AZIMUTHAL COIL DISPLACEMENTS AS AFFECTED BY LORENTZ-FORCE DISTRIBUTION, PRESTRESS, AND STRESS-STRAIN RELATIONSHIP

PART 1: GENERAL ANALYSIS, AND APPLICATION TO MATERIALS HAVING A LINEAR STRESS-STRAIN BEHAVIOR

Robert B. Meuser, Consultant
August 22, 1988

INTRODUCTION

In order to obtain a better understanding of how various parameters affect coil stresses, displacements, and field aberrations a simple, one-dimensional analysis was undertaken. In this study, it is assumed that there are no shear forces between the elements of the coil or between the coil and its surroundings. A variety of Lorentz body force distribution functions and stress-strain relationships -- linear, non-linear, non-equation-of-state -- are to be investigated.

In this first part of the series, an analytical approach is used. This approach is limited to rather simple, idealized systems, but ones that are easy to understand and which lend themselves to exploration of the effects of variations in the system parameters. Later, numerical methods will be applied to the analysis of more complex and more realistic systems. (The results of the analytical approach also provide a basis for checking the numerical approach.)

Much of this first installment is based on work done between 1974 and 1980, most of which was reported in LBL Engineering Notes. It has been cleaned up and generalized a bit. Presented here are:

Analytical study of displacements for a general stress-strain relationship, not necessarily a linear one or one that can be described by an equation of state,

Specialization of the above to a linear stress-strain relationship,

Examples of both, with brief discussion of the results.

ANALYTICAL MODEL

A simple one-dimensional model of the coil is employed. The model is applicable to the circular cross-section coils of SSC magnets, cylindrical layer elements of such coils provided there are no shear forces on the interfaces, coils of rectangular cross section such as the TAC superferric magnets, and perhaps others. The transformation from a quadrant of a dipole magnet coil layer to the model is as indicated in Fig. 1.

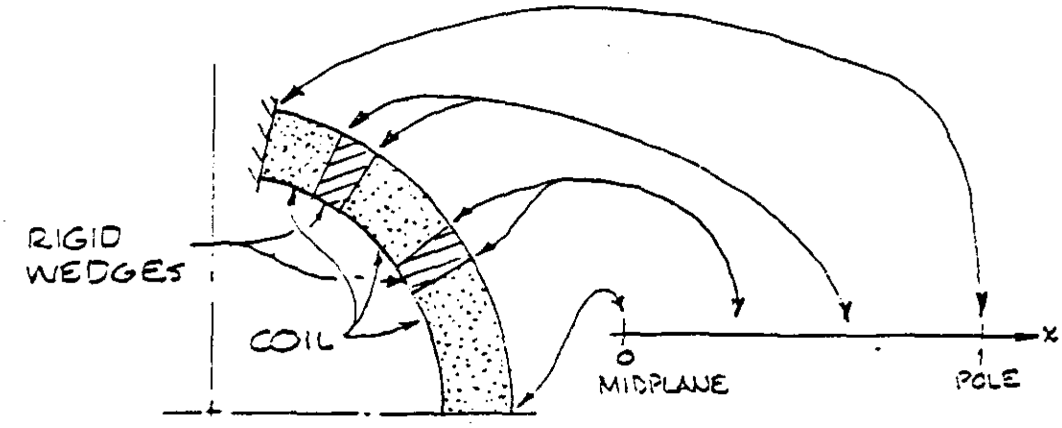


Figure 1. Correspondence between position on coil and x coordinate.

Unit length is defined as the distance between fixed ends, and x as the distance from the end toward which the Lorentz forces are directed. For a circular multipole magnet, x=0 at the midplane, and x=1 at the pole piece. The wedges are assumed to be much stiffer than the conductors, and so are removed from consideration. Positive stresses and strains are compressive; positive displacements and Lorentz forces are in the -x direction.

