AZIMUTHAL COIL DISPLACEMENTS AS AFFECTED BY LORENTZ-FORCE DISTRIBUTION, PRESTRESS, AND STRESS-STRAIN RELATIONSHIP

PART 3: THIN WINDING WITH CURRENT DENSITY VARYING AS cos mθ *

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Introduction

Inasmuch as all multipole magnets having a coil cross section that is at least roughly circular are approximations to a cosine mθ configuration (one in which the lineal current density varies as cos mθ) , it seems useful to calculate the displacements and associated field aberrations for an idealized thin cosine-theta (ITCT) configuration. Such a configuration has been shown to give results that are with 10 or 20% of those for real coils when applied to stored energy and inductance, total Lorentz stress, and field strength per ampere turn; it would be a suprise if it didn't also give a good approximation for at least the lower-order field-aberration multipoles resuting from azimuthal displacements.

In a 1980 report (Ref. 1), I presented calculations for what seemed at the time to be a good model of the ITCT configuration. The elastic modulus was assumed to be uniform over the circumference. But that assumption doesn't account for the "wedges". In real magnets, the current per conductor is uniform and it is the conductor density that varies as cosine mθ; the space between conductors is filled with wedges. The uniform-modulus model would be applicable only if the modulus of the wedges matched that of the conductors. If the wedges are infinitely stiff, then the effective modulus varies as 1/cos mθ.

Results for both the uniform modulus and 1/cos mθ modulus variation are presented. (It probably wouldn't be too difficult to consider the case of a finite wedge modulus; perhaps later.) The general analysis presented in Part 1 of this series considers a modulus that varies with stress, but not one that varies with azimuth, so some modification is necessary.

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* Parts 1 and 2 of this series were published as SSC-N-539 and -541, respectively.

NOMENCLATURE

(Additional nomenclature is defined in the text.)

a	coil radius
b	iron inside radius
h	coil radial thickness
ρ	reference radius for multipole coefficients
x	fractional circumferential distance from midplane toward pole
F	total Lorentz stress, positive when in $-x$ direction
$\hat{F}$	critical value of F; that for which $\sigma_1 = 0$ and $\delta_1 = 0$
$f(x)$	local Lorentz force distribution function
$g(x)$	integral of $f(x)$ between 0 and x
g	$g(1)$
$k(x)$	integral of $g(x)$ between 0 and x
k	$k(1)$
$\sigma(x)$	local stress, compression is positive
σ_1	$\sigma(1)$, σ at pole
	prestress
$\epsilon(x)$	local strain, compression is positive
$\delta(x)$	local displacement, with respect to end at $x=0$, and with respect to prestressed condition, positive toward midplane
δ_1	$\delta(1)$
$\bar{\delta}$	average value of δ over one half pole
n	number of pole pairs for field multipole coefficients
m	number of pole pairs for magnet
E	Young's modulus
μ_0	permeability of free space
A_n	real or skew field multipole coefficient in teslas
B_n	imaginary or normal field multipole coefficient in teslas
B_m	fundamental field multipole coef. for magnet in teslas
B_a	fundamental field multipole just inside coil
$a(n)$	A_n/B_m
$b(n)$	B_n/B_m
T	see text
G_n, G_m	"
K_n, K_m	"
P_m	"

Analysis

Displacements

Again we represent the local Lorentz body force per unit volume as

$$F\, f(x)$$

where, for the present case,

$$f(x) = \frac{\pi}{2} \sin \pi x \qquad (6.01)$$

$$g(x) \equiv \int_0^x f(x)\,dx = \frac{1}{2}(1 - \cos \pi x) \qquad (6.02)$$

and

$$g(1) = 1$$

as required. Also

$$k(x) \equiv \int_0^x g(x) = \frac{1}{2}\left(x - \frac{1}{\pi} \sin \pi x\right) \qquad (6.03)$$

