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INCORPORATION OF TOROIDAL BOUNDARY

CONDITIONS INTO PROGRAM POISSON\*

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Abstract

A technique is developed for introduction of a boundary condition applicable to relaxation computations for magnetic problems with axial symmetry and with no sources (currents, or magnetized material) external to the boundary. The procedure as described in this note is restricted to cases in which the (toroidal) boundary will surround completely the region of physical interest but will not encompass the axis of rotational symmetry. The technique accordingly provides the opportunity of economically excluding from the relaxation process regions of no direct concern in the immediate neighborhood of the symmetry axis and hence can have useful application to annular magnetic devices with axial symmetry.

The procedure adopted makes use internally of the characteristic form of the vector-potential function, in a source-free region, when expressed in toroidal coördinates. The relevant properties of associated Legendre functions of half-integral degree are summarized in this connection and their introduction into the program POISSON is outlined. Results of some test cases are included, to illustrate the application of this technique for configurations with median-plane symmetry.

I. MOTIVATION

In circular particle accelerators, with the possible exception of those of the greatest size, one cannot entirely neglect the curvature of the structure

and of the guide field. In such cases the use of cylindrical coordinates for the solution of magnetostatic problems would be appropriate, and if in certain local regions the  $\phi$  dependence can be ignored the independent variables  $\rho$  and  $Z$  become two in number.

Such magnetostatic problems are soluble, by relaxation programs such as POISSON, in  $\rho, Z$  cylindrical coordinates. As is the case with other applications of relaxation methods, however, there must be concern regarding a suitable termination of the problem at the boundary of the mesh. (The condition that normally is required is one consistent with the absence of any "sources" in the region exterior to such a boundary.) In analyzing the magnetic fields of circular particle accelerators, one may wish to restrict the region of examination to that near the working aperture and surrounding magnet structure, while excluding a very substantial area closer to (and including) the axis of rotational symmetry for the entire structure.

For the reason just indicated, one accordingly is led to consider the use of toroidal coordinates, in constructing the boundary to a relaxation mesh for use in analyzing the magnetic fields of circular devices (such as accelerators and spectrometers), and in formulating the boundary conditions that then may be usefully imposed at such boundaries. We pursue such issues in the following Sections--commencing with a review of the characteristics of toroidal coordinates and continuing with an examination of related magnetostatic issues that will permit formulation of a boundary condition analogous in spirit to those devised previously at this Laboratory for application to other configurations. \1/\*

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\*References and notes are given at the end of this report in Section VII, p. 33.

## II. TOROIDAL COÖRDINATES

Toroidal coördinates can be defined, in a manner illustrated by Arfken<sup>12/</sup>,  
by

$$\left. \begin{aligned} X &= \frac{a \operatorname{Sinh} \eta \cos \phi}{\operatorname{Cosh} \eta - \cos \xi} \\ Y &= \frac{a \operatorname{Sinh} \eta \sin \phi}{\operatorname{Cosh} \eta - \cos \xi} \\ Z &= \frac{a \sin \xi}{\operatorname{Cosh} \eta - \cos \xi} \end{aligned} \right\} \rightarrow \rho = (X^2 + Y^2)^{1/2} = \frac{a \operatorname{Sinh} \eta}{\operatorname{Cosh} \eta - \cos \xi}$$

The inverse transformation is given by

$$\operatorname{Tanh} \eta = \frac{2a\rho}{\rho^2 + Z^2 + a^2} = \frac{2a\rho}{R^2 + a^2} = 2 \frac{\rho/a}{(R/a)^2 + 1}$$

$$\tan \xi = \frac{2aZ}{\rho^2 + Z^2 - a^2} = \frac{2aZ}{R^2 - a^2} = 2 \frac{Z/a}{(R/a)^2 - 1}$$

$$\tan \phi = Y/X,$$

with

$$R = (\rho^2 + Z^2)^{1/2}.$$

The metric coefficients are found to be as follows:

$$h_{\xi} = \frac{\partial s}{\partial \xi} = \frac{a}{\operatorname{Cosh} \eta - \cos \xi}$$

