SSC-N-322

COMPUTATION OF FIRST ORDER TUNE SHIFTS

by
Etienne Forest
February 1987

I. Floquet Variables of a Linear System

In this treatment, we will assume that our unperturbed machine is linear. Mathematically, it means that one turn around the machine at position "s" can be represented by a six-dimensional matrix:

$$M_S = \left[\begin{array}{cccccc} & & & & 0 & \upsilon_1 \\ & & \mathbf{N}_S & & 0 & \upsilon_2 \\ & & & & 0 & \upsilon_3 \\ & & & & 0 & \upsilon_4 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{A.1a}$$

$$N = 4 \times 4 \text{ betatron matrix} \tag{A.1b}$$

The matrix M_S acts on the canonical set of variables:

$$\vec{A} = (x,\ p_x,\ y,\ p_y, \ell,\ d) \tag{A.2a}$$

$$d = -\frac{p - p_0}{p_0} = -\delta \tag{A.2b}$$

Knowing M_S it is easy to calculate the fixed point $\vec{f_S}$. It is given by:

$$f_s = -(I - N_s)^{-1}\,\vec{\upsilon}\,\delta = \vec{\eta}\,\delta \tag{A.3}$$

Let us construct the following transformation of f_S:

$$\vec{r}^{\,New} = \vec{r} - \vec{\eta}\,\delta \ ; \ \vec{r} = (x,\ p_x,\ y,\ p_y)\,. \tag{A.4a}$$

$$\ell^{New} = \ell - \eta_1 p_x + \eta_2 x - \eta_3 p_y + \eta_4 y \tag{A.4b}$$

$$d^{New} = d \tag{A.4c}$$

Denoting by $\vec{\xi}$ the vector $(\vec{r}^{New}, \ell^{New}, d^{New})$, we can express the new matrix T_S, parameterizing the motion of $\vec{\xi}$:

$$\vec{\xi}^1 = B^{-1} M_S B\, \vec{\xi}^0 \tag{A.5a}$$

$$T_S = B^{-1} M_S B \tag{A.5b}$$

$$B^{-1} \vec{z} = \vec{\xi} \tag{A.5c}$$

Quite clearly, the matrix T_S has the following form:

$$T_S = \begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & \mathbf{N}_S & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \tilde{\alpha} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.6a}$$

$$\tilde{\alpha} = -\vec{\omega} \bullet \vec{\eta} + \alpha \tag{A.6b}$$

Finally, we proceed to diagonalize N. Since the machine is stable, it must be true that

$$\exists\, A \quad \text{such that} \quad A_s^{-1} N_s A_s = R \tag{A.7a}$$

$$R = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & \begin{smallmatrix} 1 & \tilde{\alpha} \\ 0 & 1 \end{smallmatrix} \end{bmatrix}; \quad R_i = \begin{bmatrix} \cos \mu_i & \sin \mu_i \\ -\sin \mu_i & \cos \mu_i \end{bmatrix} \tag{A.7b}$$

The original matrix M_S can be written as follows:

$$M_S = B\, A\, R\, A^{-1} B^{-1} \tag{A.8}$$

Here, A and B are s-dependent, while R is not. In fact, B carries information which brings the particle to the local closed orbit, and A contains the betatron lattice functions.

In the next section we introduce the Lie operators associated to M_S.

Part II: Generalization to Non-Linear Systems through Lie Operators

The matrices B, A and R have a certain Lie operator representation. The representation of B and A will not be needed here, however, we will use the representation of R, given by:

$$\mathcal{R} = \exp(:-\vec{\mu}\cdot\vec{J} - \frac{\tilde{\alpha}}{2}\delta^2 :) \tag{A.9a}$$

$$J_i = \frac{q_i^2 + p_i^2}{2}\,,\ [\phi_i, J_j] = \delta_{ij}\,. \tag{A.9b}$$

Notice that we denote Lie transformation by script letters and their associated matrix representation by uppercase letters. (If they are linear mappings!) The equation of the mappings corresponding to (A.8) is just

$$\mathcal{M}_S = \mathcal{B}^{-1}\mathcal{A}^{-1}\mathcal{R}\,\mathcal{A}\,\mathcal{B} \tag{A.10}$$

Notice that Lie transformations are in the reverse order from the matrices.