NOMENCLATURE

x dimensionless distance from end toward which Lorentz forces are directed

F	total Lorentz stress, positive when in -x direction
$f(x)$	local Lorentz force distribution function
$g(x)$	integral of $f(x)$ between 0 and x
$k(x)$	integral of $g(x)$ between 0 and x
$\sigma(x)$	stress, compression is positive
σ_p	prestress, compression is positive
$\epsilon(x)$	strain, compression is positive
$\delta(x)$	displacement with respect to end at x=0 from prestressed condition. Note: displacements are positive when in -x direction
E	Young's modulus (applicable only for linear stress-vs-strain)

Subscripts: Subscripts 0 and 1 indicate "at x=0" and "at x=1" respectively.

Hats:

´	subcritical condition, no displacement at x=1
^	critical condition, no displacement and no stress at x=1
`	supercritical condition, no stress at x=1

GENERAL STRESS-STRAIN RELATIONSHIP

General Loading Condition

The local (-x)-directed component of the Lorentz body force per unit volume is represented as

$$F\,f(x) \tag{1.01}$$

where F is a constant. We define

$$q(x) = \int_0^x f(x)\,dx \tag{1.02}$$

and scale things so that

$$g(1)=1 \tag{1.03}$$

Then F is the total body force per unit area, which we refer to as the "total Lorentz stress".

Equilibrium considerations (Fig. 2) yield relations between the local stress $\sigma(x)$; the stress at the midplane end x=0, σ_0 ; the stress at the pole end x=1, σ_1 ; the total Lorentz stress F; and the distribution function g(x):

$$\sigma(x) = \sigma_0 - Fq(x) = \sigma_1 + F[1 - q(x)] \tag{1.04}$$

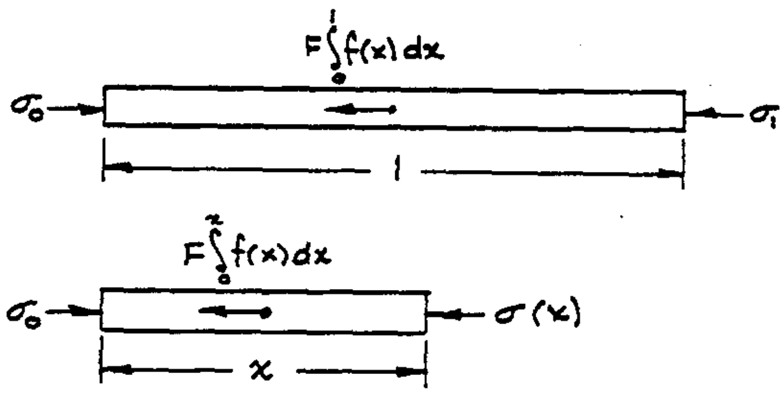


Figure 2. Equilibrium of stresses.

Inasmuch as non-linear and non-equation-of-state material properties are permitted, we express the strain as a general function of stress,

$$\epsilon(x) = \epsilon[\sigma(x)] \tag{1.05}$$

where it is to be understood that path dependence is included if applicable.

The local displacement with respect to the fixed end at x=0, relative to the prestressed condition is

$$\delta(x) = \int_0^x \left[\epsilon(x) - \epsilon(\sigma_p)\right] dx = \int_0^x \epsilon(x)dx - \epsilon(\sigma_p)x \qquad (1.06)$$

<u>Subcritical Condition</u> (We use "´" to designate this condition.)