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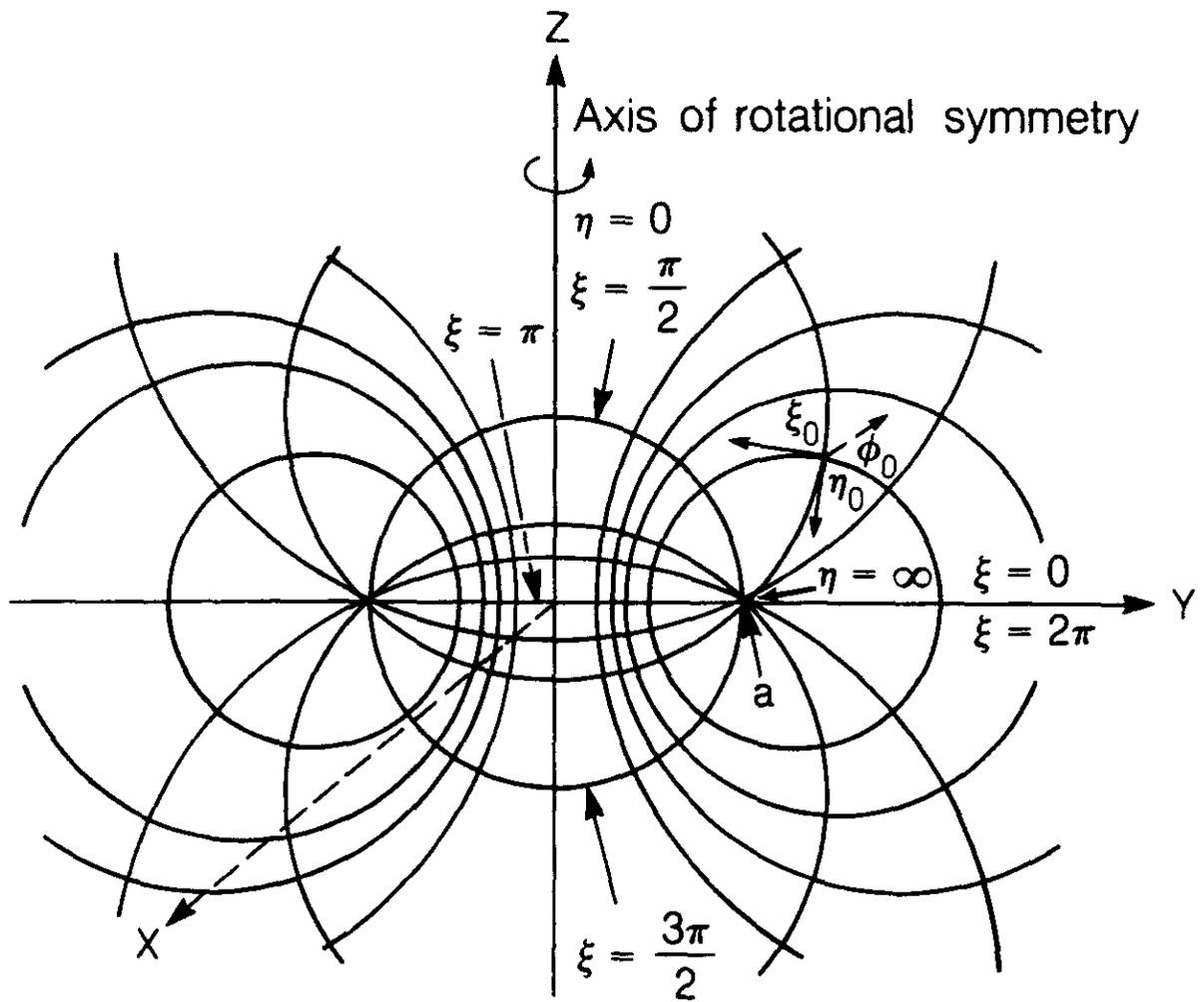
$$h_{\phi} = \frac{\partial s}{\partial \phi} = \frac{a \operatorname{Sinh} \eta}{\operatorname{Cosh} \eta - \cos \xi}$$

We shall be interested chiefly in the geometrical characteristics of these coördinates in a planar section of constant  $\phi$ .

(i) Curves (surfaces) of constant  $\eta$  are circles (toroids) of radii  $a \operatorname{Csch} \eta$  centered at  $\rho = a \operatorname{Ctnh} \eta$ ,  $Z = 0$ . Curves of constant  $\eta$  thus extend, in the mid-plane  $Z = 0$ , between the limits  $\rho = a \operatorname{Tanh} \frac{\eta}{2}$  and  $\rho = a \operatorname{Ctnh} \frac{\eta}{2}$ .

(ii) Curves (surfaces) of constant  $\xi$  are circles (spheres) of radii  $a \operatorname{csc} \xi$  centered on the  $Z$ -axis at  $Z = a \operatorname{ctn} \xi$ .

The projected curves thus exhibit a similarity to those generated in a plane by a conformal transformation illustrated by Smythe.<sup>13/</sup>



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Fig. 1 Toroidal Coordinates ( $\xi$ ,  $\eta$ ,  $\phi$ )

### III. The Differential Equation for the Potential

Through use of the metric coefficients cited in Sect. II, one can write explicitly in toroidal coördinates Laplace's differential equation for a scalar potential function. As has been shown in some detail by MacRobert,<sup>14/</sup> solutions then may be found in which this potential function has the form of a factor  $(\text{Cosh } \eta - \cos \xi)^{\frac{1}{2}}$  times the product of separate functions of the coördinates  $\eta$ ,  $\xi$ , and  $\phi$ . For such solutions, the functions of  $\xi$  and of  $\phi$  are each just circular functions of their respective arguments, and the functions of  $\eta$  are Legendre functions (or associated Legendre functions) of half-integral degree and argument  $z = \text{Cosh } \eta$ .

In the present work, however, we are specifically interested in POISSON computations of magnetic field, for cases with axial symmetry, and wish to make use of a vector-potential component  $A_\phi$  (or  $A^* = \rho A_\phi$ ) to characterize this field. The homogeneous equation  $\nabla \times [\nabla \times \vec{A}] = 0$  for  $A = A_\phi \hat{e}_\phi$  then may be written

$$\frac{\partial}{\partial \eta} \left[ \frac{h_\xi}{h_\eta h_\phi} \frac{\partial(h A_\phi)}{\partial \eta} \right] + \frac{\partial}{\partial \xi} \left[ \frac{h_\eta}{h_\xi h_\phi} \frac{\partial(h A_\phi)}{\partial \xi} \right] = 0$$

or, with insertion of the metric coefficients cited in Sect. II,

$$\frac{\partial}{\partial \eta} \left[ \frac{\text{Cosh } \eta - \cos \xi}{\text{Sinh } \eta} \frac{\partial}{\partial \eta} \left( \frac{\text{Sinh } \eta}{\text{Cosh } \eta - \cos \xi} A_\phi \right) \right] + \frac{\partial}{\partial \xi} \left[ \frac{\text{Cosh } \eta - \cos \xi}{\text{Sinh } \eta} \frac{\partial}{\partial \xi} \left( \frac{\text{Sinh } \eta}{\text{Cosh } \eta - \cos \xi} A_\phi \right) \right] = 0$$

wherein the dependent variable  $A_\phi$  is to be regarded as a function of  $\eta$  and  $\xi$ , but independent of  $\phi$ .

