SSCL-N-750

The Design of Bellows for Cryogenically-Cooled Superconducting Magnets

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Abstract

Sixteen geometries of expansion bellows were analyzed, using MSC/NASTRAN. The cases were carefully chosen to span the practical ranges of diameter and thickness. The results are plotted against dimensionless parameters giving the maximum stress due to expansion, the axial stiffness, the maximum stress due to pressure, and the axial forces caused by pressure. Design criteria based upon plastic analysis are suggested and explained. The significance of bellows thrust is explained. A means of anchoring bellows to eliminate their tendency to squirm is proposed.

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TABLE

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1.0 GEOMETRY OF A BELLOWS CONVOLUTION

A bellows is a series of flexible convolutions intended to contain pressure while allowing for the differential thermal expansion, or contraction, of other adjoining rigid pressure vessel components. The number of convolutions in a bellows is determined by the amount of expansion to be accommodated. A bellows may consist of a single convolution, in which case it is more commonly called an expansion joint, or of many convolutions, making it more of a flexible hose. But the key to the understanding of a bellows is the understanding of the convolution, which is shown in Figure 1.

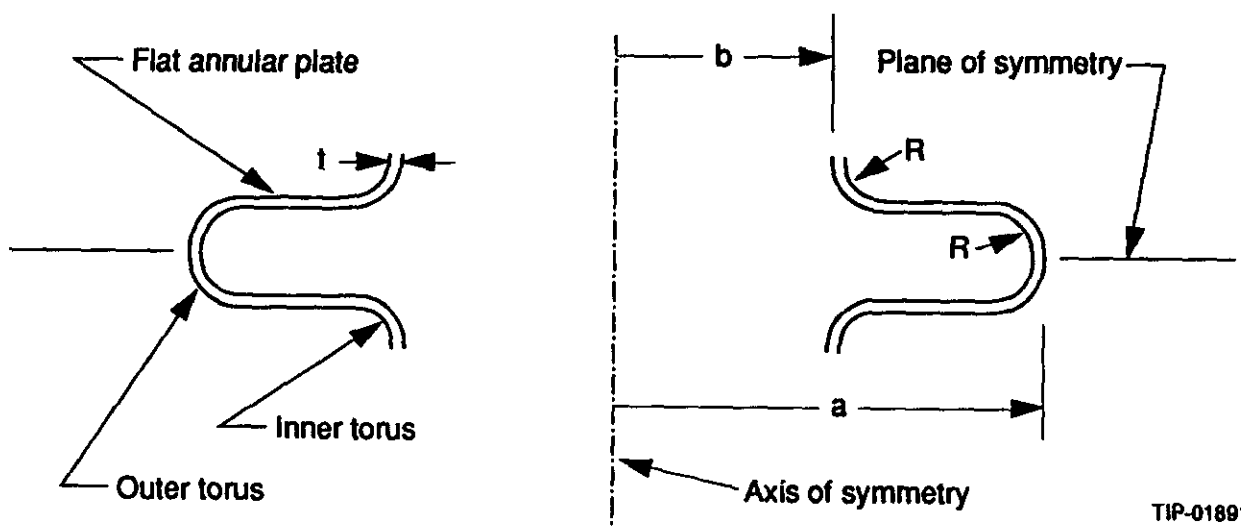


Figure 1. A Bellows Convolution.

Each bellows convolution is composed of two toroidal shell segments: the inner torus, of negative curvature, and the positively curved outer torus. These tori are connected by flat annular plates. The axial flexibility of the flat annular plate is greater than that of the toroidal shell segments. Therefore, there is a desire to make the toroidal radii, R, as small as possible.

Most bellows are formed by expanding thin, cylindrical tubes. There is a practical limit to how small the toroidal radii can be formed. This limit seems to be about three times the thickness. For the purposes of this study the radii, R, of both the inner and outer tori were taken to be four times the thickness, ($R = 4t$). Note that in both cases the 4t radius is taken from the inside of the curved wall.

The ratio of the bellows outside diameter over the bellows inside diameter, which is the same as the radius ratio a/b, would probably not be less than 1.1, except in the case of exceedingly thin walls and that would produce a very stiff bellows. On the other extreme, it is unlikely that a bellows would have a diameter ratio as great as two because there usually isn't that much space to accommodate a protrusion of that size. Therefore, a/b in this study ranged from 1.1 to 1.9. The thickness ratios, $\beta = b/t$, ranged from 25 to 200.

Because there must be some portion of a flat, annular plate between the inner and outer tori, the criterion $\beta > 10/(\alpha-1)$ must be met, causing a lower limit to the thickness ratio for some diameter ratios. The cases studied are presented in Table 1.

Table 1. Finite Element Cases Used for MSC/NASTRAN Study.

$\alpha = a/b \Rightarrow$	1.1	1.3	1.5	1.7	1.9
$\beta = b/t$ $\Downarrow$					
200	Case 1	Case 2	Case 3	Case 4	Case 5
100	*	Case 6	Case 7	Case 8	Case 9
50	*	Case 10	Case 11	Case 12	Case 13
25	*	*	Case 14	Case 15	Case 16

*The criterion, $\beta > 10/(\alpha-1)$, is not satisfied.

2.0 FINITE ELEMENT ANALYSIS

A finite element analysis was done because closed-form solutions for the general shell theory analysis, including bending of the shell wall, are unavailable for a bellows convolution. While such solutions do exist for the annular plate,[1] those for the tori[2] are valid only when the meridional angle is significantly greater than zero. But in a bellows the meridional angle of the tori does indeed go all the way to zero.

Even when an analytical solution does exist, as is the case for the annular plate portion, it is algebraically involved. One may as well use the finite element solution. In the Appendix, the closed-form solution for the flat annular plate is fully developed. It leads to 12 algebraic influence coefficients which must be evaluated in terms of boundary condition constants. Nevertheless, that closed form solution was productively used in this study to check the accuracy of the finite element analysis, and as the basis for deriving the dimensionless parameters.

The finite element analysis was conducted using the MSC•PAL 2 software[3] on a Macintosh II computer. Axisymmetric ring elements were used. In addition to axisymmetry, there is also symmetry between the upper and lower halves of the convolution, as shown in Figure 1.

Thirty rings elements were used to model half of the bellows convolution. The results of the study were compiled into four dimensionless parameters, where

$$\sigma = \text{stress},$$

$$\Delta = \text{expansion},$$

$$n = \text{number of convolutions, and}$$

$$P = \text{pressure}.$$

The four parameters are:

$$\text{Stress due to expansion: } \frac{n\sigma\pi b^2}{Et\Delta},$$

$$\text{Bellows stiffness: } \frac{nK\pi b^2}{Et^3},$$

$$\text{Stress due to pressure } \frac{\sigma t^2}{\pi b^2 P}, \text{ and}$$

$$\text{Force due to pressure: } \frac{F}{\pi b^2 P}.$$

As a check, the MSC/NASTRAN program was used to calculate the stresses and deflections for the case of a pressure vessel expansion joint on which careful experimental data has been published.[4] The NASTRAN solution confirmed the published stiffness within 2% and the stresses within 5%. The NASTRAN solution very accurately represents reality for an axisymmetrically convoluted shell like a bellows.