and so

$$k_1 \equiv k(1) = \frac{1}{2} \qquad (6.04)$$

Effective modulus = E For this case the critical prestress is (Ref. 1)

$$\hat{F} = \frac{\sigma_p}{1 - k_1} = 2\sigma_p \qquad (6.05)$$

which was shown to apply to all symmetrical lorentz-force distribution functions. The local displacement is given by (Ref. 1)

$$\delta(x) = \frac{F}{E}\left[k_1 x - k(x)\right] = \frac{F}{E} \cdot \frac{1}{2\pi} \sin \pi x \qquad (6.06)$$

and the average displacement is

$$\bar{\delta} = \int_0^1 \delta(x)\,dx = \frac{F}{E} \cdot \frac{1}{\pi^2} \qquad (6.07)$$

Effective modulus = E/cos mθ Equilibrium considerations give

$$\sigma(x) = \sigma_1 + F\left[1 - g(x)\right]$$

$$= \sigma_1 + \frac{1}{2} F \left(1 + \cos \pi x\right) \qquad (6.10)$$

The change in strain from the prestress condition is, then

$$\epsilon(x) - \epsilon(\sigma_p) = \frac{\sigma(x) - \sigma_p}{E} \cos\frac{\pi}{2}x$$

$$= \frac{1}{E}\left[(\sigma_1 + \tfrac{1}{2}F - \sigma_p)\cos\frac{\pi}{2}x + \tfrac{1}{2}F\cos\pi x\cos\frac{\pi}{2}x\right] \quad (6.12)$$

and so the local displacement is

$$\delta(x) = \int_0^x \left[\epsilon(x) - \epsilon(\sigma_p)\right]dx$$

$$= \frac{1}{E}\left[(\sigma_1 + \tfrac{1}{2}F - \sigma_p)\frac{2}{\pi}\sin\frac{\pi}{2}x + \frac{F}{2\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x\right)\right] \quad (6.13)$$

At the pole end, x=1, the displacement must be zero. That condition yields

$$(\sigma_1 + \tfrac{1}{2}F - \sigma_p)\frac{2}{\pi} = -\frac{1}{3\pi}F \quad (6.14)$$

which substituted back into Eq. 6.13 gives

$$\delta(x) = \frac{F}{E}\cdot\frac{1}{6\pi}\left(\sin\frac{\pi}{2}x + \sin\frac{3\pi}{2}x\right) \quad (6.15)$$

The average displacement is

$$\bar{\delta} = \int_0^1 \delta(x)\,dx = \frac{F}{E}\cdot\frac{4}{9\pi^2} \quad (6.16)$$

For the critical condition, $F=\hat{F}$ and $\sigma_1 = 0$, which gives

$$\hat{F} = \frac{3}{2}\sigma_p \quad (6.17)$$

Field aberrations

The field in the aperture can be expressed in the form

$$B_r = \sum (r/\rho)^{n-1} (\ A_n \cos n\theta - B_n \sin n\theta) \quad (6.21)$$

$$B_\theta = \sum (r/\rho)^{n-1} (-A_n \sin n\theta - B_n \cos n\theta) \quad (6.22)$$

where r, θ are the coordinates of the point at which the field components B_r, B_θ are evaluated, ρ is a reference radius, and n is the number of pole pairs associated with the particular aberration component. A_n and B_n are the "skew" and "normal" multipole coefficients, respectively.

The multipole coefficients produced by a single filament ar r,θ carrying current I are

$$A_n + iB_n = K_n I(\sin n\theta + i\cos n\theta) \qquad (6.23)$$

where

$$K_n \equiv \frac{\mu_0}{2\pi}\cdot\frac{1}{a}\left(\frac{\rho}{a}\right)^{n-1} G_n \qquad (6.24)$$

$$G_n \equiv 1 + (a/b)^{2n} \qquad (6.25)$$

$$G_m \equiv 1 + (a/b)^{2m} \text{ (for use later)} \qquad (6.26)$$

We are concerned here with only "normal" magnets, which have a line midway between a pair of poles oriented horizontally, and only ones having folding symmetry about the pole centerlines. For such magnets

$$A_n = 0 \quad \text{for all } n$$

$$B_n = 0 \quad \text{for } n \neq m\,(1,3,5,\ldots)$$

$$B_n = K_n I \cos n\theta \quad \text{for } n = m\,(1,3,5,\ldots) \qquad (6.27)$$