Guided by the form known to be appropriate for the scalar-potential solutions to Laplace's equation in toroidal coordinates, we may proceed heuristically to achieve a separation of variables in the present case once  $A_\phi$  is divided by the factor  $(\text{Cosh } \eta - \cos \xi)^{1/2}$ . We accordingly write the vector-potential component  $A_\phi$  in the form

$$A_\phi = (\text{Cosh } \eta - \cos \xi)^{1/2} \cos n\xi G(\eta).^*$$

With this substitution, the differential equation assumes the form

$$\frac{\partial}{\partial \eta} \left[ \frac{\text{Cosh } \eta - \cos \xi}{\text{Sinh } \eta} \frac{\partial}{\partial \eta} \left( \frac{\text{Sinh } \eta \cos n\xi}{\sqrt{\text{Cosh } \eta - \cos \xi}} G(\eta) \right) \right] + \frac{\partial}{\partial \xi} \left[ \frac{\text{Cosh } \eta - \cos \xi}{\text{Sinh } \eta} \frac{\partial}{\partial \xi} \left( \frac{\text{Sinh } \eta \cos n\xi}{\sqrt{\text{Cosh } \eta - \cos \xi}} G(\eta) \right) \right] = 0 \quad ,$$

and (following some intermediate algebraic work) the  $\xi$  dependence then is found to disappear (as hoped), and there remains only the ordinary differential equation for the factor  $G(\eta)$  itself:

$$\frac{1}{\text{Sinh } \eta} \left[ \frac{d}{d\eta} (\text{Sinh } \eta \cdot G) \right] - \left[ (n - \frac{1}{2})(n + \frac{1}{2}) + \frac{1}{\text{Sinh}^2 \eta} \right] G = 0 \quad ,$$

or, with  $z = \text{Cosh } \eta$  serving as the independent variable (and  $\nu = n - 1/2$ ),

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\* Only the use of a factor  $\cos n\xi$  (in preference to a factor  $\sin n\xi$ ) is indicated here, since we shall ultimately wish to specialize to cases with median-plane symmetry such that the function  $A_\phi$  is even with respect to the variable  $\xi$ .

$$\frac{d}{dz} \left[ (z^2 - 1) \frac{dG}{dz} \right] - \left[ \frac{1}{z^2 - 1} + (n - \frac{1}{2})(n + \frac{1}{2}) \right] G = 0$$

or

$$\frac{d}{dz} \left[ (z^2 - 1) \frac{dG}{dz} \right] - \left[ \frac{1}{z^2 - 1} + \nu(\nu + 1) \right] G = 0 .$$

Solutions to the differential equation for G can be written as directly proportional to associated ( $m = 1$ ) Legendre functions of degree  $\nu = n - 1/2$  and argument  $z = \text{Cosh } \eta$ . We shall employ in the work to follow only the functions of the first kind,  $P_{\nu=n-1/2}^1(z = \text{Cosh } \eta)$  or quantities proportional thereto, in order to avoid singularities developing at remote locations (or as the argument  $z$  approaches unity from above).<sup>5/</sup> With the index  $n$  confined to integer values (to insure a single-valued dependence upon the coordinate  $\xi$ ), we thus are confined to terms of the form

$$\text{For } A_{\phi} : (\text{Cosh } \eta - \cos \xi)^{1/2} P_{\nu=n-1/2}^1(z = \text{Cosh } \eta) \cos n\xi$$

or

$$\text{For } A^* : \frac{\text{Sinh } \eta}{(\text{Cosh } \eta - \cos \xi)^{1/2}} P_{\nu=n-1/2}^1(z = \text{Cosh } \eta) \cos n\xi$$

that contain as factors Legendre functions of half-integral degree ( $\nu = -1/2, 1/2, 3/2, \dots$ ) and which we choose to be even about the mid-plane  $\xi = 0$ .

Comment:

It is of interest to note that the vector potential for a single current-carrying circular loop at  $\rho=a$ , that Smythe<sup>6/</sup> has shown expressed in terms of complete elliptic integrals, can be equivalently expressed<sup>7/</sup> in terms of a single term of the form shown above for  $A_{\phi}$  (namely, a term with  $n = 0$ , or  $\nu = -1/2$ ). The  $\xi$ -dependence under such circumstances thus evidently arises solely through the factor  $(\text{Cosh } \eta - \cos \xi)^{1/2}$ .

We conclude this section by an Appendix in which we present, for later use, certain properties of the functions  $P_{\nu=n-\frac{1}{2}}^1(z = \text{Cosh } \eta)$ .

### APPENDIX TO SECTION III

#### 1. Relation of the Legendre Functions to Complete Elliptic Integrals

The Legendre functions of half-integral degree can be related to the complete elliptic integrals  $K$  and  $E$ . Explicit forms for the ordinary Legendre functions of degrees  $-1/2$  and  $1/2$  have been given by Irene Stegun<sup>10/</sup> in terms of elliptic integrals of parameters

$$m_A = \frac{z-1}{z+1} = \left(\text{Tanh } \frac{\eta}{2}\right)^2 \quad \text{or} \quad m_B = \frac{2\sqrt{z^2-1}}{z + \sqrt{z^2-1}} = 1 - e^{-2\eta}$$

We thus have the respective equivalent forms

$$P_{-\frac{1}{2}}(z = \text{Cosh } \eta) = \frac{2}{\pi} \frac{K_A}{\text{Cosh } \frac{\eta}{2}} = \frac{2}{\pi} e^{-\eta/2} K_B$$

and

$$P_{\frac{1}{2}}(z = \text{Cosh } \eta) = \frac{2}{\pi} \left[ 2 \left(\text{Cosh } \frac{\eta}{2}\right) E_A - \frac{K_A}{\text{Cosh } \frac{\eta}{2}} \right] = \frac{2}{\pi} e^{\eta/2} E_B$$

since elliptic integrals with such different parameters are related in the manner indicated by Milne-Thomson.<sup>10/</sup>