3.0 STRESSES DUE TO THE EXPANSION OF A BELLOWS

The half-convolution of the bellows modeled is shown deformed by axial expansion in Figure 2. The stresses are primarily due to the bending of the walls. The regions of greatest bending stress are the inner and outer tori, rather than the flat plate portion. In each of the 16 cases, the maximum stress was in the inner torus.

The maximum principle stresses for the case of expansion are shown in the dimensionless graph of Figure 3. Principle stresses are used since they are greater than the Von Mises stresses and thus can be conservatively added to the similar results for pressure. They are presented as a family of four curves for the four thickness ratios on a logarithmic scale. No simple curve fitting routine worked very well; therefore the curves are drawn as segmented lines with corners.

It was necessary to insert a Case 1b, with an a/b ratio of 1.2 and a b/t ratio of 200, in order to more effectively smooth out the b/t = 200 curve.

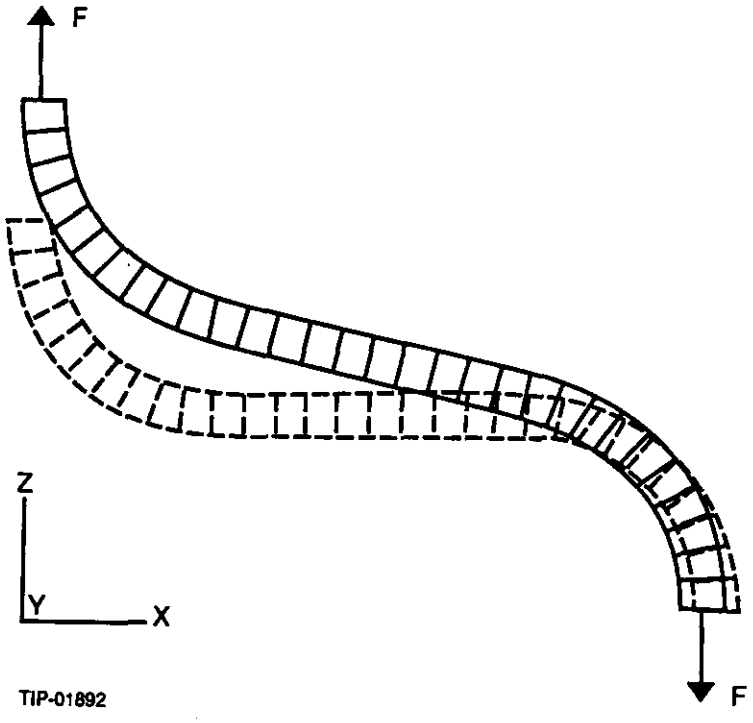


Figure 2. Expanded Convolution Half.

4.0 DESIGN CRITERIA FOR EXPANSION STRESS

Stainless steel contracts 0.3% from its room temperature length when cooled to the 4°K temperature of liquid helium. Aluminum contracts 0.4%. This contraction is highly non-linear in that 95% occurs in lowering the temperature from 300°K to the 80°K temperature of liquid nitrogen. The expansion bellows must be designed to take up not only the contraction of the bellows itself, but also the change of the length of the rigid pressure vessel components to which it is attached. The idea is to calculate the expansion required, (or the differential expansion if both aluminum and stainless are involved), and to select an a/b ratio, b/t ratio, and the number of convolutions so as to keep the stress level from Figure 3 within an acceptable value.

For a bellows restrained with a sleeve, the stress due to expansion can be permitted to be as high as three times the yield strength of the material. This seems preposterous at first glance, but it is based upon now well-accepted elastic-plastic criteria fully integrated into the Section VIII, Division Two, rules of the ASME pressure vessel code.[5] The three-times-yield criterion might be

used without sleeves if careful buckling analysis is done to ensure that the slenderness of the bellows is not so great as to cause the bellows to buckle after several cycles of plastic action. If three-times-yield is allowed for expansion stresses, the bellows will generate its own compressive preload that could squirm the bellows. This quite valid design philosophy is left to another paper.

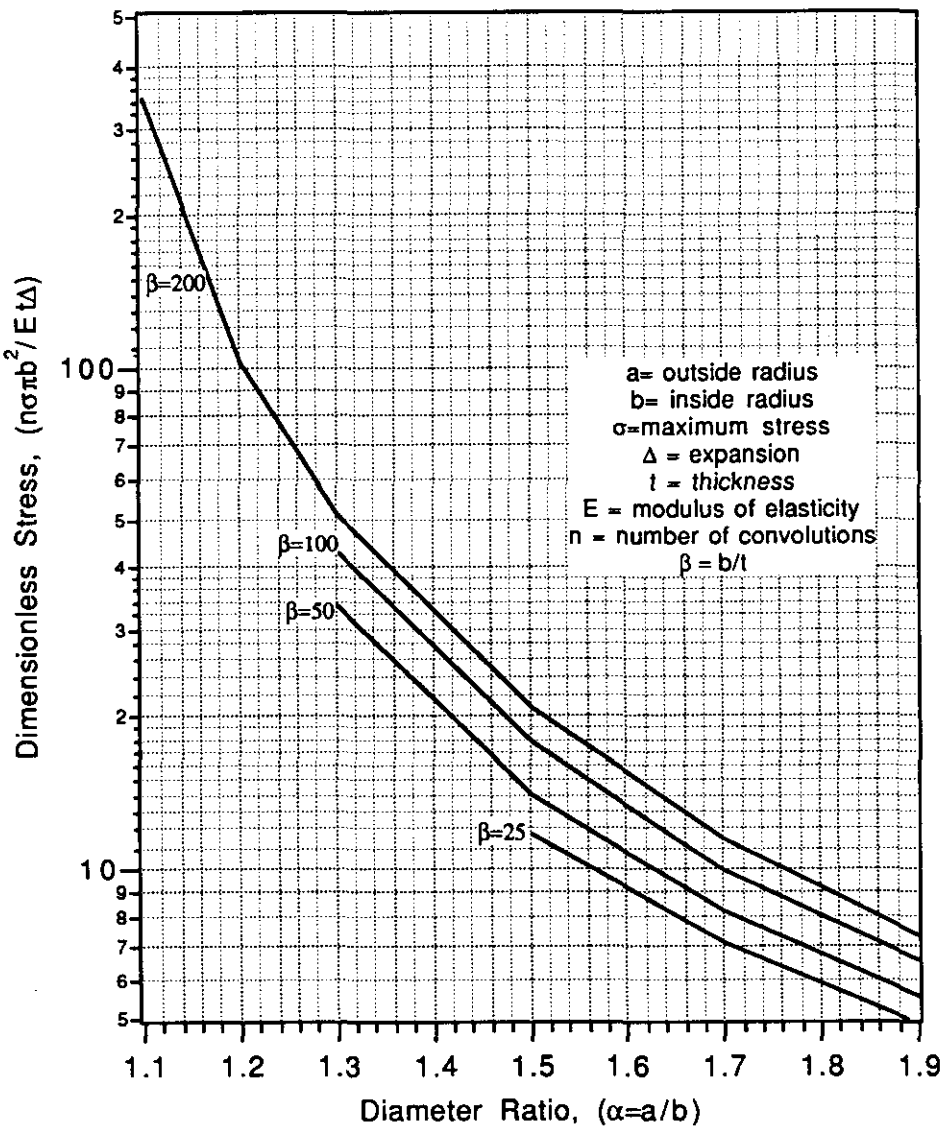


Figure 3. Maximum Principle Stress Due to Expansion.

