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STRAIN ENERGY MINIMIZATION IN SSC MAGNET WINDING

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Abstract

Differential geometry provides a natural family of coordinate systems, the Frenet frame, in which to specify the geometric properties of a magnet winding. By a modification of the Euler-Bernoulli thin rod model, the strain energy is defined with respect to this frame. Then it is minimized by a direct method from the calculus of variations. The mathematics, its implementation in a computer program, and some analysis of an SSC dipole by the program will be described.

Introduction

The 50mm bore-radius of the Superconducting Super Collider (SSC) main ring constrains its superconducting cables to tight bends at the ends of the magnets, thereby justifying increased mathematical attention to the cable-strain minimization problem. This paper, a successor to reference [1], describes progress at Fermilab in a particular approach to this problem.

We have only rough estimates of constitutive relations between stress and strain so we concentrate on strain alone which is purely geometric and susceptible to exact specification and analysis. (The pronoun "we" in this note will refer to those members of Rodger Bossert's Magnet and Tooling Development group working under Jeff Brandt in the Technical Support Section at Fermilab.)

The method to be described is implemented by an interactive computer program, BEND. It employs a variational method to present its user with cable configurations having a low total elastic strain energy as calculated with respect to a reasonable mathematical model. Because it is not the total strain but the localized points of higher strain which first endanger a cable, detailed information is presented about the high-strain points in the configuration. The user can adjust intuitively meaningful parameters to relieve strain at these points. The program then presents the new user-modified configuration with its presumably new high-strain points. The process is repeated until the user feels that significant improvement is no longer possible.

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Then files describing the optimized block of cables can be output, formatted for input into Greg Lee's AutoEnd program[2] interfacing with the rest of the Fermilab magnet design and fabrication system which includes standard stress analysis programs[3] and our computer aided design and numerically-controlled machining systems.[4,5] We also interface with the Lawrence Berkeley Laboratory field calculation programs.[6]

In the next section a classical model of cable-like objects is modified to include the nonclassical geometric constraints imposed upon a superconducting cable by winding it around the end of a magnet. Then are outlined methods for solving the resulting equations. In the last section the mathematical model is extended from single cables to packed blocks of cables.

The Rectifying Developable Method

We base our approach on the Euler-Bernoulli theory of a thin homogeneous rod in a plane, modified by Kirchhoff to include a twist out of the plane into space (see Chapters 18 and 19 in [7]). The cable cross section, constant along its length when unstrained, is a bilaterally symmetric trapezoid. The elastic properties of the cable are completely characterized by three constants, a_1, a_2 and a_3. The first two are the flexural rigidities of the cable about axes in the plane of the trapezoid, through the midpoint of its base and perpendicular and parallel respectively to the base. The third is the torsional rigidity about an axis through the midpoint and perpendicular to the plane of the trapezoid.

These midpoints form a curve in space specified by the vector function $\vec{R}(s)$ of arc length s. To complete the geometry of the cable one more function of s would suffice, an angle of rotation of the trapezoid about the tangent to $\vec{R}$. Instead it is convenient to attach to each trapezoid a right-handed orthonormal frame F (see page 32 of [8]) of vectors

- $\vec{F}_1 = d\vec{R}/ds$.
- $\vec{F}_2$ in the plane of the trapezoid and perpendicular to its line of symmetry.
- $\vec{F}_3 = \vec{F}_1 \times \vec{F}_2$.

Now we have embedded in the cable a coordinate frame with respect to which its properties are expressible in a natural way. The curvatures, κ_1, κ_2, and torsion τ, functions of s corresponding respectively to the rigidities a_1, a_2, and a_3, are defined with respect to this intrinsic coordinate system by

- $\vec{F}_2 \cdot d\vec{F}_1 = \kappa_1 ds$,
- $\vec{F}_1 \cdot d\vec{F}_3 = \kappa_2 ds$,
- $\vec{F}_3 \cdot d\vec{F}_2 = \tau ds$.

Given the curvatures and torsion we can reverse the definition and obtain the configuration of the cable by solving the system of ordinary differential equations

$$dF = \Omega F \tag{1}$$

where $\Omega = \{\omega_{ij}\}_{i,j=1,2,3}$ is a skew-symmetric matrix of one-forms (page 33 of [8]) defined by

$$\omega_{ij} = \vec{F}_i \cdot d\vec{F}_j.$$

The language of differential forms simplifies the formalism and its future sophistication to include more complicated and realistic stress-strain tensors. Flanders' textbook[8] is an excellent introduction to differential forms and is available as an inexpensive Dover reprint. In this paper we will extend the domain of definition of F and Ω from the one-dimensional manifold $\vec{R}$ to the three-dimensional manifold occupied by a block of cables.