For this condition

$$F = \acute{F} \leq \hat{F} \qquad (1.11)$$

$$\delta_1 = \acute{\delta}_1 = 0 \qquad (1.12)$$

$$(\sigma_1 = \acute{\sigma}_1 \geq 0)$$

Upon application of the Lorentz forces F the stress becomes that given by Eq. 1.04:

$$\acute{\sigma}(x) = \acute{\sigma}_0 - \acute{F}q(x) = \acute{\sigma}_1 + \acute{F}\left[1 - q(x)\right] \qquad (1.13)$$

so the corresponding strain is

$$\acute{\epsilon}(x) = \epsilon\left[\acute{\sigma}(x)\right] \qquad (1.14)$$

According to Eq. 1.06 the local displacement is

$$\acute{\delta}(x) = \int_0^x \acute{\epsilon}(x)dx - \epsilon(\sigma_p)x \qquad (1.15)$$

Applying the condition of Eq. 1.12 to Eq 1.15 we obtain

$$\int_0^1 \acute{\epsilon}(x)dx = \epsilon(\sigma_p)x \qquad (1.16)$$

This, evaluated from Eq. 1.14 and 1.13, gives the state of stress, from which the local displacement is evaluated using Eq. (1.15).

<u>Critical Condition</u> (We use the hat "^" to designate this condition.)

We define this condition as that for incipient net displacement at the pole end, that is,

$$F = \hat{F} \qquad (1.21)$$

$$\delta_1 = \hat{\delta}_1 = 0 \qquad (1.22)$$

$$\sigma_1 = \hat{\sigma}_1 = 0 \qquad (1.23)$$

Application of the condition of Eq. 1.23 to Eq. 1.04 yields

$$\hat{\sigma}(x) = \hat{F}\left[1 - q(x)\right] \qquad (1.24)$$

for which the strain is

$$\hat{\epsilon}(x) = \epsilon\left[\hat{\sigma}(x)\right] \tag{1.25}$$

From Eq. 1.13 we obtain the local displacement

$$\hat{\delta}(x) = \int_0^x \hat{\epsilon}(x)dx - \epsilon(\sigma_p)x \tag{1.26}$$

Applying the condition of Eq. 1.22 to 1.26 gives

$$\int_0^1 \epsilon(x)dx = \epsilon(\sigma_p) \tag{1.27}$$

This, evaluated using Eq. 1.25 and 1.24, gives the critical total Lorentz stress, from which the local displacement is evaluated using Eq. 1.26.

Supercritical Condition (We use " ` " to designate this condition.)

This condition is defined by

$$F = \grave{F} \geq \hat{F} \tag{1.31}$$

$$(\grave{\delta}_1 \geq 0)$$

$$\grave{\sigma}_1 = 0 \tag{1.32}$$

Applying the condition of Eq. 1.32 to 1.04 yields

$$\grave{\sigma}(x) = \hat{F}\left[1 - q(x)\right] \tag{1.33}$$

for which the strain is

$$\grave{\epsilon}(x) = \epsilon\left[\grave{\sigma}(x)\right] \tag{1.34}$$

The local displacement is, then, From Eq. 1.06

$$\grave{\delta}(x) = \int_0^x \left[\grave{\epsilon}(x) - \epsilon(\sigma_p)\right]dx = \int_0^x \grave{\epsilon}(x)dx - \epsilon(\sigma_p)x \tag{1.35}$$

LINEAR STRESS-STRAIN RELATIONSHIP

General

For a linear stress-strain relationship, Eq. 1.05 becomes

$$\epsilon(x) = \frac{1}{E}\sigma(x)$$

which, for the stresses given by Eq. 1.04, becomes

$$\epsilon(x) = \frac{1}{E}\left\{\sigma_1 + F\left[1 - q(x)\right]\right\} \tag{2.01}$$

and also

$$\epsilon(\sigma_p) = \frac{1}{E}\sigma_p \qquad (2.02)$$

Then

$$\int_0^x \epsilon(x)\,dx = \frac{1}{E}\left\{\sigma_i x + F\left[x - k(x)\right]\right\} \qquad (2.03)$$

where

$$k(x) \equiv \int_0^x q(x)\,dx \qquad (2.04)$$

Application of Eq. 2.02 and 2.03 to Eq. 1.06 yields

$$\delta(x) = \frac{1}{E}\left\{-(\sigma_p - \sigma_i)x + F\left[x - k(x)\right]\right\} \qquad (2.05)$$

and so

$$\delta_1 = \frac{1}{E}\left[-(\sigma_p - \sigma_i) + F(1 - k_1)\right] \qquad (2.06)$$