The plastic criterion to be developed in this paper is to allow the expansion stress from Figure 3 to reach 1.5 times the yield strength of the material. This is similar to the Code criterion associated with the case of "primary bending." It can be used even when the bellows is quite slender and not laterally restrained by a sleeve. It is based upon a favorable residual stress

distribution produced by the first loading cycle, keeping the material elastic on subsequent loading cycles. It works as follows:

If a bellows is extended excessively while stress is measured at the most highly stressed point on the inner torus, a curve as shown in Figure 4 results. (Since there are no such things as stress gauges, strain gauges would have to be used, and the readings back-calculated into apparent stress.)

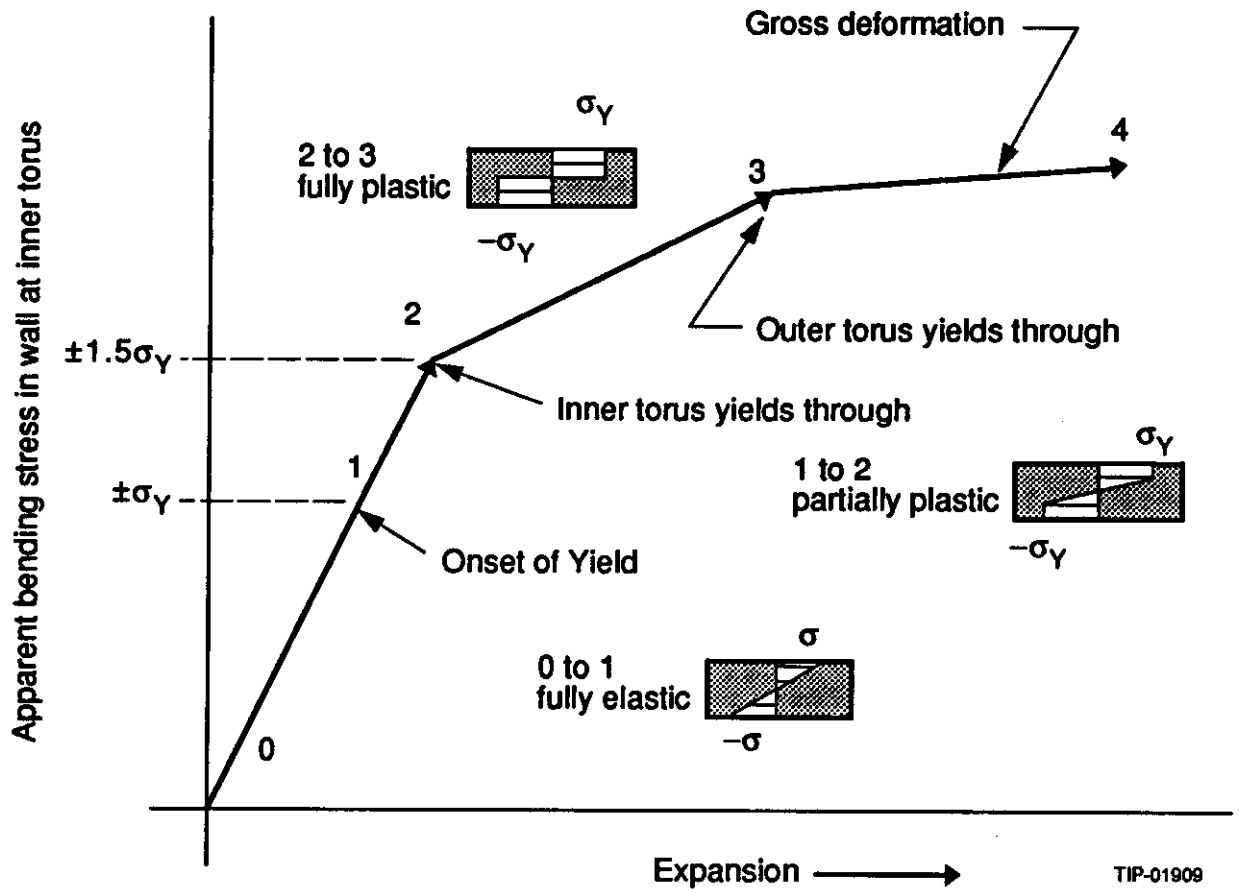


Figure 4. Apparent Stress at the Inner Torus Due to Excessively Expanding a Bellows.

From 0 to 1 the stress increases linearly with expansion because the material is below the yield point and is everywhere elastic. The bending stress distributions produced through the thickness of the wall are shown by the accompanying stress blocks. Beyond point 1 of Figure 4, the material begins to yield and become plastic at the surfaces. When this happens the actual stress of the material can no longer increase and is stopped at the yield stress. But, the core remains elastic and the stress profile as shown for points 1 to 2 develops.

As the expansion increases, the elastic core decreases and region of plastification intrudes more and more into the elastic core. But, so long as an elastic core remains, there is very little deviation form the original straight loading line, 0-1, in Figure 4.

At point 2, there is a sharp change in the loading curve as the wall of the bellows at the inner torus yields through and becomes fully plastic. The stress blocks shown for points 2 to 3 are now rectangular. The wall of the bellows has no further reserve strength. By the theory of plasticity, the apparent stress at point 2 can be shown to be 1.5 times the yield. From 2 to 3, the bellows convolution is transferring a greater proportion of the load to the outer torus. The bellows will no longer seem to be as stiff, though it continues to require additional force in order to expand it further.

When point 3 is finally reached, both the outer torus and the inner torus have yielded through and full "collapse" can take place. Collapse means that there is very little increase in resisting force, as the expansion continues to get larger. At this point there are two yield zones acting like plastic hinge lines, extending around each convolution of the bellows. The difference in apparent stress between points 2 and 3 represents the additional reserve plastic strength due to the redistribution of the bending moment from the inner to the outer torus.

Gross deformation is not in itself a failure condition of a bellows. In fact, since a bellows must be rather grossly deformed in order to be manufactured, the added deformation might not even be noticeable without taking measurements, and may not even be of cosmetic importance.

When the force causing the expansion is released from any value beyond point 2, a residual stress, as shown in Figure 5, will be present. This is a favorable residual stress distribution. Now it is possible to restress the material to an apparent stress of 1.5 of the yield strength without any further plastic action due to the fact that some of the material in exactly the right places has been prestressed in the reverse direction to 0.5 of the yield strength.

It is a rare cryogenic component that will see even a thousand cycles of cooling and warming throughout its lifetime. Therefore, the endurance limit associated with high cycle fatigue does not come into play. Due to expansion, bellows fail by low-cycle fatigue, which is the cracking by the successive embrittlement which is seen when bending a paper clip back and forth.

In order to have low-cycle fatigue, one must have alternating plasticity. Alternating plasticity is successive cycles of stress back and forth, beyond yield in tension to beyond yield in compression. This cannot occur if the bellows is extended only in one direction and back to zero repeatedly, and the absolute value of stress is kept below $1.5\sigma_y$.

Since thin 304 stainless steel sheet can have a yield greater than 120,000 psi at room temperature, and even higher at cryogenic temperatures, design expansion stresses of 200 ksi are entirely reasonable. But one must accept the fact that yielding will occur during the early cycles.