To obtain the corresponding forms for the associated functions  $P_{-\frac{1}{2}}^1(z = \text{Cosh } \eta)$  and  $P_{\frac{1}{2}}^1(z = \text{Cosh } \eta)$  one may make use of the relation<sup>10/</sup>

$$P_{\nu}^1(z) = \sqrt{z^2-1} \frac{d}{dz} P_{\nu}(z)$$

with the result

$$P_{-1/2}^1(z = \text{Cosh } \eta) = \frac{1}{\pi} \frac{E_A - K_A}{\text{Sinh } \frac{\eta}{2}} = \frac{1}{\pi} \frac{e^{\eta/2} E_B - e^{-\eta/2} (\text{Cosh } \eta) K_B}{\text{Sinh } \eta}$$

$$P_{1/2}^1(z = \text{Cosh } \eta) = \frac{1}{\pi} \frac{(\text{Cosh } \eta) E_A - K_A}{\text{Sinh } \frac{\eta}{2}} = \frac{1}{\pi} \frac{e^{\eta/2} (\text{Cosh } \eta) E_B - e^{-\eta/2} K_B}{\text{Sinh } \eta}$$

From such results, additional associated Legendre functions of higher degree can be evaluated sequentially by application of the recursion relation<sup>11/</sup>

$$P_{n+3/2}^1(z = \text{Cosh } \eta) = \frac{2(n+1)(\text{Cosh } \eta) P_{n+1/2}^1(z) - (n+3/2) P_{n-1/2}^1(z)}{n+1/2}$$

## 2. Small-Argument Form

A limiting form for the functions  $P_{\nu=n-1/2}^1(z = \text{Cosh } \eta)$  in the limit  $\eta \rightarrow 0$  has been cited in Note<sup>15/</sup> - - namely

$$P_{\nu=n-1/2}^1(z = \text{Cosh } \eta) \rightarrow \frac{\nu(\nu+1)}{2} \eta = \frac{(n-1/2)(n+1/2)}{2} \eta$$

## 3. Large-Argument Approximations (z and n approaching infinity)

For large values of the argument, the functions  $P_{\nu}^{\mu}(z)$  become quite large when  $\nu > -1/2$ , as is shown by the limiting form<sup>12/</sup>

$$P_{\nu}^{\mu}(z) \rightarrow \frac{2^{\nu}}{\sqrt{\pi}} \frac{\Gamma(\nu+1/2)}{\Gamma(1+\nu-\mu)} z^{\nu} \text{ for } \nu > -1/2$$

so that, in particular (for half-integral degree and order  $m = 1$ ),

$$P_{\nu=n-1/2}^1(z) \rightarrow \frac{2^{2n-5/2}}{\pi} \frac{[(n-1)!]^2}{(2n-2)!} z^{n-1/2} \quad n = 1, 2, 3, \dots$$

An expression of this nature may prove useful for "renormalization" of such Legendre functions in order to obtain alternative functions of more convenient magnitude for computational work.

It remains to comment on the large argument behavior of the function  $P_{-1/2}^1(z = \text{Cosh } \eta)$ . An estimate of this function may be obtained from the elliptic integral form cited earlier

$$P_{-1/2}^1(z = \text{Cosh } \eta) = \frac{1}{\pi} \frac{E_A - K_A}{\text{Sinh } \frac{\eta}{2}} = \left[ \text{with } m_A = (\text{Tanh } \frac{\eta}{2})^2 \right]$$

$$\rightarrow - \frac{\ln(4 \text{Cosh } \frac{\eta}{2}) - 1}{\pi \text{Sinh } \frac{\eta}{2}} \cong - \frac{1}{\pi} \frac{\ln z}{\sqrt{2z}}$$

This function accordingly commences with a small absolute value [ $P_{-1/2}^1(z = \text{Cosh } \eta) \cong -\frac{\eta}{8}$ ] when the argument is small, and approaches zero again for large arguments. The function reaches a maximum magnitude ( $P_{-1/2}^1 \cong -0.1739638462$ ) for  $\eta$  near 2.5285 ( $z \cong 6.307$ ). There thus appears to be no computational reason to consider renormalization of this particular function (in contrast to those of higher half-integral degree).

cos

The characteristics of the functions  $P_{\nu=n-1/2}^1$  summarized in this Appendix have been spot-checked by computational tests, and certain values also have been confirmed by reference to published tables.<sup>13/</sup> The ability of forms cited for the vector potential earlier in this Section to satisfy Laplace's equation likewise has been checked numerically by finite-difference approximations to  $\text{curl } [\text{curl } \vec{A}] = 0$ .

#### IV. APPLICATION

In application, we shall use the forms for  $A^* = \rho A_\phi$  that we have found in earlier Sections to guide the means of extending this function from an "inner" boundary curve,  $\eta = \eta_{in}$ , to points on a surrounding "outer" boundary curve. This "outer", or surrounding, boundary curve may conveniently be taken also to be a curve (surface) on which the toroidal coordinate  $\eta$  has a constant value ( $\eta = \eta_{out}$ ).

It appears computationally desirable, however, to regard the function  $A^*$  as represented not in the form of a series that contains as explicit factors the Legendre functions  $P_{\nu=n-\frac{1}{2}}^1(z = \text{Cosh } \eta)$ , but that introduces in their place factors that represent such functions renormalized as follows:

The function  $P_{-\frac{1}{2}}^1$  does not require renormalization, since its value remains finite (and in fact tends toward zero) as the argument becomes infinite. <sup>\14/</sup>

The functions  $P_\nu^1$  of higher half-integral degree are renormalized, through division by the asymptotic form for  $P_\nu^1$ , to provide the working function for computational use. <sup>\14/</sup>

