^{\Omega} {}_1\vec{\mathbf{U}}^k (\vec{\mathbf{z}}-).$$

The form of the matrix M_R, which has already been decoupled between the transverse coordinates and the longitudinal coordinates, guarantees that an additional transformation by operating M_R^{-1} on the map would not destroy the two requirements (δ is an invariant and non-linear part of the map and is independent of P_δ) for parameterizing the nonlinear Lie generators. So we made another transformation by operating $\mathcal{M}_R^{-1}$ on the map and obtained

$${}_2m : \vec{\mathbf{z}} = (\mathcal{M}_R^{-1}\, {}_1m) : \vec{\mathbf{z}} = \vec{\mathbf{z}} + \sum_{k=2}^{\Omega} {}_1\vec{\mathbf{U}}^k (M_R^{-1} \vec{\mathbf{z}}),$$

or simply,

$$ {}_2m : \vec{\mathbf{z}} = \vec{\mathbf{z}} + \sum_{k=2}^{\Omega} {}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-). \tag{1} $$

Note that $\mathcal{M}_R^{-1}$ operates on the global variables, and its matrix representation M_R^{-1} operates on the local variables. ${}_2U_5^k(\vec{\mathbf{z}}) = 0$ and ${}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-)$ is independent of P_δ for $k \geq 1$.

3.0 DRAGT-FINN FACTORIZATION

We reformed Equation (1) as follows:

$$ {}_2m : \vec{\mathbf{z}} = \vec{\mathbf{z}} + {}_2\vec{\mathbf{U}}^2(\vec{\mathbf{z}}-) + \sum_{k=3}^{\Omega} {}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-), \tag{2} $$

where $\vec{\mathbf{z}}- = [\vec{\mathbf{x}}, \delta]$ indicates that ${}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-)$ is independent of P_δ.

To obtain the second-order Lie generator, since ${}_2U_5^2(\vec{\mathbf{z}}-) = 0$, we let

$$ [f_2(\vec{\mathbf{z}}-), \vec{\mathbf{z}}] = {}_2\vec{\mathbf{U}}^2(\vec{\mathbf{z}}-). $$

Although δ and P_δ are treated as a pair of canonical coordinate and momentum, δ is actually a parameter. To obtain $f_2(\vec{\mathbf{z}}-)$, an integration path was chosen from $\vec{0}_6$ to $(\vec{0}_4, \delta, 0)$ along the δ coordinate and then to $\vec{z}$ with particular care. Note that the second-order Lie generator, $f_2(\vec{\mathbf{z}}-)$, is a third-order polynomial of $\vec{\mathbf{z}}-$. Since ${}_2\vec{\mathbf{U}}^2(\vec{\mathbf{z}}-)$ is independent of P_δ, so is $f_2(\vec{\mathbf{z}}-)$. Since

$$ \exp(: f_2(\vec{\mathbf{z}}-) :)\vec{\mathbf{z}} = \vec{\mathbf{z}} + [f_2(\vec{\mathbf{z}}-), \vec{\mathbf{z}}] + [f_2(\vec{\mathbf{z}}-), [f_2(\vec{\mathbf{z}}), \vec{\mathbf{z}}]]/2! + \dots, $$

we let

$$ {}_2m : \vec{\mathbf{z}} = \exp(: f_2(\vec{\mathbf{z}}-) :)\vec{\mathbf{z}} + \sum_{k=3}^{\Omega} \Delta\, {}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-), \tag{3} $$

where

$$ \sum_{k=3}^{\Omega} \Delta\, {}_2\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-) = {}_2m : \vec{\mathbf{z}} - \exp(: f_2(\vec{\mathbf{z}}-) :)\vec{\mathbf{z}}. $$