A perturbation of $\delta\theta$ produces a perturbation of the multipole coefficient in amount ΔB_n.

$$\Delta B_n = -K_n n \sin n\theta \; \delta(\theta) \qquad (6.28)$$

We apply this to an element of the cosine $m\theta$ winding by replacing I by

$$J a\, d\theta = J_0 a \cos m\theta\, d\theta \qquad (6.31)$$

and integrating. Then for one half pole

$$B_n = K_n J_0 a \int_0^{\pi/2m} \cos n\theta \cos m\theta\, d\theta \qquad (6.32)$$

$$\Delta B_n = -K_n J_0 a n \int_0^{\pi/2m} \sin n\theta \cos m\theta\, \delta(\theta)\, d\theta \qquad (6.33)$$

Integration of the expression for B_n yields the result that

$$B_n = 0 \quad \text{for } n \neq m \qquad (6.34a)$$

$$B_m = K_m J_0 a \frac{\pi}{4m} \qquad (6.34b)$$

It is convenient to normalize the field aberration multipoles resulting from displacements to the fundamental multipole.

$$\Delta b_n \equiv \frac{\Delta B_n}{B_m} = -\frac{K_n}{K_m}\frac{4}{\pi}mn\int_0^{\pi/2m} \sin n\theta \cos m\theta \,\delta(\theta)\, d\theta \qquad (6.35)$$

where $\delta(\theta)$ is the displacement at angle θ. To convert $\delta(x)$ to $\delta(\theta)$ we replace x by $2m\theta/\pi$, and multiply by −1 since positive $\delta(\theta)$ is toward the pole whereas positive $\delta(x)$ is toward the midplane.

<u>For (effective E)=E</u> The displacement $\delta(\theta)$ is

$$\delta(\theta) = -\frac{F}{E}\frac{1}{2\pi} \sin 2m\theta \qquad (6.36)$$

which, substituted into Eq. 6.35 yields

$$\Delta b_n = \frac{F}{E}\frac{K_n}{K_m}\frac{2m}{\pi^2 n}\int_0^{\pi/2m} \sin n\theta \cos m\theta \sin 2m\theta \, d\theta$$

Upon integration this becomes

$$\Delta b_n = \begin{cases} \dfrac{F}{E}\dfrac{K_n}{K_m} m \dfrac{n/m}{4\pi} & \text{for } n/m = 1,3 \\ 0 & \text{for } n/m = 5,7,9,\ldots \end{cases} \qquad (6.37)$$

<u>For (effective E)=E/cos mθ</u> The displacement $\delta(\theta)$ is

$$\delta(\theta) = -\frac{F}{E}\frac{1}{6\pi}(\sin m\theta + \sin 3m\theta) \qquad (6.38)$$

which, substituted into Eq. 6.35 yields

$$\Delta b_n = \frac{F}{E}\frac{K_n}{K_m}\frac{2mn}{3\pi^2}\int_0^{\pi/2m} \sin n\theta \cos m\theta (\sin m\theta + \sin 3m\theta)\, d\theta$$