Now, let us introduce a multipole kick at s. The kick is given by:

$$\Delta\vec{p} = [\tilde{A}\,(x,y)\,,\,\vec{p}] \tag{A.11a}$$

$$\frac{\partial\tilde{A}(x,y)}{\partial x} = -\frac{qL}{p_0}B_y(x,y) \tag{A.11b}$$

$$\frac{\partial\tilde{A}}{\partial y} = \frac{qL}{p_0}B_x(x,y) \tag{A.11c}$$

This kick has the following representation:

$$\mathcal{K} = \exp(:\tilde{A}\,(x,y):) \tag{A.12}$$

The map of the ring, followed by the kick, is just:

$$\mathcal{M}_S^T = \mathcal{M}_S\,\mathcal{K} \tag{A.13}$$

Now, we will attempt to diagonize $\mathcal{M}_S^T$, in the same way we did with the linear part $\mathcal{M}_S$. That is suppose there exist $\mathcal{C}$ such that

$$\mathcal{M}_S^T = \mathcal{C}^{-1}\exp(:-F(\vec{J},\delta):)\,\mathcal{C}, \tag{A.14}$$

We can extract the tunes from Eq. (A.14). We define a new angle $\vec{\psi}$ and a new action through the rules:

$$\vec{\psi} = \mathbf{C}^{-1}\vec{\phi} \tag{A.15a}$$

$$\vec{I} = \mathbf{C}^{-1}\vec{J}, \tag{A.15b}$$

Using Eq. (A.15), we have

$$\begin{aligned}\mathcal{M}_S^T\vec{\psi} &= \mathbf{C}^{-1}\exp(:-F:)\,\mathbf{C}\mathbf{C}^{-1}\vec{\phi}\\ &= \mathbf{C}^{-1}\exp(:-F:)\,\vec{\phi}\\ &= \mathbf{C}^{-1}(\vec{\phi} + \frac{\partial F(J,\delta)}{\partial J})\\ &= \vec{\psi} + \frac{\partial F(\vec{I},\delta)}{\partial \vec{I}}\end{aligned} \tag{A.16}$$

Our problem amounts to the finding of $\mathbf{C}$. We start by rewriting $\mathcal{M}_S^T$:

$$\begin{aligned}\mathcal{M}_S^T &= \mathcal{B}^{-1}\mathcal{A}^{-1}\mathcal{R}\,\mathcal{A}\,\mathcal{B}\exp(:\tilde{A}:)\\ &= \mathcal{B}^{-1}\mathcal{A}^{-1}\mathcal{R}\exp(:\mathcal{A}\,\mathcal{B}\,\tilde{A}(x,y):)\,\mathcal{A}\,\mathcal{B}\\ &= \mathcal{B}^{-1}\mathcal{A}^{-1}\mathcal{R}\exp(:\overline{A}(x,y):)\,\mathcal{A}\,\mathcal{B}\end{aligned} \tag{A.17a}$$

$$\begin{aligned}\overline{A}(x,y) &= \tilde{A}(\mathcal{A}\,\mathcal{B}\,x, \mathcal{A}\,\mathcal{B}\,y)\\ &= \tilde{A}(B_{1j}A_{jk}Z_k, B_{3j}A_{jk}Z_k)\\ &= \tilde{A}(A_{1k}r_k + \eta_1\delta, A_{3k}r_k + \eta_3\delta)\end{aligned} \tag{A.17b}$$

Let us write $\mathbf{C}$ as:

$$\mathbf{C} = \exp(:\Gamma:)\,\mathcal{A}\,\mathcal{B} \tag{A.18}$$

Combining Eq. (A.14) and Eq. (A.17), we must have:

$$\exp(:\Gamma:)\,\mathcal{R}\exp(:\tilde{A}:)\exp(:-\Gamma:) = \exp(:-F(\vec{J},\delta):) \tag{A.19}$$