Then we obtained

$$ {}_3m : \vec{\mathbf{z}} = \{\exp(- : f_2(\vec{\mathbf{z}}-) :)\, {}_2m\} : \vec{\mathbf{z}} = \vec{\mathbf{z}} + {}_3\vec{\mathbf{U}}^3(\vec{\mathbf{z}}-) + \sum_{k=4}^{\Omega} {}_3\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-). \tag{4} $$

Equation (4) is similar to Equation (2). So we let $[f_3(\vec{\mathbf{z}}-), \vec{\mathbf{z}}] = {}_3\vec{\mathbf{U}}^3(\vec{\mathbf{z}}-)$ to obtain the third-order Lie generator $f_3(\vec{\mathbf{z}}-)$, and follow a process similar to the one we used to obtain Equation (4). Now we obtain

$$ {}_4m : \vec{\mathbf{z}} = \{\exp(- : f_3(\vec{\mathbf{z}}-) :)\, {}_3m\} : \vec{\mathbf{z}} = \vec{\mathbf{z}} + {}_4\vec{\mathbf{U}}^4(\vec{\mathbf{z}}-) + \sum_{k=5}^{\Omega} {}_4\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-). \tag{5} $$

Through iteration, we finally obtain all the Lie generator $f_i(\vec{\mathbf{z}}-)$ and

$$ {}_{i+1}m : \vec{\mathbf{z}} = \{\exp(- : f_i(\vec{\mathbf{z}}-) :)\, {}_i m\} : \vec{\mathbf{z}} = \vec{\mathbf{z}} + {}_{i+1}\vec{\mathbf{U}}^{i+1}(\vec{\mathbf{z}}-) + \sum_{k=i+2}^{\Omega} {}_{i+1}\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-), \tag{6} $$

and so

$$ \begin{aligned} {}_{\Omega+1}m : \vec{\mathbf{z}} &= \{\exp(- : f_\Omega(\vec{\mathbf{z}}-) :)\, {}_\Omega m\} : \vec{\mathbf{z}} = \vec{\mathbf{z}} + {}_{\Omega+1}\vec{\mathbf{U}}^{\Omega+1}(\vec{\mathbf{z}}-) + \sum_{k=\Omega+2}^{\Omega} {}_\Omega\vec{\mathbf{U}}^k(\vec{\mathbf{z}}-) \\ &= \vec{\mathbf{z}} + \sigma(\Omega+1) \approx \vec{\mathbf{z}} = I\vec{\mathbf{z}}. \end{aligned} $$

Therefore, neglecting terms with order higher than Ω, we have (in terms of global variables)

$$ m = \mathcal{M}_1^{-1}\mathcal{M}_1'^{-1}\mathcal{M}_R\, {}_2m\, \mathcal{M}_1'\mathcal{M}_1 \approx \mathcal{M}_1^{-1}\mathcal{M}_1'^{-1}\mathcal{M}_R\, m_f\, \mathcal{M}_1'\mathcal{M}_1, \tag{7} $$

where

$$ m_f = \exp(: f_2(\vec{\mathbf{z}}-) :)\exp(: f_3(\vec{\mathbf{z}}-) :)\ \ldots\ \exp(: f_\Omega(\vec{\mathbf{z}}-) :). \tag{8} $$

4.0 KICK FACTORIZATION BASES

Followed Irwin's Work,[2] we first decided the total number of kicks necessary. Recall that each of the Dragt-Finn factors, $f_i(\vec{\mathbf{z}}-)$, is independent of P_δ, that is, its longitudinal dimension (δ, P_δ) has already been in a kick form. Thus, we need only rotation bases in the transverse dimension. Since the same rotation bases are used for each of the Dragt-Finn factors, $f_i(\vec{\mathbf{z}}-)$, the minimum required number of kicks is determined by the highest-order of Dragt-Finn factor, $f_\Omega(\vec{\mathbf{z}}-)$, which is given by:[2]

$$ K \geq \begin{cases} (\Omega+3)^2/4 & \text{if } \Omega \text{ is odd} \\ (\Omega+2)(\Omega+4)/4 & \text{if } \Omega \text{ is even.} \end{cases} $$

