SSCL-N-749

DECOHERENCE OF THE KICKED BEAMS II

S.Y. Lee

Department of Physics
Indiana University
Bloomington, IN 47405

February 1991

SSCL-N-749

DECOHERENCE OF THE KICKED BEAMS II

S.Y. Lee

Department of Physics
Indiana University
Bloomington, IN 47405

February 1991

Abstract

The decoherence of betatron oscillations of the kicked beam is calculated analytically in two degrees of freedom by assuming a Gaussian distribution. Decoherence due to higher order chromaticity effect is also discussed.

1.0 INTRODUCTION

From basic linear beam dynamics, a beam, kicked transversely from its closed orbit, will execute betatron oscillations. These betatron oscillations can be observed by a beam position monitor, which measures the centroid of the particle distribution. For a linear betatron motion, the observed betatron oscillation is harmonic. Such procedures are commonly used to measure the betatron tunes. However, if beam particles have betatron tune spreads due to chromaticity or to the effects of nonlinear multipole elements in the accelerator, then the observed centroid of the beam will be decohered due to the accumulated betatron phase spread of particles. Knowledge of these decoherence effects can be used more reliably to obtain the betatron tunes and to understand nonlinear beam dynamics. When detailed phase space maps (Poincaré maps) are needed to study the nonlinear betatron motion, the decoherence process must be understood in deducing the effect of nonlinear motion.

In the past, Meller et al.[1] have derived analytic formulas for the decoherence in the presence of betatron tune spreads due to chromaticity and to the betatron amplitude of one degree of freedom. Their results have been used successfully in analyzing the data of the nonlinear beam dynamics experiments E778.[2] Recently, experiments in nonlinear dynamics, extended to two degrees of freedom, have been proposed and approved in the IUCF Cooler Ring. Extending the discussion of decoherence effects to two degrees of freedom is therefore important. In this note, I shall study the betatron decoherence of kicked beams due to the tune spread in two transverse degrees of freedom in the absence of linear coupling. I shall also study the effect of higher order chromaticity. Fortunately most of these effects can be expressed in terms of an analytic formula when a Gaussian particle distribution is assumed. The discussion includes decoherence due to chromaticity and due to betatron tune spreads in two degrees of freedom.

2.0 DECOHERENCE DUE TO CHROMATICITY

Assuming a linear dependence of the betatron tunes with respect to the momentum deviation of the particle, Meller et al.[1] showed that the betatron coordinate, $x(n)$, of the kicked beam at the nth turn can be expressed as

$$\langle x(n) \rangle = x_0 F_\delta \cos(2\pi\nu_\beta n + \phi_\beta), \tag{1}$$

with

$$F_\delta = \exp\left(-\frac{\alpha^2}{2}\right); \quad \alpha = \frac{2\xi\sigma_\delta \sin \pi\nu_s n}{\nu_s}, \tag{2}$$

for a Gaussian particle distribution. Here x_0 is the initial kicked betatron amplitude, ν_s is the synchrotron tune, ξ is the chromaticity, and σ_δ is the rms momentum spread, $\delta = \Delta p/p$. Thus the rms tune spread is $\xi\sigma_\delta$. Equation (1) shows that the measured centroid of the particle distribution is oscillating with a Gaussian form factor with frequency of $\frac{1}{\nu_s}$ turns. The modulation does not affect the tune measurement. The Fourier spectrum of Equation (1) is also Gaussian.

The frequency of the amplitude modulation of the betatron oscillation is independent of the particle distribution function used in obtaining Equation (2). For a uniform particle distribution function in the phase space, we obtain

$$F_\delta = \frac{2J_1(\sqrt{2}\alpha)}{\sqrt{2}\alpha} = {}_0F_1(2; -\frac{\alpha^2}{2}), \tag{3}$$

where J_1 is the Bessel function, ${}_0F_1$ is the generalized hypergeometric function, and α is given by Equation (2). Again the betatron oscillation recoheres every $\frac{1}{\nu_s}$ turns around the accelerator. In general the decoherence factor depends on the longitudinal distribution function of the beam in the accelerator. Methods to calculate the decoherence factor are given in Reference 1 and will be discussed in the next few sections.