To evaluate such working functions, that we here shall denote by the dimensioned variable  $ASP(k)$  with  $k = 1, 2, 3, \dots$ , to replace the Legendre functions  $P_{\nu=k-\frac{1}{2}}^1(z = \text{Cosh } \eta)$ , we may first compute the interim quantities  $AP1 = P_{-\frac{1}{2}}^1(z)$  and  $AP2 = P_{\frac{1}{2}}^1(z)$  from the elliptic-integral formulas cited earlier. <sup>\15/</sup> Likewise one may form the quantity

$$AP3 = 4.0 z AP2 - 3.0 AP1,$$

which is identical to the Legendre function  $P_{\frac{3}{2}}^1(z)$  by virtue of the recursion relation for such functions. <sup>\11/</sup> Through application of the asymptotic forms for the functions of degree  $+1/2$  and greater, the working functions  $ASP(k)$  may now be identified as

$$\text{ASP}(1) = \text{AP1} \text{ [being simply } P_{-\frac{1}{2}}^1(z)\text{]}$$

$$\text{ASP}(2) = \frac{\pi}{\sqrt{2z}} \text{AP2}$$

$$\text{ASP}(3) = \frac{\pi}{4z\sqrt{2z}} \text{AP3}$$

and additional renormalized functions can be formed through use of the recursion relation

$$\text{ASP}(k) = \text{ASP}(k-1) - \frac{(2k-3)(2k-7)}{16(k-2)(k-3)z^2} \cdot \text{ASP}(k-2)$$

for  $k > 3$ . For terms of identical degree  $\nu = k - 3/2$  (or identical index,  $k$ ) but different arguments, it then follows that the Legendre function factors will be in the ratio

$$\frac{P_{\nu=n-\frac{1}{2}}^1(z_2)}{P_{\nu=n-\frac{1}{2}}^1(z_1)} = \left(\frac{z_2}{z_1}\right)^{\alpha_k} \cdot \frac{\text{ASP}(k=n+1) \text{ for } z_2}{\text{ASP}(k=n+1) \text{ for } z_1}$$

where  $\alpha_k = \begin{cases} 0 & \text{for } n = 0 (k = 1) \\ \nu = n - \frac{1}{2} = k - \frac{3}{2} & \text{for } n \geq 1 (k \geq 2) \end{cases}$  so that in this instance the factor  $(z_2/z_1)^{\alpha_k}$  becomes unity and can, in effect, be ignored

To specify in toroidal coördinates a suitable inner boundary, from POISSON computation of a magnetostatic problem with rotational symmetry, we first select a suitable region of interest in  $\rho, Z$  space such that one is assured that there are no "sources" exterior to this region. Such a circular region of constant  $\eta$ , centrally located about the mid-plane, may be specified<sup>16/</sup> by means of the radial coördinates (measured for the axis of rotational symmetry)

$$\rho_{1,a} = a \text{Tanh } \frac{\eta_{in}}{2} \quad \text{and} \quad \rho_{1,b} = a \text{Ctnh } \frac{\eta_{in}}{2}$$

of the points of intersection of such a circle with the mid-plane ( $Z = 0$ ). Alternatively, one could specify the location  $\rho_{1,o}$  of the center and the radius  $R_1$  of such a circle.

From the first type of specification, it follows that

$$a = (\rho_{1,a}\rho_{1,b})^{1/2}, \quad \eta_{in} \text{ is given by } \text{Tanh } \frac{\eta_{in}}{2} = (\rho_{1,a}/\rho_{1,b})^{1/2},$$

$$R_1 = a \text{ Csch } \eta_{in} = a \frac{\text{Ctnh } \frac{\eta_{in}}{2} - \text{Tanh } \frac{\eta_{in}}{2}}{2} = \frac{\rho_{1,b} - \rho_{1,a}}{2}$$

and the center is situated at

$$\rho_{1,o} = a \text{ Ctnh } \eta_{in} = a \frac{\text{Ctnh } \frac{\eta_{in}}{2} + \text{Tanh } \frac{\eta_{in}}{2}}{2} = \frac{\rho_{1,b} + \rho_{1,a}}{2}$$

Alternatively, from the second type of specification

$$a = \sqrt{\rho_{1,o}^2 - R_1^2}, \quad \eta_{in} \text{ is given by } \text{Sinh } \eta_{in} = \sqrt{\left(\frac{\rho_{1,o}}{R_1}\right)^2 - 1}$$

$$\rho_{1,a} = \rho_{1,o} - R_1, \quad \text{and } \rho_{1,b} = \rho_{1,o} + R_1.$$

To specify the surrounding "outer" circular boundary at  $\eta = \eta_{out}$  (with the same value of the parameter "a", but with the center displaced from that of the "inner" boundary), it may be convenient now merely to specify the intercept (the lesser intercept)

$$\rho_{2,a} = \rho_{1,a} - \Delta x = \rho_{1,o} - (R_1 + \Delta x)$$

of this outer curve with the mid-plane. The quantity  $\Delta x$  should be no less than the mesh spacing desired in this region. With the parameter "a" already known, it follows that  $\eta_{out}$  is then given by

$$\rho_{2,a} = \rho_{1,a} - \Delta x = \rho_{1,o} - (R_1 + \Delta x) = a \text{ Tanh } \frac{\eta_{out}}{2}$$

One also may continue to compute for this "outer" curve the other intercept

$$\rho_{2,b} = a \operatorname{Ctnh} \frac{\eta_{\text{out}}}{2} = \frac{a^2}{\rho_{2,a}}$$

the radius

$$R_2 = a \operatorname{Csch} \eta_{\text{out}} = \frac{\rho_{2,b} - \rho_{2,a}}{2} = \frac{a^2 - \rho_{2,a}^2}{2\rho_{2,a}}$$

and the location of the center at

$$\rho_{2,o} = a \operatorname{Ctnh} \eta_{\text{out}} = \frac{\rho_{2,b} + \rho_{2,a}}{2} = \frac{a^2 + \rho_{2,a}^2}{2\rho_{2,a}}$$

At any intermediate stage of the relaxation process (executed on a mesh in  $\rho, Z$  space), following some complete relaxation pass through the mesh, one will have available provisional values of the working variable ( $A^* = \rho A_\phi$ ) at mesh points on the inner boundary ( $\eta_{in}$ ) where the values of the  $\xi$  coordinate have explicit values  $\xi_i$ . It then will be the object to employ these values of  $A^*$  to revise ("update") the values of  $A^*$  at mesh points on the outer boundary,  $\eta_{out}$  ( $< \eta_{in}$ ), so that they can be used to revise internal values when continuing the relaxation process. We discuss this process of boundary-value revision in further detail below.