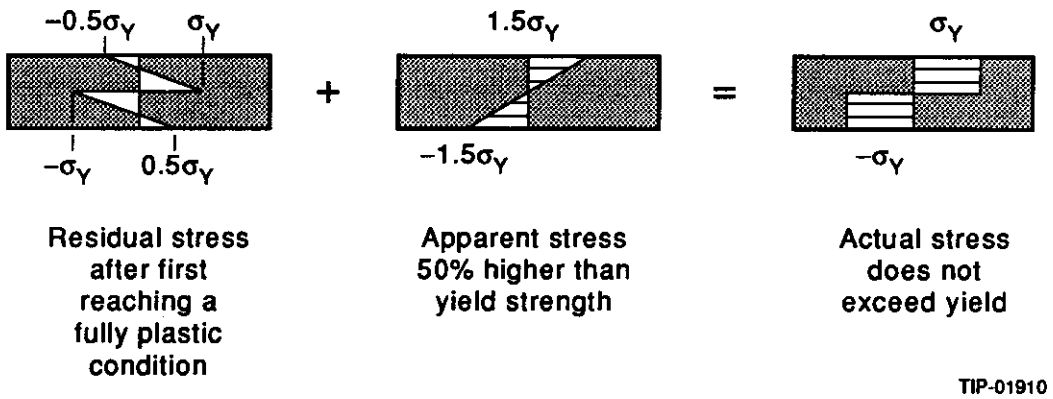


Figure 5. Stress Distribution. Due to a favorable residual stress, the bellows wall remains elastic after the first load cycle 50% beyond yield.

5.0 AXIAL STIFFNESS OF A BELLOWS

The axial stiffness, K, of a bellows is sometimes a matter of concern. This finite element study produced the stiffness curves given in Figure 6. These curves give rise to two observations.

First, for the dimensionless parameters chosen, the stiffness curves for all the ranges of b/t fall closely within a curved band. When the results from a flat annular plate (the data points in Figure 9) are plotted, they too fall within that band. The flat annular plate was one of the checks used for the finite element analysis. The support analysis used for the flat plate is given in the Appendix.

Second, the stiffness falls off dramatically as the diameter ratio increases. The curves are plotted on a logarithmic scale making them appear less pronounced. But by simply increasing the diameter ratio a small amount, one can significantly reduce the stiffness.

6.0 STRESS DUE TO PRESSURE

The deformation of a half convolution due to pressure is shown in Figure 7. In the case of pressure, there are three regions of high stress: the two tori and the mid-span of the flat plate. Again, the highest stress is at the inner torus, usually at the point where it attaches to another convolution half.

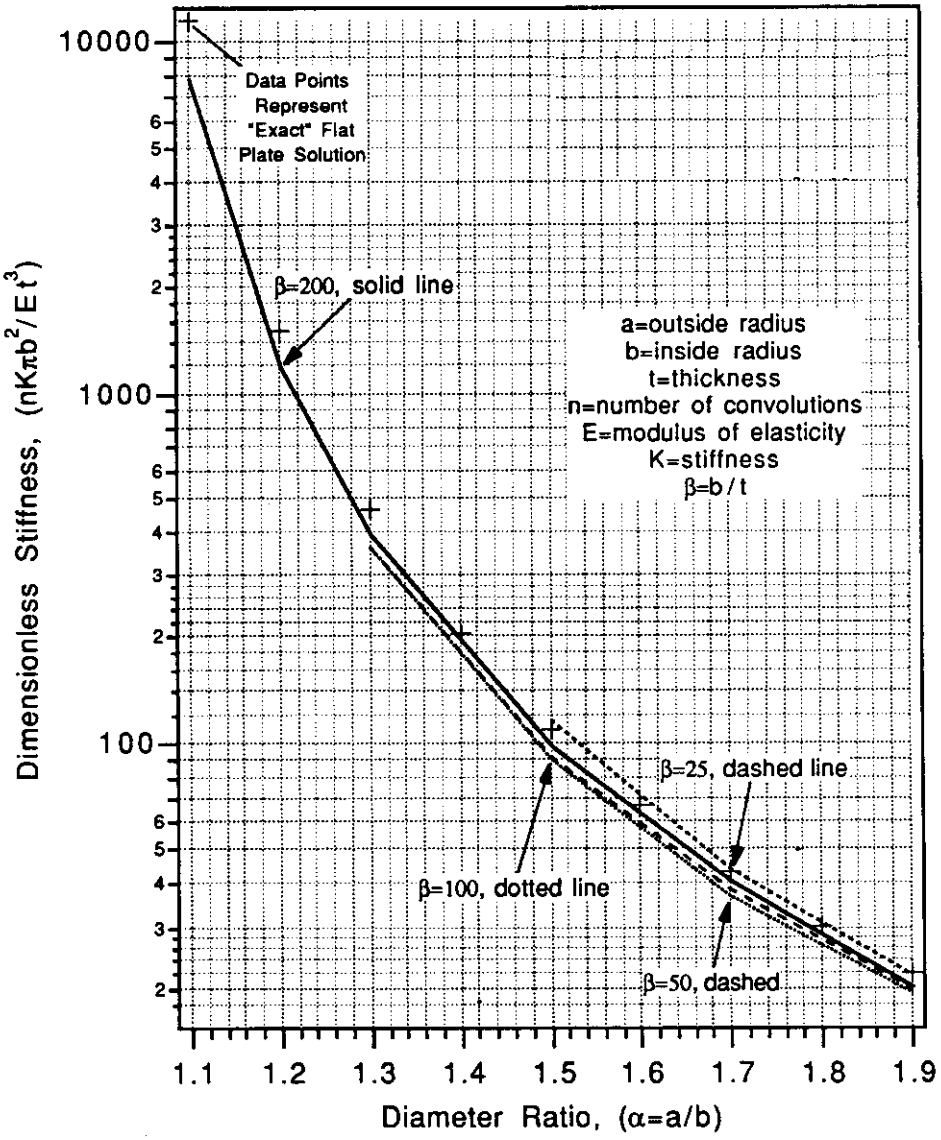


Figure 6. Stiffness of a Bellows.

The stresses due to pressure are given in dimensionless form in Figure 8. Here again the maximum principal stress is given. A family of curves represents the increase of stress with the diameter ratio.

The maximum stress due to pressure occurs at the inner radius of the bellows, just as in the case of expansion. But it is important to understand that the stress from bending at the inner torus is opposite in sign to that due to expansion. Comparing Figure 7 to Figure 2, one notes that the inner torus in Figure 7 is closing, whereas that for Figure 2 is opening. It is only at the outer torus that the stress for expansion has the same sign as that for internal pressure.