In Eq. (A.19), Γ and A are non-linear functions. Our calculation will only be linear in Γ and A. On the other hand, $\mathcal{R}$ is a linear operator and must be handled exactly. This is done by a few manipulations:

$$\begin{aligned}
\exp(:\!-F(\vec{J},\delta)\!:) &= \mathcal{R}\mathcal{R}^{-1}\exp(:\Gamma:)\,\mathcal{R}\,\exp(:\bar{A}:)\exp(:\!-\Gamma\!:) \\
&= \mathcal{R}\exp(:\mathcal{R}^{-1}\Gamma:)\exp(:\bar{A}:)\exp(:\!-\Gamma\!:) \\
&= \mathcal{R}\exp(:(\mathcal{R}^{-1}-\mathbb{1})\Gamma+\bar{A}+O(\bar{A}^2)\;....:)
\end{aligned} \tag{A.20}$$

Now, suppose we succeed in selecting Γ, such that

$$(\mathcal{R}^{-1}-\mathbb{1})\Gamma+\bar{A}=W_0(\vec{J},\delta) \tag{A.21}$$

Then, we have attained our goal since:

$$\begin{aligned}
\exp(:\!-F(\vec{J},\delta)\!:) &= \mathcal{R}\exp(:W_0(\vec{J},\delta):) \\
&= \exp(:\!-\vec{\mu}\cdot\vec{J}-\frac{\tilde{\alpha}}{2}\delta^2+W_0(\vec{J},\delta)\!:) \\
\Rightarrow\; F(\vec{J},\delta) &= \vec{\mu}\cdot\vec{J}+\frac{\tilde{\alpha}}{2}\delta^2-W_0(\vec{J},\delta)
\end{aligned} \tag{A.22}$$

W_0 as well as Γ are obtained by Fourier analyzing $\overline{A}$:

$$\begin{aligned}
\overline{A} &= \sum_{\vec{m}\neq 0}\overline{V}_{\vec{m}}(\vec{J},\delta)\exp(i\,\vec{m}\cdot\phi)+\overline{A}_0(\vec{J},\delta)\,. \\
&= \overline{A}_R+\overline{A}_0
\end{aligned} \tag{A.23}$$

Using Eq. (A.23) and (A.21) we have:

$$\overline{A}_0=W_0 \tag{A.24a}$$

$$\Gamma=\frac{1}{\mathbb{1}-R^{-1}}\overline{A}_R=\sum_{\vec{m}\neq 0}\frac{\overline{V}_{\vec{m}}(\vec{J},\delta)}{1-\exp(-i\,\vec{m}\cdot\vec{\mu})}\exp(i\,\vec{m}\cdot\vec{\phi}) \tag{A.25a}$$

The final result is just, using Eq. (A.15) and (A.16):

$$\mathcal{M}_S^T\vec{\psi}=\vec{\mu}-\frac{\partial\overline{A}_0}{\partial\vec{I}}(\vec{I},\delta)+O(\overline{A}^2) \tag{A.26a}$$

$$I=\mathcal{C}^{-1}\vec{J}=\vec{J}-\frac{\partial\Gamma}{\partial\vec{\phi}}+O(\overline{A}^2) \tag{A.26b}$$

Finally, to first order, we can add all the contributions to Eq. (A.26a) coming form various locations around the ring. Of course the appropriate $\vec{\eta}$ and A_{ij} must be used to compute $\overline{A}$.

In this paper we specialized to systems with mid-plane symmetry. In that case, the following relations hold:

$$A = \begin{bmatrix} \sqrt{\beta_x} & 0 & 0 & 0 & 0 & 0 \\ \frac{-\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y} & 0 & 0 & 0 \\ 0 & 0 & \frac{-\alpha_y}{\sqrt{\beta_y}} & \frac{1}{\sqrt{\beta_y}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(A.27a)}$$

$$\vec{\eta} = (\eta_1, \eta_2, 0, 0) \qquad \text{(A.27b)}$$