We may imagine the values of the provisional vector potential on the boundary curve  $\eta_{in}$  to be developed in the form of a Fourier series

$$a A_\phi(\eta_{in}, \xi) = \sqrt{\operatorname{Cosh} \eta_{in} - \cos \xi} \sum_{k=1} C_k \cos(k-1)\xi \quad \left( \begin{array}{l} \text{for situations of} \\ \text{even symmetry,} \\ \text{with respect to} \\ \xi, \text{ about } z = 0. \end{array} \right)$$

or, for the working variable  $A^* = \rho A_\phi$ , in the form of the series

$$A^*(\eta_{in}, \xi) = \frac{\operatorname{Sinh} \eta_{in}}{\sqrt{\operatorname{Cosh} \eta_{in} - \cos \xi}} \sum_{k=1} C_k \cos(k-1)\xi, \quad \frac{\rho}{a} = \frac{\operatorname{Sinh} \eta_{in}}{\operatorname{Cosh} \eta_{in} - \cos \xi} \quad \text{since}$$

The coefficients of such a development can be obtained by a weighted fit of

$$\text{values of } A^*(\eta_{in}, \xi_j) \left[ \frac{\text{Sinh } \eta_{in}}{\sqrt{\text{Cosh } \eta_{in} - \cos \xi_j}} \right]^{-1} \text{ at some or all of the mesh}$$

points at locations  $\xi_j$  on the inner boundary  $\eta_{in}$ .

The transfer to the outer boundary ( $\eta_{out}$ ) of the implications of such a development then follows from recognition of the  $\eta$ -dependent factors

$$P_{n-\frac{1}{2}}^1(\text{Cosh } \eta) \text{ or } P_{k-\frac{3}{2}}^1(\text{Cosh } \eta) \quad [k = n + 1]$$

that should be associated with factors  $\cos n\xi$  [=  $\cos(k - 1)\xi$ ] in such a development. For the purposes of forming ratios, we may conveniently make use of the scaled (or "re-normalized") quantities ASP (introduced earlier in this Section), for which

$$\frac{P_{\nu=n-\frac{1}{2}}^1(z_2)}{P_{\nu=n-\frac{1}{2}}^1(z_1)} = \left( \frac{z_2}{z_1} \right)^{\alpha_k} \frac{\text{ASP}(k = n + 1) \text{ for } z_2}{\text{ASP}(k = n + 1) \text{ for } z_1}$$

with

$$\alpha_k = \begin{cases} 0 \text{ for } n = 0 \text{ (} k = 1 \text{)} & \left[ \text{so that in this instance, the factor } (z_2/z_1)^{\alpha_k} \right. \\ & \left. \text{becomes unity and can, in effect, be ignored} \right] \\ \nu = n - \frac{1}{2} = k - \frac{3}{2} \text{ for } n \geq 1 \text{ (} k \geq 2 \text{)} . \end{cases}$$

The results extended to points  $\eta_{out}, \xi_j$  on the outer boundary circle thus provide the values

$$A^*(\eta_{out}, \xi_j) = \frac{\text{Sinh } \eta_{out}}{\sqrt{\text{Cosh } \eta_{out} - \cos \xi_j}} \sum_{k=1} C_k \left( \frac{\text{Cosh } \eta_{out}}{\text{Cosh } \eta_{in}} \right)^{\alpha_k} \cdot \frac{\text{ASP}(k) \text{ for } z = \text{Cosh } \eta_{out}}{\text{ASP}(k) \text{ for } z = \text{Cosh } \eta_{in}} \cos(k - 1)\xi_j$$

(with  $\alpha_k = 0$  or  $k - 3/2$  for  $k = 1$  or  $k \geq 2$ , respectively).

It is recognized that with the coefficients  $C_k$  expressed (through the mechanism of an inverted matrix) in terms of the values  $A^*(\eta_{in}, \xi_j)$ , the result cited immediately above constitutes a linear (homogeneous) transformation from such values to the required values  $A^*(\eta_{out}, \xi_j)$ .

Matrix Notation:

Given values of the function  $A^*(n_{in}, \xi_i)$  for points  $\xi_i$  on the boundary  $n_{in}$ , we wish to make a weighted least-squares fit (with weights  $w_i$ ) of

$$\frac{A^*(n_{in}, \xi_i)}{\left( \frac{\text{Sinh } n_{in}}{\sqrt{\text{Cosh } n_{in} - \cos \xi_i}} \right)} \quad \text{to the truncated series } \sum_{k=1} C_k \cos(k-1)\xi_i ;$$

i.e., we adjust the coefficients  $C_k$  so as to minimize

$$\frac{1}{2} \sum_i \left\{ w_i \left[ \sum_{k=1} C_k \cos(k-1)\xi_i - \frac{A^*(n_{in}, \xi_i)}{\left( \frac{\text{Sinh } n_{in}}{\sqrt{\text{Cosh } n_{in} - \cos \xi_i}} \right)} \right]^2 \right\}$$

[Regarding suggested forms for the weight factors  $w_i$ , see the Section included on p.5 of LBL-18798/UC-28 pertaining to weights used in connection with circular functions  $F(v)$ .]