3.0 HIGHER ORDER CHROMATIC EFFECTS

For a high energy particle accelerator, the linear chromaticity is normally corrected to a zero value. Higher order chromatic effect may then become important. It is known that the second order chromaticity depends on the half-integer stopband of the accelerator. I will discuss the decoherence effect later.

Let us consider the tune spread due to the second order chromaticity, where the tune spread is given by

$$\Delta\nu(n) = \xi_2 \left[\frac{a\sigma_\delta}{\sigma_a} \sin(2\pi\nu_s n + \phi_s)\right]^2 . \tag{4}$$

Here a is the synchrotron oscillation amplitude of the particle, σ_a is the rms synchrotron amplitude of the distribution, σ_δ is the rms momentum deviation of the distribution, and ξ_2 is the second order chromaticity. The betatron phase spread is then given by

$$\begin{aligned} \Delta\phi_\beta(a, \phi_s, n) &= 2\pi \int_0^n \Delta\nu(m) dm \\ &= \pi\xi_2 \frac{\sigma_\delta^2}{\sigma_a^2} a^2 n + 2\pi\xi_2 \frac{\sigma_\delta^2}{\sigma_a^2} a^2 \sin(2\pi\nu_s n)\cos(2\pi\nu_s n + 2\phi_s). \end{aligned} \tag{5}$$

Thus the transverse position of a particle with synchrotron amplitude a and synchrotron phase ϕ_s at the nth turn in the beam is given by

$$x(n) = x_0 \cos(2\pi\nu_\beta n + \phi_\beta + \Delta\phi_\beta(a, \phi_s, n)). \tag{6}$$

The centroid of the beam measured from the beam position monitor is given by

$$\langle x(n)\rangle = x_0 \int \cos(2\pi\nu_\beta n + \phi_\beta + \Delta\phi_\beta(a, \phi_s, n))\rho(a, \phi_s) da d\phi_s. \tag{7}$$

When the distribution function $\rho(a, \phi_s)$ of the beam is assumed to be a Gaussian, i.e.,

$$\rho(a, \phi_s) = \frac{1}{2\pi\sigma_a^2} a \exp\left(-\frac{a^2}{2\sigma_a^2}\right), \tag{8}$$

then Equation (7) can be solved easily as

$$\langle x(n) \rangle = x_0 F_\delta \cos(2\pi\nu_\beta n + \phi_\beta + \Delta\phi), \tag{9}$$

with

$$\begin{aligned} F_\delta &= \left[\left(1 - 4\pi^2\xi_2^2\sigma_\delta^4 n^2 + 16\pi^2\xi_2^2\sigma_\delta^4 \sin^2 2\pi\nu_s n\right)^2 + 16\pi^2\xi_2^2\sigma_\delta^4 n^2\right]^{-\frac{1}{2}}, \\ \Delta\phi &= \frac{1}{2}\arctan\frac{4\pi\xi_2\sigma_\delta^2 n}{1 - 4\pi^2\xi_2^2\sigma_\delta^4 n^2 + 16\pi^2\xi_2^2\sigma_\delta^4 \sin^2 2\pi\nu_s n}. \end{aligned} \tag{10}$$

Since ν_s is a small number, we expect that

$$F_\delta \approx \frac{1}{1 + 4\pi^2\xi_2^2\sigma_\delta^4 n^2}.$$

Equation (9) indicates that the amplitude of the measured betatron amplitude decoheres with an amplitude function F_δ and that the phase is also modulating with the phase function $\Delta\phi$ of Equation (10). The measured betatron tune is obtained from the Fourier transform of Equation (9), i.e.,

$$\nu_\beta^{\text{measured}} \approx \nu_\beta + \xi_2\sigma_\delta^2. \tag{11}$$

The characteristic Fourier amplitude will be exponential.

When the linear chromaticity and the second order chromaticity are simultaneously present in the accelerator, then the ϕ_s integral in Equation (7) can be performed to obtain sums of Bessel functions. The radial integral becomes too complicated to be represented by special functions. However numerical solutions can be obtained easily.