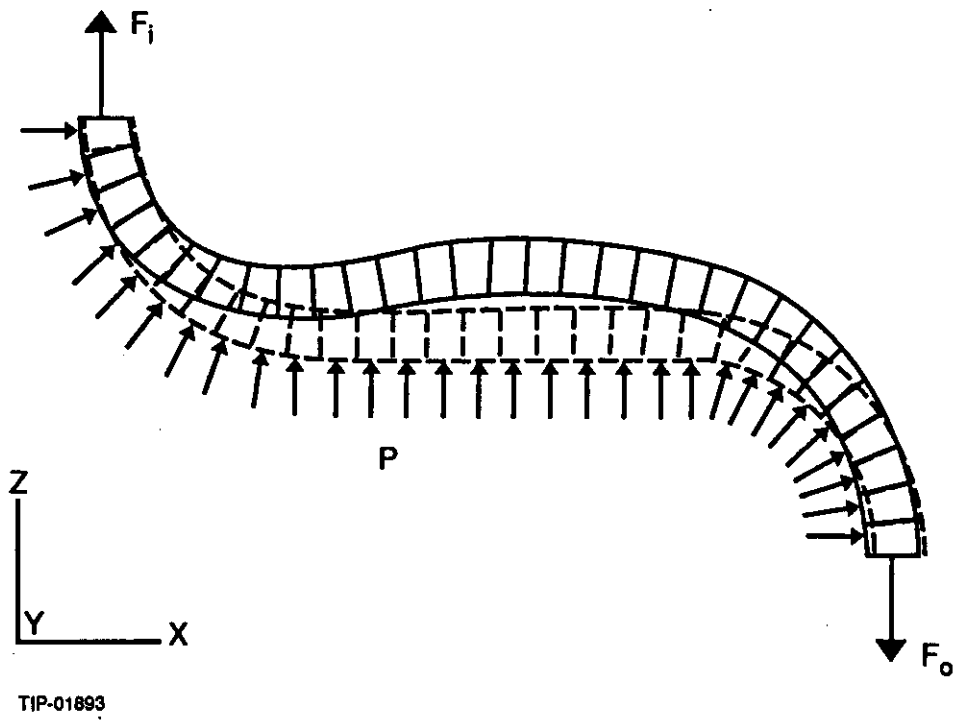


Figure 7. Pressurized Convolution Half.

One should also note when in comparing Figures 7 and 2 that pressure tends to bulge the annular plate. Expansion, while forcing the plate into a shallow conical shape, does not tend to bulge the plate. The effects of pressure on the bellows are very different from those of expansion.

But the most important difference is that for the same thickness, increasing the diameter ratio from 1.1 to 1.9 increases the stress about 30 times. However, the same diameter increase reduces the expansion stress about 40 times. What one does to reduce expansion stress increases the stress due to pressure, and vice versa.

7.0 DESIGN CRITERIA FOR PRESSURE

The stress due to pressure falls into the category of primary bending stress by the ASME code. Primary bending is bending that is not self-equilibriating, as is the case with expansion. Failure due to primary bending is seen in the outward bulging of the flat plate walls of the convolution. In the case of pressure the stress should be kept within $1.5\sigma_y$, which allows for the reserve bending strength as explained previously.

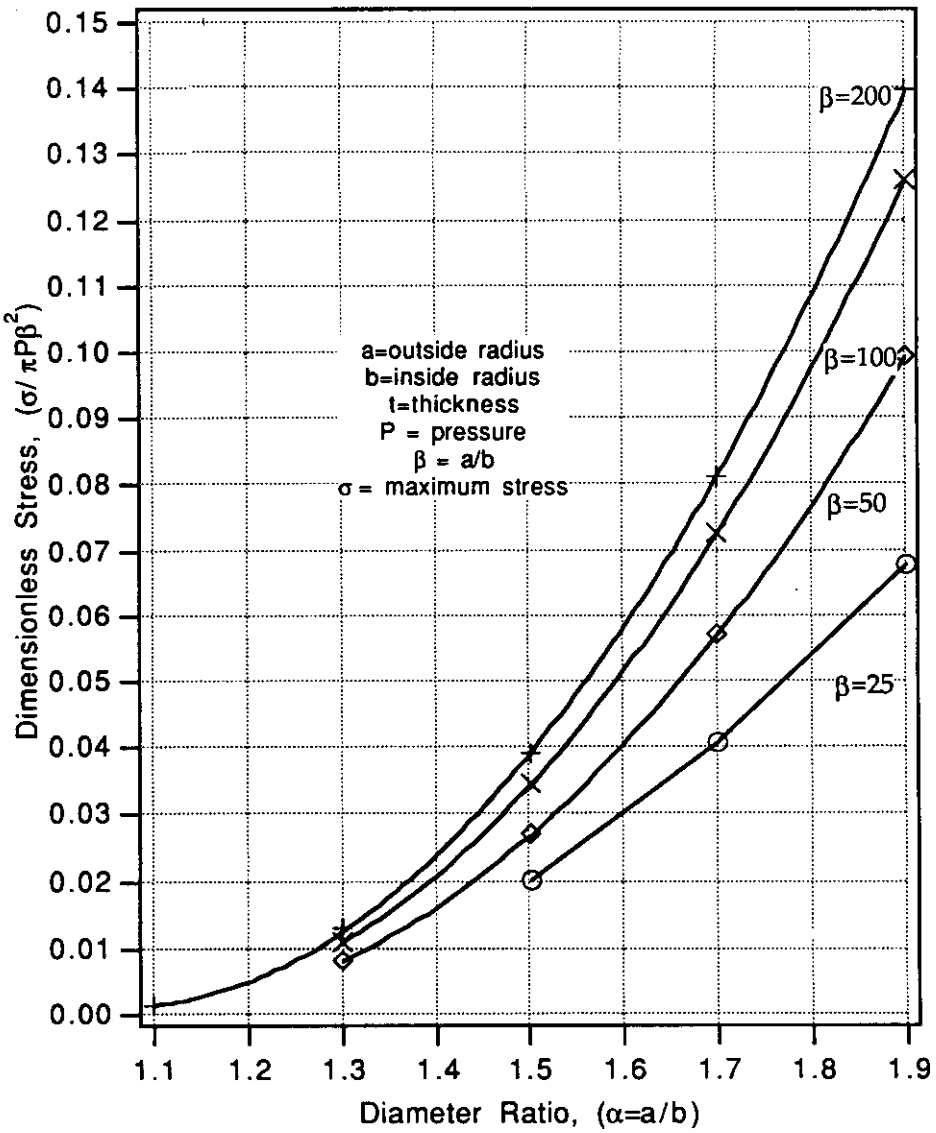


Figure 8. Stress Due to Pressure.

However, it is important to understand that bellows, if restrained against squirm, are very forgiving as pressure containing elements. As the flat annular plates of the convolutions bulge to the point of gross deformation, they actually assume a better shape for pressure containment. They are unlikely to burst even when greatly overpressured. The worst that will probably happen is that they bulge to the point that the convolutions come together and the bellows loses its flexibility.

Because catastrophic bursting is unlikely, it makes no sense to impose pressure vessel safety factors on a bellows. Here again, stressing bellows to 1.5 times the yield is more than conservative.

The interaction of pressure and cyclic expansion is theoretically possible in the form of incremental failure. In such a failure scenario, each cycle of expansion produces an increment of bulging due to pressure. Presumably, after many cycles of expansion while the pressure is held steady, the bellows convolutions are so deformed that the bellows no longer works. The ASME code deals with this situation by cascading the criteria.[5]

In other words, the absolute value for stress due to pressure should be less than 1.5 times the yield; the same holds true for stress due to expansion. This will insure that the two absolute values added together will be no greater than 3 times the yield. At those stress levels, incremental failure cannot take place because the shell wall will "shake down" to elastic action with repeated cycles.