Clever rascal that I am, I obtain the solution

$$\Delta b_n = \frac{F}{E}\frac{K_n}{K_m} m \frac{\frac{n}{m}}{3\pi^2}\left[\frac{\sin(\frac{n}{m}-2)\frac{\pi}{2}}{\frac{n}{m}-2} - \frac{\sin(\frac{n}{m}+2)\frac{\pi}{2}}{\frac{n}{m}+2} + \frac{\sin(\frac{n}{m}-4)\frac{\pi}{2}}{2(\frac{n}{m}-4)} - \frac{\sin(\frac{n}{m}+4)\frac{\pi}{2}}{2(\frac{n}{m}+4)}\right]$$

which looks more like a problem than a solution. However the sine terms have only values +1 or −1, alternating with n/m in strange ways, and so this boils down to

$$\Delta b_n = \frac{F}{E}\cdot\frac{K_n}{K_m}\cdot m\cdot\frac{n/m}{\pi^2}\,\frac{16}{(n/m)^4 - 20(n/m)^2 + 64}\,(-1)^{(\frac{n}{m}+1)/2} \qquad (6.39)$$

the (−1)... is simply a shenanigan to make the signs come out right.

It is useful to express the total Lorentz stress in terms of the fundamental aperture field. In Reference 4 the following formula for the total azimuthal force in one half pole is developed.

$$Fh = 2\,p_m\,\frac{a}{m}\,\frac{1}{G_m} \qquad (6.40)$$

where

$$p_m = \frac{B(a)^2}{2\mu_0} \qquad (6.41)$$

B(a) is the fundamental field just inside the winding and is given by

$$B(a) = B_m(a/\rho)^{m-1} \qquad (6.42)$$

p_m is the magnetic pressure, or energy per unit volume, corresponding to B(a).

The leading terms in Eq. 6.34 then become

$$\frac{F}{E}\cdot\frac{K_n}{K_m}m = 2\left(\frac{B_m^2}{2\mu_0}\right)\frac{a}{hE}\,\frac{G_n}{G_m^2}\left(\frac{\rho}{a}\right)^{n-3m+2} \qquad (6.43)$$

Numerical results and discussion

Figure 1 shows the displacement as a function of position in the coil for the two cases. A major part of the factor-of-2 ratio of maximum ordinates is a result of the overall average stiffness ratio for the two cases, a factor of $\pi/2$. The tail of the lower curve at the upper end is horizontal because the stiffness goes to infinity at the pole end x=1.

Table 1 presents the field multipole functions for the two cases, calculated from Eq. 6.37 and 6.39. Note that for the uniform-effective-modulus case there are aberrations only for the fundamental and the first higher-order multipole. For the case where the effective modulus is E/cos mθ, there are multipoles of all order, although the ones beyond the fundamental and the first higher-order one are very small. The factor-of-2 ratio of first higher-order multipoles is, again, largely a result of the greater average stiffness for the E/cos mθ case.

Table 2 presents the multipoles for coil parameters that approximate those of the SSC dipoles.