Once the total number, K, of rotation bases is chosen, the next step is to choose the rotation bases. We used random number generator to generate the two sets of rotation angles, θ_{xk} and θ_{yk}, $k = 1, 2, \ldots, K$. Note that $0 \leq \theta_{xk} < \pi$. If two of the angles are too close in a set, that set of angles was regenerated until no two angles are too closely based on a pre-determined criterion. We then calculated the rotation bases, which were given by

$$\begin{aligned} C_{xk} &= \cos\theta_{xk}, \\ S_{xk} &= \sin\theta_{xk}, \\ C_{yk} &= \cos\theta_{yk}, \\ S_{yk} &= \sin\theta_{yk}. \end{aligned}$$

New local base coordinates were then formed by letting

$$\begin{aligned} x_k &= C_{xk}\, x + S_{xk}\, P_x \\ y_k &= C_{yk}\, y + S_{yk}\, P_y. \end{aligned}$$

The task of this section is to construct from the Dragt-Finn factorization given in Equation (8) a kick factorization of the form

$$m_g = \exp(:g_1(\vec{\mathbf{x}}_1+):)\exp(:g_2(\vec{\mathbf{x}}_2+):)\ldots\exp(:g_k(\vec{\mathbf{x}}_k+):)\ldots\exp(:g_K(\vec{\mathbf{x}}_K+):) \qquad (9)$$

such that

$$m_g = m_f + \sigma(\Omega + 1) \approx m_f$$

where m_f is given in Equation (8) and $\vec{\mathbf{x}}_k+$ is defined as $\vec{\mathbf{x}}_k+ = (x_k, y_k, \delta)$. The "+" emphasizes that the third dimension, δ, although it appears, has no effect in determining the rotation bases.

5.0 KICK FACTORIZATION OF THE SECOND-ORDER DRAGT-FINN FACTOR

The task is to find ${}_2g_k(\vec{\mathbf{x}}_k+)$, for $k = 1, 2, \ldots, K$ such that

$$\sum_{k=1}^{K} {}_2g_k(\vec{\mathbf{x}}_k+) = f_2(\vec{\mathbf{z}}-),$$

that is,

$$\sum_{k=1}^{K} \sum_{n+m+o=3} \beta_k^{nmo}\, x_k^n\, y_k^m\, \delta^o = \sum_{n+m+o=3} \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{rs}^{nmo}\, x^r\, P_x^{n-r}\, y^s\, P_y^{m-s}\, \delta^o$$

if we take the explicit power series for ${}_2g_k(\vec{\mathbf{x}}_k+)$ and $f_2(\vec{\mathbf{z}}-)$. Therefore, the purpose is to find all the β_k^{nmo} for $k = 1, 2, \ldots, K$, and all possible combinations of n, m, o such that

$n + m + o = 3$ (note that both ${}_2g_k(\vec{\mathbf{x}}_k+)$ and $f_2(\vec{\mathbf{z}}-)$ are 3rd order polynomials). Simply speaking, for each possible combination of n, m, o, the task is to find all the β_k^{nmo} for $k = 1, 2, \ldots, K$ such that

$$\sum_{k=1}^{K} \beta_k^{nmo}\, x_k^n\, y_k^m = \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{rs}^{nmo}\, x^r\, P_x^{n-r}\, y^s\, P_y^{m-s}. \tag{10}$$

Note that the third dimension, δ, is a common factor on both sides of the "=" sign, so it has been taken out of the equation. Therefore we have exactly the same equation as Irwin's except an extra superscript, "o", appears for β_k^{nmo} and α_{rs}^{nmo} to indicate that there are more coefficients than with a non-parameterized case.