This minimization objective leads to the set of algebraic equations that can be written, in matrix notation,  $\sum_{\ell} M_{k,\ell} C_{\ell} = V_k$ , where  $M$  is the symmetric matrix with  $k, \ell$  elements

$$M_{k,\ell} = \sum_i w_i \cos(k-1)\xi_i \cos(\ell-1)\xi_i$$

$$\text{and } V_k = \sum_i w_i \cos(k-1)\xi_i \cdot \frac{A^*(n_{in}, \xi_i)}{\left( \frac{\text{Sinh } n_{in}}{\sqrt{\text{Cosh } n_{in} - \cos \xi_i}} \right)}$$

Accordingly, the solution may be written in terms of the elements of the inverse matrix, as

$$C_{\ell} = \sum_k (M^{-1})_{\ell,k} V_k$$

$$= \sum_i w_i \left( \sum_{k=1} (M^{-1})_{\ell,k} \cos(k_i - 1)\xi_i \right) \left( \frac{A^*(\eta_{in}, \xi_i)}{\frac{\text{Sinh } \eta_{in}}{\sqrt{\text{Cosh } \eta_{in} - \cos \xi_i}}} \right) .$$

Then values of  $A^*(\eta_{out}, \xi_j)$  may be computed, for locations  $\xi_j$  on the outer boundary ( $\eta_{out}$ ):

$$A^*(\eta_{out}, \xi_j) = \frac{\text{Sinh } \eta_{in}}{\sqrt{\text{Cosh } \eta_{out} - \cos \xi_j}}$$

$$\cdot \sum_{\ell=1} C_{\ell} + \frac{\text{ASP}_{\ell}(\eta_{out})}{\text{ASP}_{\ell}(\eta_{in})} \left( \frac{\text{Cosh } \eta_{out}}{\text{Cosh } \eta_{in}} \right)^{\alpha_{\ell}} \cos(\ell - 1)\xi_j ,$$

$$\text{where } \alpha_{\ell} = \begin{cases} 0 & \text{for } \ell = 1 \\ \ell - 3/2 & \text{for } \ell \geq 2 \end{cases} .$$

With substitution of the expression written, at the bottom of the preceding sheet, for  $C_{\ell}$ , there results the working equation (for use in updating values of  $A^*$  on the outer boundary):

$$A^*(\eta_{out}, \xi_j) = \sum_i E_{j,i} A^*(\eta_{in}, \xi_i) ,$$

where the "working matrix" (a rectangular matrix) is composed of the elements

$$\begin{aligned}
E_{j,i} &= \sum_{\substack{\ell=1 \\ k=1}} \frac{\text{Sinh } n_{\text{out}}}{\sqrt{\text{Cosh } n_{\text{out}} - \cos \xi_j}} \frac{\sqrt{\text{Cosh } n_{\text{in}} - \cos \xi_i}}{\text{Sinh } n_{\text{in}}} \\
&\cdot \frac{\text{ASP}_{\ell}(n_{\text{out}})}{\text{ASP}_{\ell}(n_{\text{in}})} \left( \frac{\text{Cosh } n_{\text{out}}}{\text{Cosh } n_{\text{in}}} \right)^{\alpha_{\ell}} w_i(M_{\ell,k}^{-1}) \cos(\ell - 1)\xi_j \cos(k - 1)\xi_i \\
&= \frac{\text{Sinh } n_{\text{out}}}{\sqrt{\text{Cosh } n_{\text{out}} - \cos \xi_j}} \frac{\sqrt{\text{Cosh } n_{\text{in}} - \cos \xi_i}}{\text{Sinh } n_{\text{in}}} \\
&\cdot w_i \sum_{\ell=1} \left[ \frac{\text{ASP}_{\ell}(n_{\text{out}})}{\text{ASP}_{\ell}(n_{\text{in}})} \left( \frac{\text{Cosh } n_{\text{out}}}{\text{Cosh } n_{\text{in}}} \right)^{\alpha_{\ell}} \cos(\ell - 1)\xi_j \left[ \sum_{k=1} (M_{\ell,k}^{-1}) \cos(k - 1)\xi_i \right] \right].
\end{aligned}$$

## V. COMPUTATIONAL AIDS

In previous portions of this report reference has been made to evaluations expressed in terms of complete elliptic integrals of the first and second kinds. Coefficients for rather accurate evaluation of such elliptic integrals have been provided by C. Hastings, Jr. and cited by L.M. Milne-Thomson.<sup>\17/</sup> A somewhat more extensive sequence of such coefficients has been kindly furnished to us by Mrs. Barbara (Harold) Levine of this Laboratory and these coefficients have recently been built into three of the VAX REAL\*8 Programs cited below (ELIPM, ASHLE, and RINGF). The Programs mentioned below may be of use for illustrating or testing relationships introduced in the present report.

### ELIPM:

This program computes values of the complete elliptic integrals K and E after entering the numerical value of the parameter  $m (= k^2)$ . A working variable in the program is the complementary parameter,  $1-m$ .

### RINGC:

This Program provides the vector potential and field components of a single-turn circular loop carrying a current of 1 Ampere.

### SPOLE

This Program similarly provides values of the vector potential etc. for an assembly of several azimuthally-wound current-carrying circular coils.<sup>\18/</sup>

### ASHLE:

This Program computes values of associated Legendre functions of order  $m = 1$  and half-integral degree ( $\nu = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \frac{37}{2}$ ) in terms of  $\eta (= \text{Cosh}^{-1}z)$ . Elliptic-integral evaluations are employed to evaluate the

functions  $P_{-1/2}^1(z)$  and  $P_{1/2}^1(z)$ , followed by use of the appropriate recursion relation to compute functions of higher half-integral degree (see Appendix to Sect. III).

RINGF:

This Program similarly computes the scaled ("re-normalized") associated Legendre functions, introduced as ASP(k) in Sect. IV, in terms of  $\eta (= \text{Cosh}^{-1}z)$ .