The notion of safety factor is not really applicable in the case of expansion elements. Safety factors in pressure vessels are intended to produce thicker shell walls, thereby lowering stress. But while a thicker bellows wall will lower the stress due to pressure, it will increase the stress due to expansion. Whatever helps for pressure hurts when expansion enters the picture.

Instead of safety factors, it is advisable to use ductile, high-strength materials up to and beyond yield to the full plastic strength of the shell wall. There is no justification to do otherwise, even from the standpoint of micro-leaks. Micro-leaks, if they are going to be present, would already have been produced by the severe forming processes necessary to manufacture a bellows. The only reason for keeping bellows stresses low is the avoidance of corrosion fatigue. But corrosion fatigue is not likely to be a problem in the presence of helium.

8.0 BELLOWS FORCE DUE TO PRESSURE

A bellows exerts an axial force when pressurized. Figure 9 shows this force as a dimensionless number in terms of the diameter ratio. If the bellows is experiencing positive internal pressure, the force at its inner edge is compressive. In Figure 9 it is called the thrust force. If the bellows is experiencing a vacuum, the force at the inner edge is tension.

Likewise, there will be a reverse force at the outer edge of the convolution. Under positive pressure that force will be tensile, and is called the draw force in Figure 9.

In Figure 9, one line is drawn which represents the flat annular plate solutions for the inner edge forces. A second one is drawn representing the outer edge forces. The data points, in the case of the graph, are the results of the finite element analyses of the bellows convolution. They fall almost exactly on the annular plate curve. This is true for all thickness ratios studied.

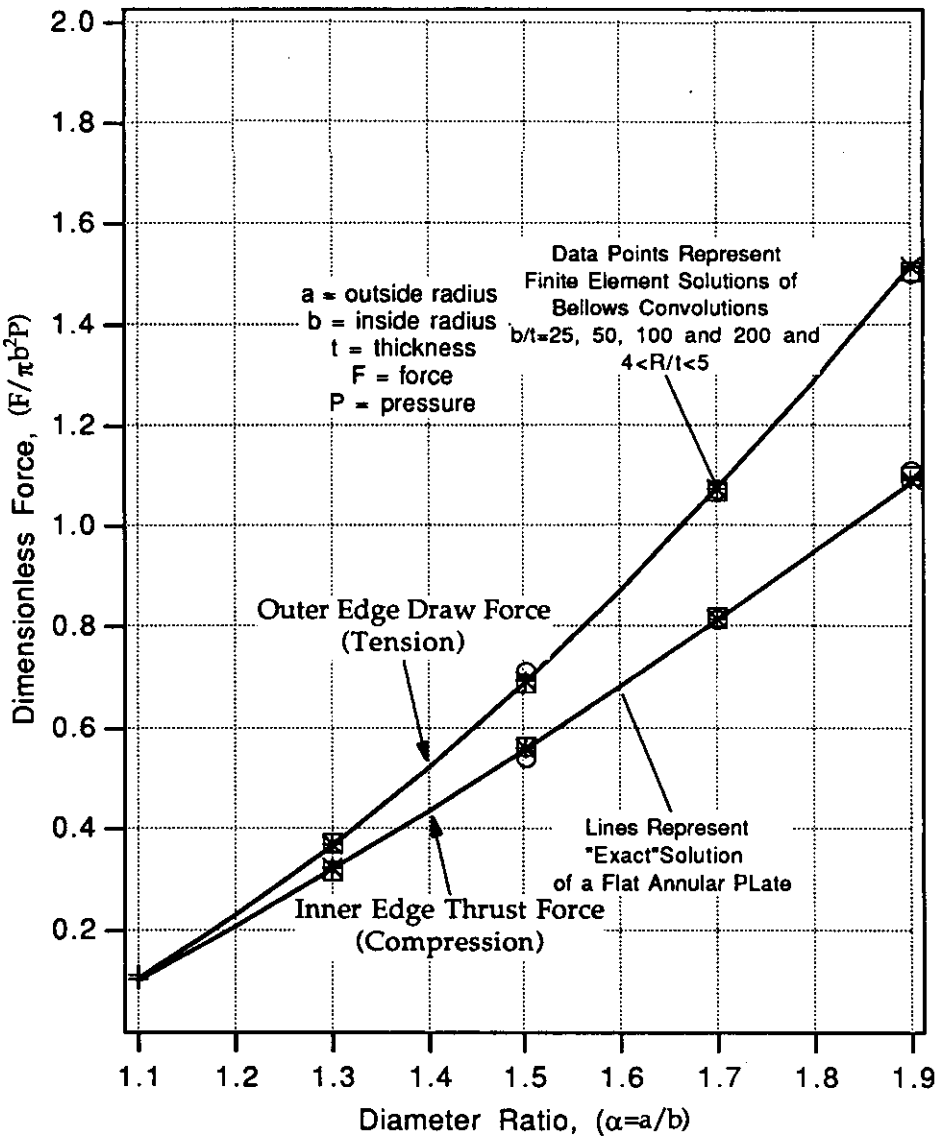


Figure 9. Forces on Adjoining Elements Due to a Pressurized Bellows.

9.0 SIGNIFICANCE OF THE THRUST OF A BELLOWS

The term *thrust* is used to mean the push from a bellows. But, that term is not used by everyone to mean exactly the same thing. In this paper thrust, F_i, is used to mean the force provided by the inner edge of a pressurized bellows. It is given directly by the lower curve of Figure 9. It does *not* include the hydrostatic force, $\pi b^2 P$, developed over the inside area of the bellows. Figure 10 illustrates the difference.

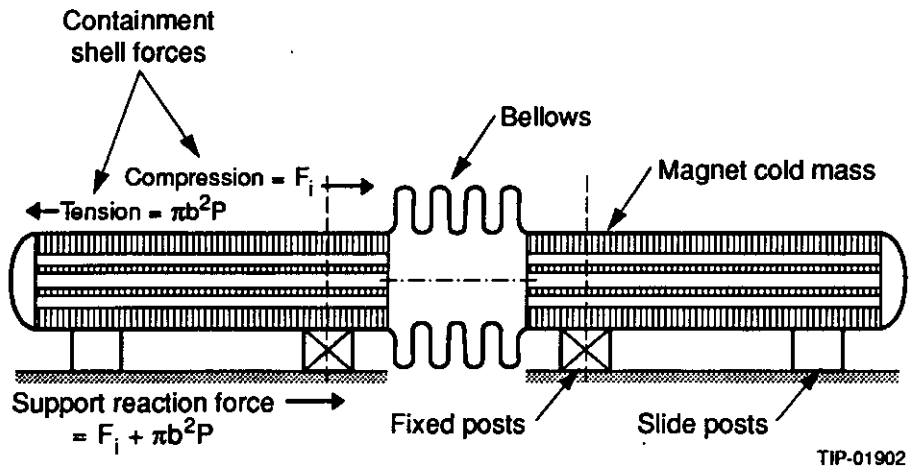


Figure 10. Two Magnet Cold Masses Connected by Bellows.

The two magnet cold masses are supported horizontally on four posts. One post on each magnet allows the cold mass to slide and expand horizontally. The other post is fixed to the containment shell in order to transfer the horizontal load to the cryostat shell.

The bellows convolutions must be restrained by the force, F_i, which is obtained from Figure 9. This force will put the containment shells into an axial *compression* of F_i inboard of the two fixed posts.