4.0 DECOHERENCE DUE TO THE BETATRON TUNE SPREAD

The betatron tune spreads due to the betatron amplitudes also decohere the betatron motion of the kicked beam. Meller et al.[1] have derived an analytic formula of the betatron oscillations for the kicked beam in one degree of freedom. The measured betatron oscillation is given by

$$\langle x(n)\rangle = -x_k F_x \sin\left(2\pi\nu_x n + \phi_x + \frac{x_k^2}{2\sigma_x^2}\frac{\theta}{(1+\theta^2)}\right) , \tag{12}$$

where

$$x_k = \beta_x \Delta x'; \quad \theta = 4\pi\Delta\nu_x n; \quad \phi_x = 2\arctan\theta; \tag{13}$$

$$F_x = \frac{1}{1+\theta^2}\exp\left(-\frac{x_k^2}{2\sigma_x^2}\frac{\theta^2}{(1+\theta^2)}\right) . \tag{14}$$

Here x_k is the kicked amplitude, $\Delta\nu_x = k_{xx}\sigma_x^2$ is the rms tune spread in one degree of freedom. From Equation (12), we obtain the betatron tune of the kicked beam as

$$\nu_x^{\text{measured}} \approx \nu_x + 4k_{xx}\sigma_x^2 + k_{xx}x_k^2 . \tag{15}$$

The measured Fourier spectrum is initially a Gaussian and then evolves towards an exponential form.

5.0 DECOHERENCE IN TWO DEGREES OF FREEDOM

The betatron motion of the kicked particles can be described by

$$\begin{aligned} x(n) &= a_x\cos(2\pi\nu_x n + \phi_x + \Delta\phi_x); \quad \alpha_x x + \beta_x x' = -a_x\sin(2\pi\nu_x n + \phi_x + \Delta\phi_x), \\ z(n) &= a_z\cos(2\pi\nu_z n + \phi_z + \Delta\phi_z); \quad \alpha_z z + \beta_z z' = -a_z\sin(2\pi\nu_z n + \phi_z + \Delta\phi_z), \end{aligned} \tag{16}$$

where a_x, a_z are the betatron amplitudes, ν_x, ν_z are the betatron tunes, ϕ_x, ϕ_z are betatron phases of the particle, and $\Delta\phi_x$, $\Delta\phi_z$ are the betatron phase spread

given by

$$\Delta\phi_x = 2\pi(k_{xx}a_x^2 + k_{xz}a_z^2)n; \quad \Delta\phi_z = 2\pi(k_{xz}a_x^2 + k_{zz}a_z^2)n. \tag{17}$$

Here the tune spread is second order in the betatron amplitude due mainly to sextupoles in an accelerator. The betatron coordinates, measured from the beam position monitor, are then given by

$$\begin{aligned}\langle x(n)\rangle &= \int a_x \cos(2\pi\nu_x n + \phi_x + \Delta\phi_x)\rho(a_x,\phi_x)\rho(a_z,\phi_z)da_x d\phi_x da_z d\phi_z,\\ \langle z(n)\rangle &= \int a_z \cos(2\pi\nu_z n + \phi_z + \Delta\phi_z)\rho(a_z,\phi_z)\rho(a_z,\phi_z)da_z d\phi_z da_z d\phi_z,\end{aligned} \tag{18}$$

where the distribution function of the kicked beam is given by

$$\begin{aligned}\rho(a_x,\phi_x) &= \frac{a_x}{2\pi\sigma_x^2}\exp\left(-\frac{a_x^2+x_k^2}{2\sigma_x^2}\right)\cdot\exp\left(\frac{x_k a_x}{\sigma_x^2}\sin\phi_x\right);\\ \rho(a_z,\phi_z) &= \frac{a_z}{2\pi\sigma_z^2}\exp\left(-\frac{a_z^2+z_k^2}{2\sigma_z^2}\right)\cdot\exp\left(\frac{z_k a_z}{\sigma_z^2}\sin\phi_z\right);\end{aligned} \tag{19}$$