Critical Condition

For this condition

$$F = \hat{F} \qquad (2.11)$$

$$\sigma_i = \hat{\sigma}_i = 0 \qquad (2.12)$$

$$\delta_1 = \hat{\delta}_1 = 0 \qquad (2.13)$$

Application to Eq. 2.06 gives the critical total Lorentz stress

$$\hat{F} = \frac{\sigma_p}{1 - k_1} \qquad (2.14)$$

and Eq. 2.05 then gives the local displacement

$$\hat{\delta}(x) = \frac{\hat{F}}{E}\left[k_1 x - k(x)\right] \qquad (2.15)$$

Subcritical Condition

$$F = \acute{F} \le \hat{F} \tag{2.21}$$

$$\delta_1 = \acute{\delta}_1 = 0 \tag{2.22}$$

Application to Eq. 2.06 gives

$$\sigma_p - \sigma_1 = \acute{F}(1 - k_1) \tag{2.23}$$

and so, from Eq. 2.05

$$\acute{\delta}(x) = \frac{\acute{F}}{E}\left[k_1 x - k(x)\right] \tag{2.24}$$

Comparison with Eq. 2.15 gives

$$\acute{\delta}(x) = \frac{\acute{F}}{\hat{F}}\,\hat{\delta}(x) \tag{2.25}$$

Note that the shape of the displacement function for the subcritical condition is identical to that for the critical condition, and that it is scaled linearly with the total Lorentz stress F.

Supercritical Condition

$$F = \grave{F} \ge \hat{F} \tag{2.31}$$

$$\sigma_1 = \grave{\sigma}_1 = 0 \tag{2.32}$$

Application to Eq. 2.05 gives

$$\grave{\delta}(x) = \frac{1}{E}\left\{-\hat{F}\,(1 - k_1)x + \grave{F}\left[x - k(x)\right]\right\} \tag{2.33}$$

A little manipulation and comparison with Eq. 2.15 gives

$$\grave{\delta}(x) = \frac{1}{E}\left[(\grave{F} - \hat{F})(1 - k_1)x + \frac{\grave{F}}{\hat{F}}\,\hat{\delta}(x)\right] \tag{2.34}$$

Note that the displacement function for the supercritical condition consists of two additive parts: The first is proportional to the difference between the total Lorentz stress F and its critical value F, and proportional also to the distance from the end at x=0. The second is that for the critical condition scaled according to F/F, which is the same as that for the subcritical condition.

SOME EXAMPLES

Linear Stress-Strain Relationship

For a variety of Lorentz-force distribution functions, and for the critical condition (incipient separation at x=1), various parameters giving local stress and displacement and average displacement -- a crude indicator of field aberrations -- are presented in the following tables. The derivations are fairly straight-forward, and are not presented here. Extension from the critical to the subcritical and supercritical conditions is easily performed by using Eq. 2.25 and 2.34.

It is easily shown that for any Lorentz-force distribution that is symmetrical about x = 1/2, the critical prestress is just one-half the critical total Lorentz stress. Since the stress -- and hence the strain -- integrates the Lorentz force, and the displacement integrates the strain, it is not surprising, then, that the critical prestress and the displacement distribution are only weak functions of the Lorentz force distribution. The average displacement, involving a still further integration, is an even weaker function.

For any Lorentz-force distribution and for the subcritical condition, the stress at x=1 falls off linearly with the total Lorentz stress, and local displacements are independent of the magnitude of the prestress.

TABLE 1. Lorentz-Force Function: General Polynomial.