To solve Equation (10) for β_k^{nmo}, $k = 1, 2, \ldots, K$ (n, m, o fixed), the key is to take $x_k^n y_k^m$, $k = 1, 2, \ldots (n+1)(m+1)$ as complete bases (this can always be proved afterwards). Note that K was chosen such that $(n+1)(m+1) \leq K$ for all possible combinations of n, m, o, such that $n + m + o \leq \Omega + 1$. With the new complete bases, we can obtain

$$x^r\, P_x^{n-r}\, y^s\, P_y^{m-s} = \sum_{k=1}^{(n+1)(m+1)} C_{rsk}^{nmo}\, x_k^n\, y_k^m \tag{11}$$

for each combination of r, s, where $r = 0, 1, \ldots n$, and $s = 0, 1, \ldots m$, and

$$x_j^n y_j^m = \sum_{k=1}^{(n+1)(m+1)} q_{jk}^{nmo}\, x_k^n\, y_k^m \tag{12}$$

for each of the extra rotation bases $x_j^n y_j^m$, $j = (n+1)(m+1) + 1, \ldots K$. Substituting Equations (11) and (12) into Equation (10), we have

$$\sum_{k=1}^{(n+1)(m+1)} \beta_k^{nmo}\, x_k^n\, y_k^m + \sum_{j=(n+1)(m+1)+1}^{K} \beta_j^{nmo} \sum_{k=1}^{(n+1)(m+1)} q_{jk}^{nmo}\, x_k^n\, y_k^m$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{rs}^{nmo} \sum_{k=1}^{(n+1)(m+1)} C_{rsk}^{nmo}\, x_k^n\, y_k^m.$$

or after rearranging

$$\sum_{k=1}^{(n+1)(m+1)} \left(\beta_k^{nmo} \, x_k^n \, y_k^m + \sum_{j=(n+1)(m+1)+1}^{k} \beta_j^{nmo} \, q_{jk}^{nmo} \, x_k^n \, y_k^m \right)$$

$$= \sum_{k=1}^{(n+1)(m+1)} \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{rs}^{nmo} \, C_{rsk}^{nmo} \, x_k^n \, y_k^m . \tag{13}$$

An easy solution would be

$$\beta_k^{nmo} = \widehat{\beta}_k^{nmo} = \sum_{r=0}^{n} \sum_{s=0}^{m} \alpha_{rs}^{nmo} \, C_{rsk}^{nmo}, \qquad k = 1, 2, \ldots (n+1)(m+1)$$

and

$$\beta_k^{nmo} = 0, \qquad k = (n+1)(m+1) + 1, \ldots K. \tag{14}$$

However, such a solution would sometimes result in large artificial higher-orders in the final kick map. So we follow Irwin's minimization method to introduce a set of Lagrange multipliers λ_j, $j = (n+1)(m+1)+1, \ldots k$ in order to introduce $(k - (n+1)(m+1))$ constraints to eliminate the extra independent bases $x_j^n y_j^m$, $j = (n+1)(m+1)+1, \ldots k$ for a unique solution of β_k^{nmo}, $k = 1, 2, \ldots K$. The Lagrange multipliers are introduced to minimize the sum of the squares of β_k^{nmo}, $k = 1, 2, \ldots K$.

With the introduction of the Lagrange multiplier, an equally valid solution as opposed to Equation (14) would be

$$\beta_k^{nmo} = \widehat{\beta}_k^{nmo} - \sum_{j=(n+1)(m+1)+1}^{K} \lambda_j \, q_{jk}^{nmo} \qquad k = 1, 2, \ldots (n+1)(m+1)$$

$$\beta_k^{nmo} = \lambda_k, \tag{15}$$

that is, let the coefficients β_k^{nmo}, $k = (n+1)(m+1)+1, \ldots K$, of the extra independent rotation bases be Lagrange multipliers instead of 0 given in Equation (14). Because these Lagrange multipliers are introduced to minimize the sum of the squares of β_k^{nmo},