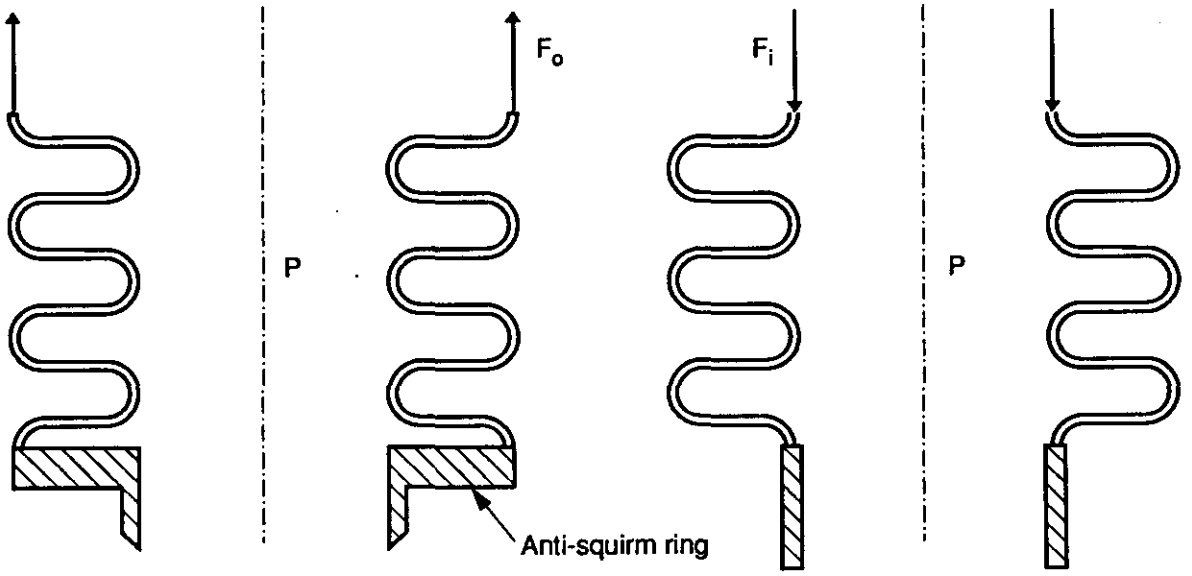
The elliptical head on the containment shell must be restrained by the force, $\pi b^2 P$. It will put the containment shells into axial *tension* outboard of the two fixed posts.

The fixed posts provide the total of the head force and the bellows thrust, $F_i + \pi b^2 P$, while *no part of the cold mass is acted upon by the total of the two forces.*

10. A PRACTICAL MEANS OF AVOIDING BELLOWS SQUIRM

Squirm is the only truly catastrophic event that can happen to a bellows. It occurs because a bellows, when constrained at its inner edge, is a self-actuating column. When pressurized, it tries to expand beyond its supports with the compression force shown in Figure 9. If the bellows is too slender, it buckles elastically and squirms off to the side. The relatively thin-walled bellows can tear where it is attached to the pipe.

Figure 11 shows a simple anti-squirm connection for a bellows that should be easy to test experimentally. Mathematically, the force, F, is tension on the outer edge of the convolution of a bellows, and compression on the inner edge. If one accepts that the cause of bellows squirm is a self-generated compression due to pressure on the very flexible column formed by the bellows, then changing the attachment force from compression to tension should stabilize the bellows.



This bellows tends to get shorter; F_o is tension and is a stabilizing force.

This bellows tends to get longer; F_i is compression and is a destabilizing force.

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Figure 11. An Anti-Squirm Connection for a Bellows on the Left.

Intuitively, one can visualize that one pitch of the bellows on the left in Figure 11 is squeezed together by the internal pressure. Therefore, the stack of pitches tries to shorten the bellows, and a tension force is needed to keep it at its original length. On the other hand, a single pitch of the bellows on the right tends to be forced open by internal pressure. Therefore, the stack of pitches tends to lengthen the bellows, and a compression force is needed to keep the bellows at its unpressurized length. Pulling on a rope is a stable process, but one needs a very stiff rope to push on. So, it is with a bellows.

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APPENDIX

FLAT ANNULAR PLATE

For a flat annular plate of outer radius, a, and inner radius, b, acted upon by a uniform pressure, P, and an edge force having a total value of F, the differential equation[6] is

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}=\frac{P}{D} \tag{1}$$

where the coordinate system is as shown in Figure A-1.

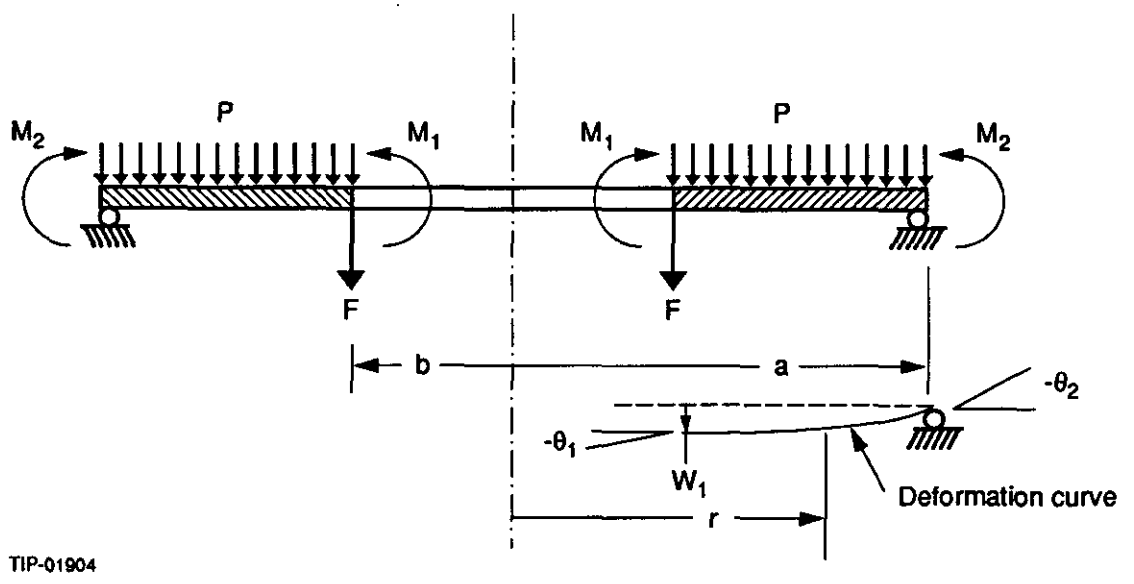

Figure A-1. Annular Plate Geometry Variables. w = downward deflection, $\frac{dw}{dr}$ = the slope θ, and M is the bending moment per unit length of edge.

$$D = \text{the plate stiffness} = \frac{Et^3}{12(1-\mu^2)}, \tag{2}$$

where E = the modulus of elasticity, and μ = Poisson's ratio.

Solving for the differential equation, one obtains

$$w = \frac{Pr^4}{64D} + \frac{K_1 r^2}{4} + K_2 \ln \frac{r}{a} + K_3 \tag{3}$$

where K_1, K_2, and K_3 are constants to be evaluated.