The meaning of the rigidities is contained in the expression

$$\frac{1}{2}\left(a_1\kappa_1(s)^2 + a_2\kappa_2(s)^2 + a_3\tau(s)^2\right) \tag{2}$$

for the strain energy density. $\vec{R}$ is the curve which, subject to given constraints, minimizes the total strain energy, the integral of (2) with respect to s over the length of the cable. One of the constraints is that it lie on a given smooth surface, S. (At present the software can handle only cylinders but we generalize here because of a remark by Shlomo Caspi of Lawrence Berkeley Laboratory that there is a real need for other surfaces.) Constraints are handled in the rectifying developable method not by first modeling a cable in space and then constraining it to S, but rather by basing the construction on variables inside the intrinsic geometry of S from the beginning. There they are unconstrained. This mathematical convenience becomes more important in the last section where we model a block of cables. In an unconstrained block each cable would slide freely against its neighbors. The block would have too many degrees of freedom whereas when constrained by S and the packing condition, the entire block has the same number of degrees of freedom as a single cable.

The original Euler-Bernoulli model thin rod model is easy to modify for a nonplanar S. Then the two flexural rigidities, a_1 and a_2, are equal and expression (2) becomes $a_1\kappa(s)^2/2$ where κ is the curvature (page 92 of [9]) of $\vec{R}$. If we add Kirchhoff's term it becomes $(a_1\kappa(s)^2 + a_3\tau(s)^2)/2$ where τ is the torsion of $\vec{R}$.

But the height of the cross sectional trapezoid of the cable, measured parallel to its line of symmetry, is much larger than its midthickness measured perpendicular to that line. So the strain κ_2 imposes a much greater longitudinal deformation on curves passing through the endpoints of the line segment of symmetry and parallel to $\vec{R}$ than does κ_1 impose on curves along the two lateral faces of the cable. a_2 is much larger than a_1. The so-called "constant perimeter" condition[10] requires that the integral of $\kappa_2(s)$ along the length of the cable be zero so that "perimeters", curves parallel to $\vec{R}$, maintain constant total lengths. The rectifying developable method carries this condition to an extreme. κ_2 is required to be identically equal to zero. The cable is modeled at this stage of the method by a long thin developable rectangle, a two-dimensional strip as if its trapezoidal cross section had zero thickness.

A surface is developable if and only if it can be flattened out into a plane without any stretching or tearing. (See the bottom of page 303 in [9].) Not only are the "perimeters" of the strip constant, so is the length of every curve in it. We constrain the cable to S by requiring $\vec{R}$ to be in S. Now we use a nice result from differential geometry: Any smooth $\vec{R}$ in space, with nonzero curvature, is contained in one and only one developable surface, its rectifying developable (see page 47 of [11]). The strip is uniquely determined by the condition that it be in the rectifying developable and that one of its sides lie along $\vec{R}$. Now the frame F is the Frenet frame of $\vec{R}$ (see page 93 of [9]). $\kappa = \kappa_1$, $\kappa_2 = 0$, and the τ of the cable equals the τ of the curve. *To find the configuration of the constrained cable in space we need only find the configuration of an unconstrained Euler-Bernoulli-Kirchhoff rod in the surface.*

To express κ and τ in terms of variables intrinsic to the surface we use still another moving frame, a special surface frame (see §130 in [9]) that can be attached to any smooth curve in S. It is obtained by rotating the Frenet frame about its first column vector, tangent to $\vec{R}$, until its second is perpendicular to S. As in [1], κ and τ can then be expressed in terms of the

intrinsic variables $\vec{R}$, $\vec{F}_1$, γ and γ' in the strain energy density expression (2), where γ is the geodesic curvature (page 284 in [9]) of $\vec{R}$.

The rectifying developable is not defined at singularities of $\vec{R}$ where κ does not exist or where it is equal to zero as along a straight section of the coil. If $\vec{R}$ is analytic the rectifying developable can be uniquely defined by continuity across the singularity. Otherwise it may be necessary to twist the strip away from developability in order to make a smooth transition. The degree of smoothness is measured by the smallness of the strain energy added by the twist. Because the rectifying developable is developable, it is a ruled surface (page 303 in [9]). It is swept out by a family of straight line segments, its rulings, parameterized by s where $\vec{R}(s)$ is the point at which the ruling intersects $\vec{R}$. Let α be the angle of intersection of the ruling with $\vec{R}$. The twist from the rectifying developable to this new strip, which we call the guiding strip, is for each s a rotation of amount $\varphi(s)$ about $\vec{F}_1(s)$ of the ruling through $\vec{R}(s)$, so $\alpha(s)$ is held constant. The guiding strip is still a ruled surface even though it is no longer developable.