Lorentz body force function: $f(x) = \sum_{i=0}^{n} A_i x^i$

where $\sum_{i=0}^{n} \frac{1}{i+1} A_i = 1$

Critical Lorentz-force function: $\sigma_p / \hat{F} = 1 - k_1 = 1 - \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} A_i$

Local-displacement function: $\hat{\delta}(x) = \frac{\hat{F}}{E} \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} A_i (x - x^{i+2})$

Average-displacement function: $\hat{\delta}_{Ave.} = \frac{\hat{F}}{E} \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} A_i \left(\frac{1}{2} - \frac{1}{i+3}\right)$

TABLE 2. Lorentz-Force Function: Linear and Quadratic.

Lorentz body force function: $f(x) = A_0 + A_1 x + A_2 x^2$

where $A_0 + \frac{1}{2}A_1 + \frac{1}{3}A_2 = 1$

Critical Lorentz-force function: $\hat{F}/\sigma_p = 1 - k_1$

Local-displacement function:, $\hat{\delta}(x) = \frac{\hat{F}}{E}\left(B_1 x + B_2 x^2 + B_3 x^3\right)$

Average-displacement function: $\hat{\delta}_{AVG} = \frac{\hat{F}}{E} K$

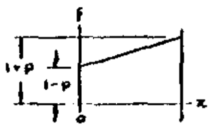
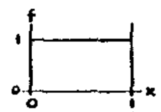
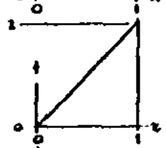
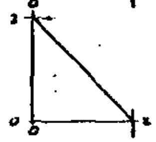
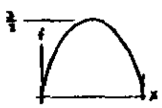
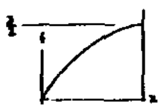
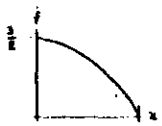
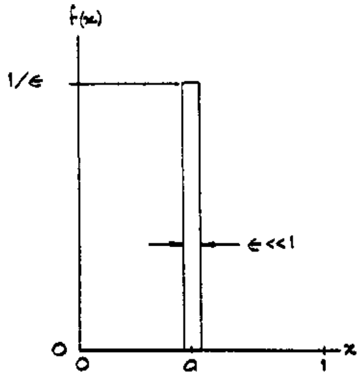
	p	A_0	A_1	A_2	B_1	B_2	B_3	$1-k_1$	K
		$1-p$	$2p$	0	$\frac{1}{6}(3+p)$	$-\frac{1}{2}(1-p)$	$-\frac{1}{3}p$	$\frac{1}{6}(3-p)$	$\frac{1}{12} = .0833$
	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{12} = .0833$
	1	0	2	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{12} = .0833$
	-1	2	-2	0	$\frac{2}{3}$	-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12} = .0833$
		0	6	-6	$\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{10} = .1000$
		0	3	$-\frac{3}{2}$	$\frac{3}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{7}{80} = .0875$
		$\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{5}{8}$	$-\frac{3}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{7}{80} = .0875$

TABLE 3. Lorentz-Force Function: Impulse.



Lorentz body force integral function: $q(x) = \begin{cases} 0, & 0 \le x \le a \\ 1, & a \le x \le 1 \end{cases}$

Critical total Lorentz stress: $\hat{F} = \sigma_p / a$

Displacement at x=a: $\hat{\delta}(a) = \frac{\hat{F}}{E} a(1-a)$

(Displacement elsewhere is linear in x.)

Average displacement: $\hat{\delta}_{AVG} = \frac{1}{2}\frac{\hat{F}}{E} a(1-a)$

Nonlinear Stress-Strain Relationship

Only one example is presented -- that for a uniform Lorentz force distribution function, and for strain quadratic in stress. The main point is to show how the variation of the stress at x=1 -- the pole end of the coil -- varies with total Lorentz stress. For a linear stress-strain relationship, The stress at x=1 falls off linearly with total Lorentz stress; it is no surprise that the stress fall-off is nonlinear for a nonlinear stress-strain relationship. But it takes a lot of nonlinearity in the stress-strain relationship to produce much in the stress fall-off.