$k = 1, 2, \ldots K$. We therefore obtain $k - (n+1)(m+1)$ constraints by setting the derivative with respect to each of the λ_k, $k = (n+1)(m+1)+1, \ldots K$, as follows:

$$\sum_{k=1}^{(n+1)(m+1)} \left(\widehat{\beta}_k^{nmo} - \sum_{l=(n+1)(m+1)+1}^{k} \lambda_l \, q_{lk}^{nmo} \right) q_{jk}^{nmo} + \lambda_j = 0, \tag{16}$$

$$\text{for } j = (n+1)(m+1)+1, \ (n+1)(m+1)+2, \ldots K.$$

Equation (16) has a total of $k - (n+1)(m+1)$ linear equations of $k - (n+1)(m+1)$ variables λ_j's. The linear system can be easily solved using standard numerical libraries. We therefore get a unique solution for β_k^{nmo} given by Equation (15). Note that other combinations of n, m, o such that $n + m + o = 3$ can be obtained by following the above process. Once we have gotten all the β_k^{nmo}'s for $k = 1, 2, \ldots K$, and all possible n, m, o, such that $n + m + o = 3$, we have gotten ${}_2g_k(\vec{\mathbf{x}}_k+)$ for $k = 1, 2, \ldots K$ such that

$$\sum_{k=1}^{K} {}_2g_k(\vec{\mathbf{x}}_k+) = f_2(\vec{\mathbf{z}}-),$$

that is, the kick factorization for the 2nd order Lie factor is finished.

6.0 KICK FACTORIZATION OF 3RD-ORDER DRAGT-FINN FACTOR

Now that we have obtained

$$\begin{aligned} {}_2m_g = &\exp(: {}_2g_1(\vec{\mathbf{x}}_1+) :) \exp(: {}_2g_2(\vec{\mathbf{x}}_2+) :) \\ &\ldots \exp(: {}_2g_k(\vec{\mathbf{x}}_k+) :) \ldots \exp(: {}_2g_K(\vec{\mathbf{x}}_K+) :), \end{aligned}$$

our next step is to expand ${}_2m_g$ into a Taylor map and then convert the Taylor map into a Dragt-Finn Factorization map up to the 3rd order. Let it be

$${}_2m_g = e^{:{}_2f_2^g(\vec{\mathbf{z}}-):} \, e^{:{}_2f_3^g(\vec{\mathbf{z}}-):} + \sigma(4).$$

Then we have

$${}_2f_2^g(\vec{\mathbf{z}}-) = f_2(\vec{\mathbf{z}}-)$$

and

$${}_3f_3(\vec{\mathbf{z}}-) = f_3(\vec{\mathbf{z}}-) - {}_2f_3^g(\vec{\mathbf{z}}-) \neq 0 \ \text{ in general.}$$

The task in this section is to find ${}^3g_k(\vec{\mathbf{x}}_k+),\ k=1,2,\ldots K$ such that

$$\sum_{k=1}^{K} {}^3g_k(\vec{\mathbf{x}}_k+) = {}_3f_3(\vec{\mathbf{z}}-).$$

This can be done by following a process similar to that discussed in Section 5. Then add ${}^3g_k(\vec{\mathbf{x}}_k+)$ to ${}_2g_k(\vec{\mathbf{x}}_k+)$ for $k=1,2,\ldots K$ to get

$$\begin{aligned} {}_3m_g = \exp(:{}_3g_1(\vec{\mathbf{x}}_1+):)\ \exp(:{}_3g_2(\vec{\mathbf{x}}_2+):) \\ \ldots \exp(:{}_3g_k(\vec{\mathbf{x}}_k+):)\ \ldots \exp(:{}_3g_K(\vec{\mathbf{x}}_K+):) \end{aligned} \tag{17}$$

where

$${}_3g_k(\vec{\mathbf{x}}_k+) = {}_2g_k(\vec{\mathbf{x}}_k+) + {}^3g_k(\vec{\mathbf{x}}_k+), \qquad k=1,2,3,\ldots K.$$