The guiding strip is now a model for the cable so we define F by rotating the Frenet frame also by an amount φ about the tangent to $\vec{R}$. Then $\vec{F}_2$ is perpendicular to the guiding strip and $\vec{F}_3$ is tangent to it. The total strain energy to be minimized is now

$$E(\gamma,\varphi) = \frac{1}{2}\int_0^{s_{final}} \left((a_1\cos^2\varphi + a_2\sin^2\varphi)\kappa^2 + a_3\tau^2\right) ds. \quad (3)$$

It is a functional of γ (including its first derivative and, to get $\vec{F}_1$ and $\vec{R}$ respectively, its first and second indefinite integrals, see §3 in [1]) and φ. It also depends on four endpoint conditions for γ and two for φ. τ is now the torsion of the guiding strip, not of $\vec{R}$.

Strain Minimization Algorithms

To minimize $E(\gamma,\varphi)$ we iterate between two suboptimizations, alternatively minimizing with respect to γ with φ fixed and φ with γ fixed. This is the Alternating Variable Optimization Method (see page 16 of [12]) except that we alternate between two orthogonal subspaces instead of all of the orthogonal coordinate axes. The proof of convergence is the same.

We have found that good numerical accuracy can be obtained with a number of points along $\vec{R}$ that is small enough to put the minimization problem well within the range of modern, general purpose, numerical optimization programs; especially so because our independent variables were chosen to put the problem in the domain of unconstrained optimization[12] where numerical methods, in particular conjugate gradient[13] and quasi-Newton[13] methods, work well with smooth objective functions like ours for which we can find a very good initial guess.

When numerical methods are applied in this way they are called "direct" methods in the calculus of variations (see Chapter 8 in [14]). Available general purpose programs are straightforward and powerful and would undoubtedly be able to perform each of the two suboptimizations or even optimize γ and φ simultaneously. However they ignore all of the special structure in our model and are not capable of giving the user intimate access to it interactively during progress of the optimization, so we use a special purpose direct method. Though less automatic and more complicated it is appropriate at a time when we are still investigating various mathematical structures to be optimized.

Applied mathematics is an art of approximation and we are still investigating the approximations to be made. We optimize φ only in a linear approximation. In a full perturbation expansion of the nonlinear problem, derivatives with respect to s of order higher than two are neglected. The solution is then just a cubic spline of a chosen monotonic function of s.

The suboptimization of γ can be achieved by solving the Euler-Lagrange equation with φ fixed. If φ is small enough to be neglected, the Lagrangian is to that extent independent of s. Then by Noether's theorem (§20 in [14]) the order of the differential equation can immediately be reduced by one. If the Lagrangian is independent of $\vec{x}$ then two more first integrals can be found. For example the surface of a right-circular cylinder has a two-dimensional family of symmetries so in reference [1] the equation is reduced to a single third order ordinary differential equation. (The first Noether reduction for our particular problem is explicitly given in the second part of problem 13 on page 52 of [14].)

Partly for historical reasons in the development of the project, and partly because the equation may be singular at an endpoint, in [1] we solve it by a special method of successive approximations, a direct method which simulates a physically reasonable relaxation of the cable into its equilibrium configuration.

Blocks of Cables

The cables in a block are packed together with slightly changing cross sections as they twist around an end. Program BEND is supposed to predict the shape of the block so that when pressure is applied during the curing process the cables

will fit together exactly with neither gaps between cables nor bulges outside of the prescribed volume.

Placement of the packed cables is determined by an orthonormal frame F attached to each point $\vec{x}$ in the block. $\vec{F}_1$ is tangent to the curve passing through it which is parallel to the axis of the cable containing $\vec{x}$. When $\vec{x}$ is on the lateral surface of a cable (not one of the parallel sides of a trapezoidal cross section), then $\vec{F}_2(\vec{x})$ is perpendicular to that surface. In the unstrained block these lateral surfaces are all planar. We include them in a family of disjoint planes whose union is the entire block. The family then constitutes a foliation of the unstrained block which is carried into a smooth foliation of the strained block by two-dimensional surfaces that are no longer planar but still give a continuous interpolation between the lateral surfaces of each cable. $\vec{F}_2$ is defined throughout the block by the requirement that it be perpendicular to the foliation.