The constants for this equation from Reference 6 can be solved, with a number of algebraic steps omitted here, for the cases where M_1, M_2 F, and P act separately upon the annular plate yielding a matrix of influence coefficients, such that

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \frac{w_1}{b} \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} \\ I_{2,1} & I_{2,2} & I_{2,3} & I_{2,4} \\ I_{3,1} & I_{3,2} & I_{3,3} & I_{3,4} \end{bmatrix} \begin{Bmatrix} M_1 b \\ M_2 b \\ Fb \\ Pb^3 \end{Bmatrix}. \tag{4}$$

Where $\eta = a/b$, the $I_{i,j}$ can be expressed in dimensionless form.

$$I_{1,1} = \frac{1}{(\eta^2 - 1)} \left[\frac{1}{(1 + \mu)} + \frac{\eta^2}{(1 - \mu)} \right]$$

$$I_{2,1} = \frac{\eta}{(\eta^2 - 1)} \left[\frac{1}{(1 + \mu)} + \frac{1}{(1 - \mu)} \right]$$

$$I_{3,1} = \frac{1}{(\eta^2 - 1)} \left[\frac{1 - \eta^2}{2(1 + \mu)} + \frac{\eta^2}{(1 - \mu)} \ln \frac{1}{\eta} \right]$$

$$I_{1,2} = -\frac{\eta^2}{(\eta^2 - 1)} \left[\frac{1}{(1 + \mu)} + \frac{1}{(1 - \mu)} \right]$$

$$I_{2,2} = -\frac{\eta}{(\eta^2 - 1)} \left[\frac{\eta^2}{(1 + \mu)} + \frac{1}{(1 - \mu)} \right] \tag{5}$$

$$I_{3,2} = \frac{\eta^2}{(\eta^2 - 1)} \left[\frac{\eta^2 - 1}{2(1 + \mu)} - \frac{1}{(1 - \mu)} \ln \frac{1}{\eta} \right]$$

$$I_{1,3} = \frac{1}{4\pi} \left[\ln \frac{1}{\eta} - \frac{1}{2} - \frac{C_1}{2} - C_2 \right]$$

$I_{2,3}$ $$= \frac{1}{4\pi}\left[-\frac{\eta}{2} - \frac{C_1\eta}{2} - C_2\right]$$

$I_{3,3}$ $$= \frac{1}{4\pi}\left[\frac{1}{2}\left(\ln\frac{1}{\eta} - 1\right) - \frac{C_1}{4} - C_2 \ln\frac{1}{\eta} + C_3\right]$$

where

$$C_1 = \frac{1-\mu}{1+\mu} - \frac{2}{(\eta^2 - 1)}\ln\frac{1}{\eta},$$

$$C_2 = -\frac{1+\mu}{1-\mu}\frac{\eta^2}{(\eta^2 - 1)}\ln\frac{1}{\eta},$$

and $$C_3 = \frac{\eta^2}{2}\left[1 + \frac{1}{2}\left(\frac{1-\mu}{1+\mu}\right) - \frac{1}{(\eta^2 - 1)}\ln\frac{1}{\eta}\right]$$

$I_{1,4}$ $$= T_4 + T_5 + T_6$$

$I_{2,4}$ $$= T_7 + T_8 + T_9$$

$I_{3,4}$ $$= T_1 + T_2 + T_3,$$

where the T_1, are constants defined as follows:

$$T_1 = \frac{(\eta^2 - 1)}{64}\left[\frac{(5+\mu)}{(1+\mu)}\eta^2 - 1\right]$$

$$T_2 = -\frac{1}{4}\left[\frac{1}{2}\left(\ln\frac{1}{\eta} - 1\right) - \frac{C_1}{4} - C_2 \ln\frac{1}{\eta} + C_3\right]$$

$$T_3 = -\frac{(3+\mu)}{16}\left[\frac{1-\eta^2}{2(1+\mu)} + \frac{\eta^2}{(1-\eta)}\ln\frac{1}{\eta}\right] \qquad (6)$$

$$T_4 = -\frac{1}{32}\left[\eta^2\left(1 + \frac{5+\mu}{1+\mu}\right) - 2\right]$$

$$T_5 = -\frac{1}{4}\left[\left(\ln\frac{1}{\eta} - \frac{1}{2}\right) - \frac{C_1}{2} - C_2\right]$$

$$T_6 = -\frac{(3+\mu)}{16}\left[\frac{1}{(1+\mu)} + \frac{\eta 2}{(1-\mu)}\right]$$

$$T_7 = -\frac{\eta^3}{32}\left(-1 + \frac{5+\mu}{1+\mu}\right)$$

$$T_8 = -\frac{1}{4}\left[-\frac{\eta}{2} - \frac{C_1\eta}{2} - \frac{C_2}{\eta}\right]$$

$$T_9 = -\frac{\eta(3+\mu)}{16}\left[\frac{1}{(1+\mu)} + \frac{1}{(1-\mu)}\right].$$

For the case where the plate is supported at both a and b and restrained from rotation at those edges as well, as shown in Figure A-2,

$$\theta_1, \theta_2 \text{ , and } w_1 = 0.$$

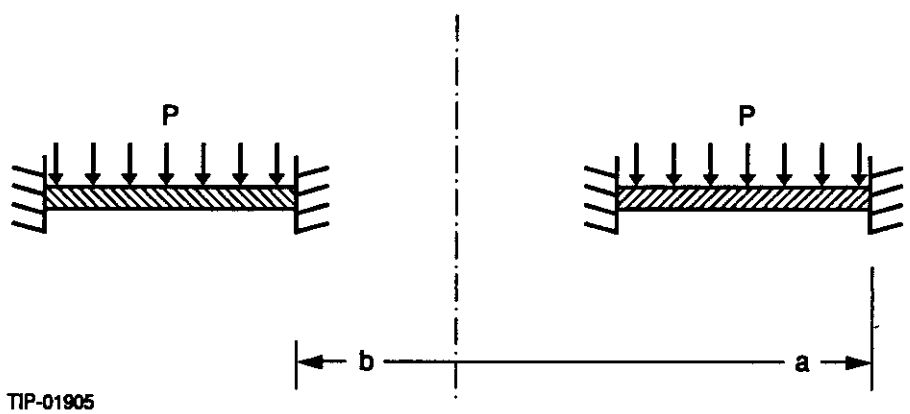

Figure A-2. The Annular Plate with Fixed Edges.

Equations (4) become:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} \\ I_{2,1} & I_{2,2} & I_{2,3} & I_{2,4} \\ I_{3,1} & I_{3,2} & I_{3,3} & I_{3,4} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ F \\ Pb^2 \end{Bmatrix}. \tag{7}$$

Letting $Pb^2 = 1$,

$$\begin{Bmatrix} I_{1,4} \\ I_{2,4} \\ I_{3,4} \end{Bmatrix} = \begin{bmatrix} I_{1,1} & I_{1,2} & I_{1,3} \\ I_{2,1} & I_{2,2} & I_{2,3} \\ I_{3,1} & I_{3,2} & I_{3,3} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ Fb \end{Bmatrix}, \tag{8}$$

which can be solved for the edge moments M_1 and M_2 and the force, F.

The matrix equation was solved for a/b ratios from 1.1 to 2. The inner edge force, F, is plotted over $\pi b^2 P$ as a dimensionless factor in Figure 9.