7.0 KICK FACTORIZATION UP TO Ω ORDER

Assume we have obtained

$$\begin{aligned} {}_im_g = \exp(:{}_ig_1(\vec{\mathbf{x}}_1+):)\ \exp(:{}_ig_2(\vec{\mathbf{x}}\ F_2+):) \\ \ldots \exp(:{}_ig_k(\vec{\mathbf{x}}_k+):)\ \ldots \exp(:{}_ig_K(\vec{\mathbf{x}}_K+):). \end{aligned} \tag{18}$$

Following a process similar to that in Section 6, we would first expand ${}_im_g$ into a Taylor map, then extract from the Taylor map a Dragt-Finn Factorization map up to $i+1$ order. Thus we would have

$$\begin{aligned} {}_im_g = \exp(:{}_if_2^g(\vec{\mathbf{z}}-):)\ \exp(:{}_if_3^g(\vec{\mathbf{z}}-):)\ \ldots \\ + \exp(:{}_if_i^g(\vec{\mathbf{z}}-):)\ \exp(:{}_if_{i+1}^g(\vec{\mathbf{z}}-):) + \sigma(i+2), \end{aligned}$$

where

$${}_if_j^g(\vec{\mathbf{z}}-) = f_j(\vec{\mathbf{z}}-) \qquad \text{for } j=2,3,\ldots i,$$

and

$${}_{i+1}f_{i+1}(\vec{\mathbf{z}}-) = f_{i+1}(\vec{\mathbf{z}}-) - {}_if_{i+1}^g(\vec{\mathbf{z}}-) \neq 0 \ \text{ in general.}$$

The task is then to follow the process in Section 5 to find ${}^{i+1}g_k(\vec{\mathbf{x}}_k+)$, $k=1,2,\ldots K$, such that

$$\sum_{k=1}^{K} {}^{i+1}g_k(\vec{\mathbf{x}}_k+) = {}_{i+1}f_{i+1}(\vec{\mathbf{z}}-).$$

Then we get

$$ {}_{i+1}m_g = \exp(:{}_{i+1}g_1(\vec{\mathbf{x}}_1+):) \exp(:{}_{i+1}g_2(\vec{\mathbf{x}}_2+):) \\ \ldots \exp(:{}_{i+1}g_k(\vec{\mathbf{x}}_k+):) \ldots \exp(:{}_{i+1}g_K(\vec{\mathbf{x}}_K+):), \tag{19} $$

where

$$ {}_{i+1}g_k(\vec{\mathbf{x}}_k+) = {}_i g_k(\vec{\mathbf{x}}_k+) + {}^{i+1}g_k(\vec{\mathbf{x}}_k+), \qquad k = 1, 2, \ldots K. $$

Iterate the above process until $i + 1 = \Omega$.

We therefore have the original closed-orbit Taylor map represented by a kick factorization map as follows:

$$ m = \mathcal{M}_1^{-1} \mathcal{M}_1'^{-1} \mathcal{M}_R \, m_g \, \mathcal{M}_1' \, \mathcal{M}_1, \tag{20} $$

where

$$ m_g = {}_\Omega m_g = \exp(:{}_\Omega g_1(\vec{\mathbf{x}}_1+):) \exp(:{}_\Omega g_2(\vec{\mathbf{x}}_2+):) \ldots \exp(:{}_\Omega g_k(\vec{\mathbf{x}}_k+):), $$

or simply written as

$$ m_g = \exp(:g_1(\vec{\mathbf{x}}_1):) \exp(:g_2(\vec{\mathbf{x}}_2+):) \ldots \exp(:g_k(\vec{\mathbf{x}}_k+):). \tag{21} $$