with

$$x_k = \beta_x\Delta x'; \quad z_k = \beta_z\Delta z', \tag{20}$$

as the kicked amplitudes. Here σ_x, σ_z are rms beam sizes. Using the Gaussian distribution of Equation (19), we obtain easily

$$\begin{aligned}\langle x(n)\rangle &= -x_k F_{xx}F_{xz}\sin\left(2\pi\nu_x n + \phi_{xx} + \phi_{xz} + \frac{x_k^2}{2\sigma_x^2}\frac{\theta_{xx}}{1+\theta_{xx}^2} + \frac{z_k^2}{2\sigma_z^2}\frac{\theta_{xz}}{1+\theta_{xz}^2}\right),\\ \langle z(n)\rangle &= -z_k F_{zx}F_{zz}\sin\left(2\pi\nu_z n + \phi_{zx} + \phi_{zz} + \frac{x_k^2}{2\sigma_x^2}\frac{\theta_{zx}}{1+\theta_{zx}^2} + \frac{z_k^2}{2\sigma_z^2}\frac{\theta_{zz}}{1+\theta_{zz}^2}\right),\end{aligned} \tag{21}$$

with

$$\theta_{xx} = 4\pi k_{xx}\sigma_x^2 n; \quad \theta_{xz} = 4\pi k_{xz}\sigma_z^2 n; \quad \theta_{zx} = 4\pi k_{zx}\sigma_x^2 n; \quad \theta_{zz} = 4\pi k_{zz}\sigma_z^2 n. \tag{22}$$

$$\phi_{ij} = 2\arctan\theta_{ij}; \quad (i,j = x,z)$$

$$F_{xx} = \frac{1}{1+\theta_{xx}^2}\exp\left(-\frac{x_k^2}{2\sigma_x^2}\frac{\theta_{xx}^2}{1+\theta_{xx}^2}\right); \quad F_{xz} = \frac{1}{1+\theta_{xz}^2}\exp\left(-\frac{z_k^2}{2\sigma_z^2}\frac{\theta_{xz}^2}{1+\theta_{xz}^2}\right),$$
$$F_{zx} = \frac{1}{1+\theta_{zx}^2}\exp\left(-\frac{x_k^2}{2\sigma_x^2}\frac{\theta_{zx}^2}{1+\theta_{zx}^2}\right); \quad F_{zz} = \frac{1}{1+\theta_{zz}^2}\exp\left(-\frac{z_k^2}{2\sigma_z^2}\frac{\theta_{zz}^2}{1+\theta_{zz}^2}\right). \tag{23}$$

Within a short time interval after the kick, the measured betatron tunes, derived from the Fourier transform of Equation (21), would be

$$\begin{aligned} \nu_x^{\text{measured}} &\approx \nu_x + 4k_{xx}\sigma_x^2 + k_{xx}x_k^2 + 4k_{xz}\sigma_z^2 + k_{xz}z_k^2\,, \\ \nu_z^{\text{measured}} &\approx \nu_z + 4k_{xz}\sigma_x^2 + k_{xz}x_k^2 + 4k_{zz}\sigma_z^2 + k_{zz}z_k^2\,. \end{aligned} \tag{24}$$

The amplitude modulation shortly after the kick is Gaussian. Thus the Fourier spectrum would also be Gaussian in shape. When the condition $|\theta_{ij}| \gg 1$ is met, the decoherence of the betatron oscillation obeys a power law. The corresponding Fourier spectrum is exponential.

CONCLUSION

The decoherence of betatron oscillations of a kicked beam is studied in the cases of (1) higher order chromaticity effect and (2) the betatron tune spread in two degrees of freedom. Most of the discussions in Reference 1 remain valid. I do not wish to repeat them here. The present study, a generalization of Reference 1, should be useful in data analysis of nonlinear beam dynamics experiments in two degrees of freedom.

REFERENCES

1. R.E. Meller, A.W. Chao, J.M. Peterson, S.G. Peggs, and M. Furman, *Decoherence of Kicked Beams*, SSC-N-360, 1987.

2. A.W. Chao et al., *Phys. Rev. Lett.* **61**, 2752, 1988.