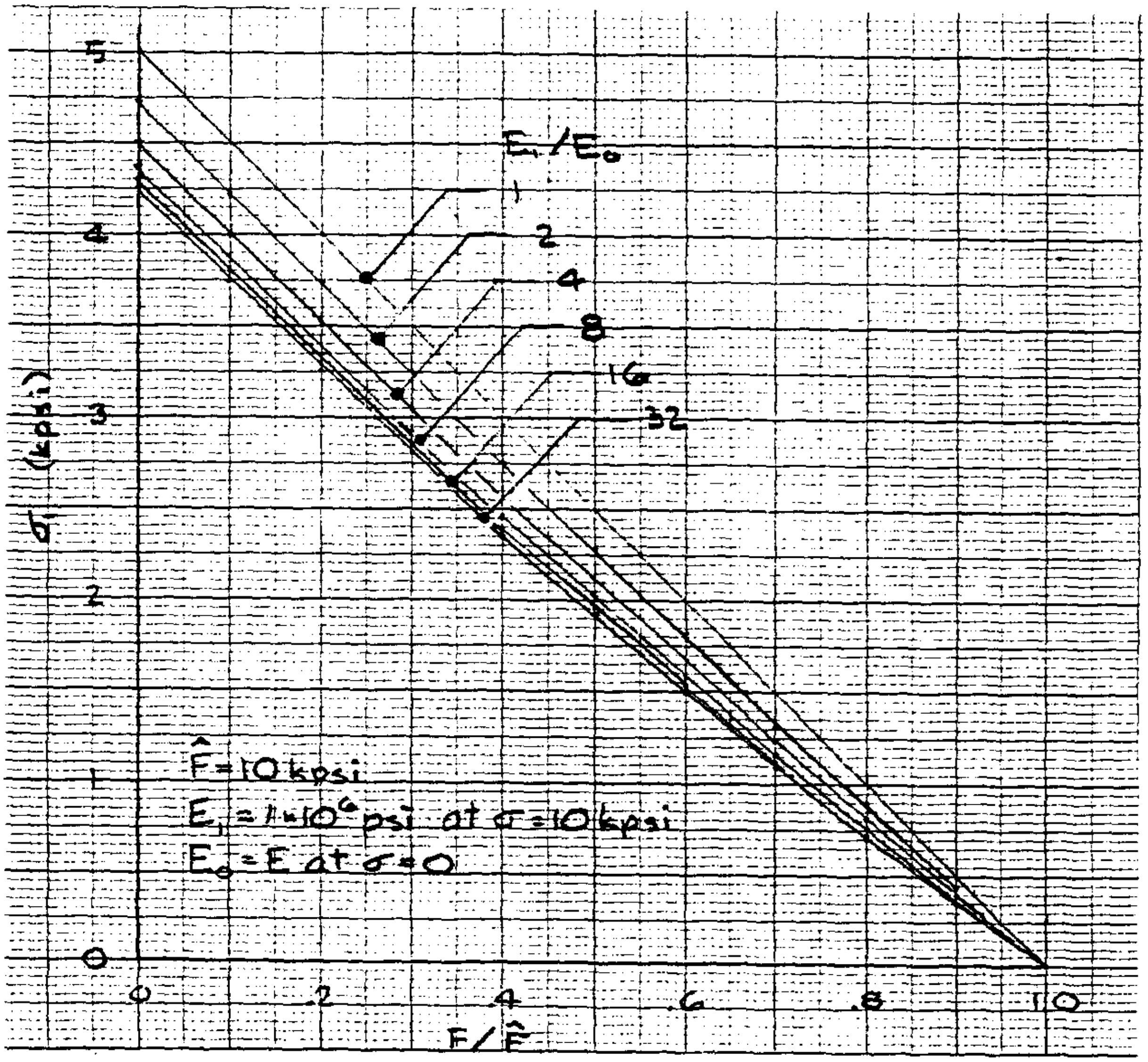


Figure 3. The effect of modulus ratio on the variation of the stress at the pole, σ_1, with total Lorentz stress, F.