At the end of the second section above, F was defined on $\vec{R}$ by twisting the Frenet frame attached to the rectifying developable until it was attached to the guiding strip. That F will first be extended from $\vec{R}$ to the rest of the S, and then from S to the entire block. Let $\eta(\vec{x}_S)$ be the angle between $\vec{F}_3(\vec{x}_S)$ and the normal to S at the point $\vec{x}_S$ in S. $\vec{F}_3$ is tangent to the foliation, so η satisfies an eikonal equation

$$|\vec{\nabla}_S\eta| = \frac{\Theta}{\Delta}\cos(\eta) + \frac{1}{\rho} \tag{4}$$

on S where $\vec{\nabla}_S$ is the gradient on S, and ρ is the signed radius of normal curvature in the direction $\vec{\nabla}_S$ through $\vec{x}_S$. $\Theta(\vec{x}_S)$ is the keystone angle of the cable (the angle between the extended nonparallel sides of the trapezoid) and $\Delta(\vec{x}_S)$ is its thickness at the edge in contact with S. The first term on the right-hand side of the equation is the rate of change of η caused by the keystoning. Its integral, ζ, with respect to arc length along a characteristic of the equation is proportional to the number of cables traversed. The second term is the rate of change of η caused by the curvature of S. η is determined by (4) and its known boundary condition on $\vec{R}$. $\vec{F}_1$ must be tangent to S and perpendicular to $\vec{\nabla}_S\eta$, so $\vec{F}_3$ and hence F is determined over all of S by η.

In BEND, equation (4) is solved by a forward difference scheme in which each cable corresponds to a single step. The functions Θ and Δ are determined by user input. The program then automatically takes into account the variation of the cross sections of the cables along their lengths and their subsequent shifting along intercable surfaces so as to maintain contact of their outer edges with the constraining surface.

Let $\vec{x}_S(s,\zeta)$ in S be on the characteristic of equation (4) which intersects $\vec{R}$ orthogonally at s. Let

$$\vec{\beta} = \cos(\alpha(s))\vec{F}_1(\vec{x}_S) + \sin(\alpha(s))\vec{F}_3(\vec{x}_S).$$

To extend the definition of F from any point on S to the point $\vec{x} = \vec{x}_S + s_1\vec{\beta}$ in the block, require $\vec{F}_1(\vec{x})$ to be parallel to $\partial\vec{x}/\partial s$ at $\vec{x}$, and $\vec{F}_3(\vec{x})$ at $\vec{x}$ to be in the plane spanned by $\vec{F}_1(\vec{x})$ and β. The normal curvatures of S are assumed small enough with respect to the size of the block that F is now well-defined throughout the block.

As a matrix of column vectors, F is the 3×3 submatrix in the upper left-hand corner of the 4×4 T-matrix of the robotics theorists (see Chapter 2 in reference [15]). These T-matrices are output from BEND for automation of the winding process. Planning for this project is being directed at Fermilab by Eric Haggard. A T-matrix is attached to each end of the straight section of cable between the spool and the block. The T-matrix at the block contains the frame F attached to the midpoint of the line segment in the ruling of first contact with the outside surface of the underlying cable already wound. The other T-matrix, a rectilinear translate of the first along the straight section of the cable, is attached to the midline of the corresponding surface of the cable at the point where it leaves the spool. Between these two frames are interpolated other frames attached to the successive links in the kinematic chain constituting the arm of a robot manipulator. Bob Bonaguro has solved the inverse kinematic problem (see Chapter 3 in [15]) to obtain these interpolating frames for one of the manipulators that was proposed at Fermilab. He has coded his solution into a program which accepts output from BEND and in turn outputs a file for input into the program controlling the robot.

Traditionally stress-strain relations in rods and beams have been derived from a model made up of long fibers parallel to their axis, elongated or shortened by flexing and twisting of the rod. Thus the elastic properties of a three-dimensional rod could be derived theoretically from postulated elastic properties of idealized one-dimensional fibers. But the superconducting fibers in our cable are by specific design not parallel to its axis, and the insulation between cables in a block introduces still other complications. Further, the packing algorithm is accurate only to within first order in the midthickness, width and keystone angle of the unstrained cable. So the varying cross section of the strained cable is at present being checked empirically. Cured ends are sliced transversely at several points along the length of the block. Then actual dimensions of the cables are compared with their predicted values to recalibrate the program in our ongoing effort to increase its accuracy. An exposition of

its use to design end parts for the 50mm SSC dipoles is being prepared for publication.[16]

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