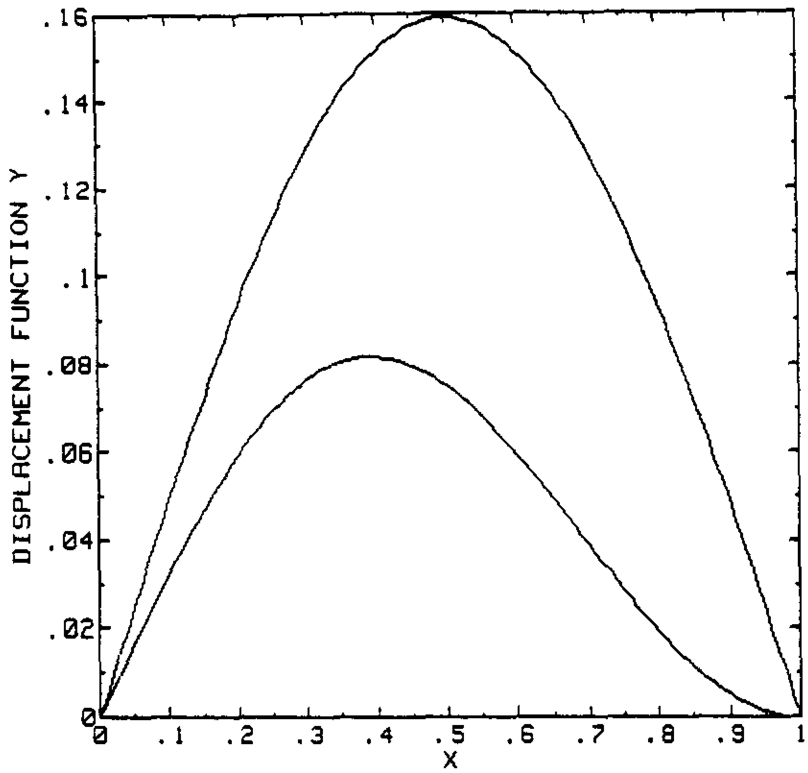

Fig. 1 Displacement functions for thin idealized cosine-mθ winding. x=0 at the midplane, x=1 at the pole. The displacement is YF/E in units of x. The upper curve is for a uniform effective elastic modulus; the lower is for one that varies as 1/cos mθ.

TABLE 1

The factor P in the table is used in the following formula to calculate field aberration multipole coefficients for a magnet of any multipole order:

$$\Delta b_n = \frac{F}{E}\frac{K_n}{K_m}\, m\, P$$

n/m	EFFECTIVE MODULUS			
	UNIFORM		E/cos θ	
1	+.079577	+7.958E-02	-.036025	-3.603E-02
3	+.238732	+2.387E-01	-.138955	-1.390E-01
5	+0.000000	+0.000E+00	-.042887	-4.289E-02
7	+0.000000	+0.000E+00	+.007642	+7.642E-03
9	+0.000000	+0.000E+00	-.002915	-2.915E-03
11	+0.000000	+0.000E+00	+.001452	+1.452E-03
13	+0.000000	+0.000E+00	-.000835	-8.348E-04
15	+0.000000	+0.000E+00	+.000526	+5.265E-04

TABLE 2

Multipole coefficients for a coil approximating the SSC dipole.

COIL MEAN RADIUS, a (mm)........ 30
YOKE INSIDE RADIUS, b (mm)...... 55.05
REFERENCE RADIUS, (mm)........ 10
COIL THICKNESS, h (mm).......... 20
YOUNG'S MODULUS, E (psi)........ 1500000
NO. OF POLE PAIRS, m............ 1
FIELD MAGNITUDE
AT REFERENCE RADIUS, Bm (T)... 6.6

FIELD ABERRATIONS in units of 1E-4 x DIPOLE component.

MULTIPOLE COMPONENT	EFFECTIVE MODULUS UNIFORM		E/cos θ	
DIPOLE	+3.084683	+3.085E+00	+1.396459	+1.396E+00
SEXTUPOLE	+.813551	+8.136E-01	+.473529	+4.735E-01
DECAPOLE	+0.000000	+0.000E+00	+.015861	+1.586E-02
14-POLE	+0.000000	+0.000E+00	-.000313	-3.134E-04
18-POLE	+0.000000	+0.000E+00	+.000013	+1.328E-05
22-POLE	+0.000000	+0.000E+00	-.000001	-7.347E-07
26-POLE	+0.000000	+0.000E+00	+.000000	+4.695E-08
30-POLE	+0.000000	+0.000E+00	-.000000	-3.290E-09

Conclusions

The multipoles calculated for the SSC dipoles are small, but not negligible. It will be interesting to see if the multipoles calculated for a real magnet are in agreement with these results, as I think they will be. And, I remind you that the results presented here are applicable to multipole magnets of all orders, not merely dipoles.

References

1. Part 1 of this series, SSC-N-539
2. Part 2 of this series, SSC-N-541
3. Circumferential Displacements if a Thin Cosine-Theta Dipole Coils, R. B. Meuser, LBL Eng. Note M5584, Sep. 1980
4. Forces in a Thin Multipole Magnet Winding, R. B. Meuser, LBL Eng. Note M5597, Oct. 1980