8.0 KICK MAP TRACK

Equation (21) can be further transformed by taking the rotation out of the Lie operators to get a global variable form, that is,

$$ m_g = (\mathcal{R}_1 e^{:\hat{g}_1(x,y,\delta):} \mathcal{R}_1^{-1})(\mathcal{R}_2 e^{:\hat{g}_2(x,y,\delta):} \mathcal{R}_2^{-1}) \ldots (\mathcal{R}_k e^{:\hat{g}_k(x,y,\delta):} \mathcal{R}_k^{-1}), $$

where $\mathcal{R}_k$ and $\mathcal{R}_k^{-1}$, $k = 1, 2, \ldots K$, are 4-dimensional (phase space) rotations. We could combine adjacent rotations:

$$ m_g = \mathcal{R}_1 \, e^{:\hat{g}_1(x,y,\delta):} \, \mathcal{R}_{12} \, e^{:\hat{g}_2(x,y,\delta):} \, \mathcal{R}_{23} \, e^{:\hat{g}_3(x,y,\delta):} \ldots \mathcal{R}_{(k-1)k} \, e^{:\hat{g}_k(x,y,\delta):} \, \mathcal{R}_k^{-1}. $$

Since $\hat{g}_k$ is independent of conjugate momentums (P_x, P_y, P_δ), to advance the momentum we simply have

$$ \begin{aligned} \Delta P_x^k &= \frac{\partial \hat{g}_k}{\partial x} \\ \Delta P_y^k &= \frac{\partial \hat{g}_k}{\partial y} \\ \Delta P_\delta &= \frac{\partial \hat{g}_k}{\partial \delta} \qquad \text{for each } k = 1, 2, \ldots K. \end{aligned} $$

Therefore,

$$
\begin{aligned}
P_x &= P_x + \Delta P_x^k(x, y, \delta) \\
P_y &= P_y + \Delta P_y^k(x, y, \delta) \\
P_\delta &= P_\delta + \Delta P_\delta^k(x, y, \delta) \qquad\qquad \text{for each kick } k = 1, 2, \ldots K
\end{aligned}
$$

where $\Delta P_x, \Delta P_y, \Delta P_\delta$ are given as explicit functional form of x, y, δ. Since at the beginning of each turn (or after each RF cavity), the off-momentum δ is updated and fixed before the next RF cavity, we can substitute δ into each of the functional forms of $\Delta P_x^k, \Delta P_y^k, \Delta P_\delta^k$ to convert them into 2-variable polynomial instead of 3-variable polynomial to save computer time. That is, after substituting δ into the polynomial in the beginning of each term we have

$$
\begin{aligned}
P_x &= P_x + \Delta P_x^k(x, y) \\
P_y &= P_y + \Delta P_y^k(x, y) \\
P_\delta &= P_\delta + \Delta P_\delta^k(x, y)
\end{aligned}
$$

in each of the kicks to update the conjugate momenta of x, y, δ. Therefore the nonlinear map m_g becomes rotate, kick, rotate, kick, ..., rotate, kick, rotate for $k+1$ 4-Dimensional (phase space) rotation and k 3-dimensional kicks. The linear part is six-dimensional. All six-dimensional coordinates are updated by 6-D matrix operation.

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REFERENCES

1. A. Dragt and J. Finn, *J. Math. Phys.*, **17**, 2215 (1976); A. Dragt et al., *Ann. Rev. Nucl. Part. Sci.*, **38**, 455 (1988).
2. J. Irwin, "A Multi-Kick Factorization Algorithm for Nonlinear Maps," SSC-228, 1989.
3. Y. Yan, *Zmap—A Differential-Algebra Map Extraction Program Using ZLIB*, Superconducting Super Collider Publication No. SSCL-299 (1990).
4. Y. Yan and C. Yan, *ZLIB—A Numerical Library for Differential Algebra and Lie Algebraic Treatment of Beam Dynamics*, Superconducting Super Collider Publication No. SSCL-300 (1990).