## VI. INTRODUCING THE BOUNDARIES INTO POISSON'S MESH GENERATOR

The use of the toroidal coordinate system in solving problems with axisymmetry requires an eccentric pair of circular arcs at the boundary of such a problem (i.e., no external sources are permitted). The specification for the center and radius of one of the arcs is a matter of choice; these values are then used to compute the center and radius of the other arc, using the procedure described below.

We have chosen to assign values for  $\rho_{1,0}$  and  $R_1$  (center and radius) of the inner boundary and compute the corresponding values,  $\rho_{2,0}$  and  $R_2$  of the outer boundary. (The values of  $\rho_{1,0}$  and  $R_1$  are arbitrary as long as there are no sources outside  $R_1$ .)

Once  $R_1$  and  $\rho_{1,0}$  are known, we calculate the focal length  $a$ ;  

$$a^2 = \rho_{1,0}^2 - R_1^2.$$

As shown in the text (page 14), the minor intersection point between a circular boundary and the abscissa is  $a \cdot \text{Tanh}(\frac{\eta}{2})$ . The distance  $\Delta x$  (Fig. 2a) between two such boundaries on the abscissa is:

$$\Delta x = \rho_{1,0} - R_1 - a \cdot \text{Tanh}(\frac{\eta_{\text{out}}}{2}).$$

Assuming that  $\Delta x$  is assigned, we calculate  $\eta_{\text{out}}$ :

$$\eta_{\text{out}} = 2 \text{Tanh}^{-1} \left[ \frac{\rho_{1,0} - (R_1 + \Delta x)}{a} \right] = \ln \left[ \frac{a + \rho_{1,0} - (R_1 + \Delta x)}{a - \rho_{1,0} + (R_1 + \Delta x)} \right]$$

We can now calculate the center and radius of the outer boundary:

$$\rho_{2,0} = \frac{a}{\text{Tanh } \eta_{\text{out}}} ; \quad R_2 = \frac{a}{\text{Sinh } \eta_{\text{out}}}$$

When the mesh generator to the program POISSON is used to generate such boundaries,  $\Delta x$  can be set to the nominal grid spacing. This will assure the existence of a finite distance between the boundaries and prevent them from

collapsing into each other. It is, however, advisable to increase the mesh density at this point, which can be easily done by choosing a  $\Delta x$  that is larger by an integer multiple of the nominal grid spacing.

### Example

To demonstrate the use of the toroidal boundary condition, we have used a set of coils in a configuration shown in Fig. 2b. We have placed 1000 A in each coil in the indicated directions and computed  $A^* = \rho A$  vs.  $\rho$  at  $z = 0$ . We further computed  $B_z$  along that same path and  $B_r$  vs.  $z$  at the mid radius between the two coils. In addition, the same functions have been computed analytically for both conventional axisymmetric and cartesian geometries. The above computations were done at an increasing focal dimension (parameter  $a$ ); however, the relative position of the coils, with respect to each other and to the mesh boundaries, remained unchanged. (In all problems, a midplane symmetry is assured by specification of a Neumann boundary condition for  $A^*$  at  $z = 0$ , and the relaxation computations were then performed only in the region  $z \geq 0$ .)

#### Case A - Coils Close to the Axis

The coils were placed at  $\rho_a = 3.25$  cm ( -1000 A),  $\rho_b = 4.25$  cm ( +1000 A), with each at  $z = 0.25$  cm. The inner boundary was centered midway between the coils at  $\rho_{1,0} = 3.75$ , with a radius of  $R_1 = 1.25$ . We assumed  $\Delta x = 0.1$  and computed  $\eta_{out} = 1.65385404$  rad., so that  $\rho_{2,0} = 3.8042$  and  $R_2 = 1.4042$ .

The close proximity of the coils to the axis of symmetry in this example permitted a solution that includes the axis of symmetry and a circular type boundary condition. Flux plots for a cartesian (circular boundary condition), axisymmetry (toroidal b.c.) and axisymmetry (circular b.c.) are shown in Fig. 3. Variations in  $A^*$  are compared in Fig. 4. These variations include a comparison between two solutions that differ in the number of mesh points that

have been used. (The cartesian case is a poor approximation and is therefore omitted from Fig. 4.) Good agreement ( $< 0.5\%$ ) in  $A^*$  is obtained between theory, circular b.c., and toroidal b.c.. The values for  $B_z$  and  $B_r$  are compared in Fig. 5.

#### Case B - Medium

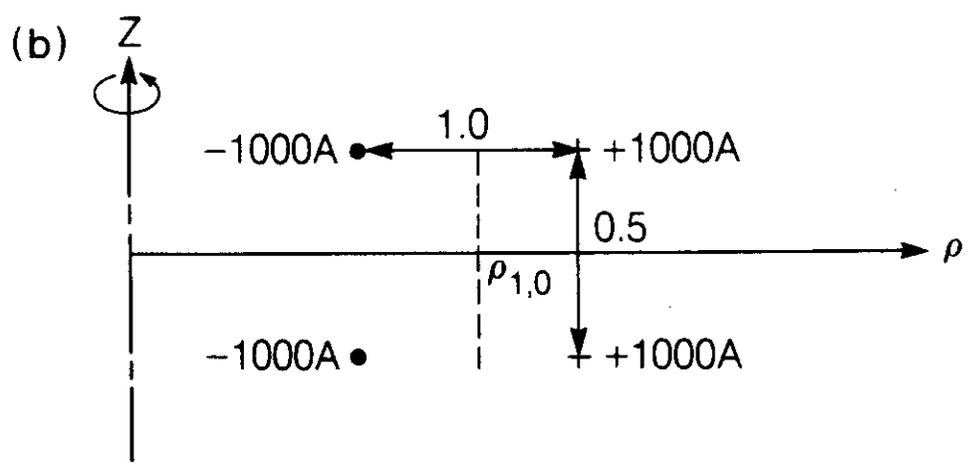
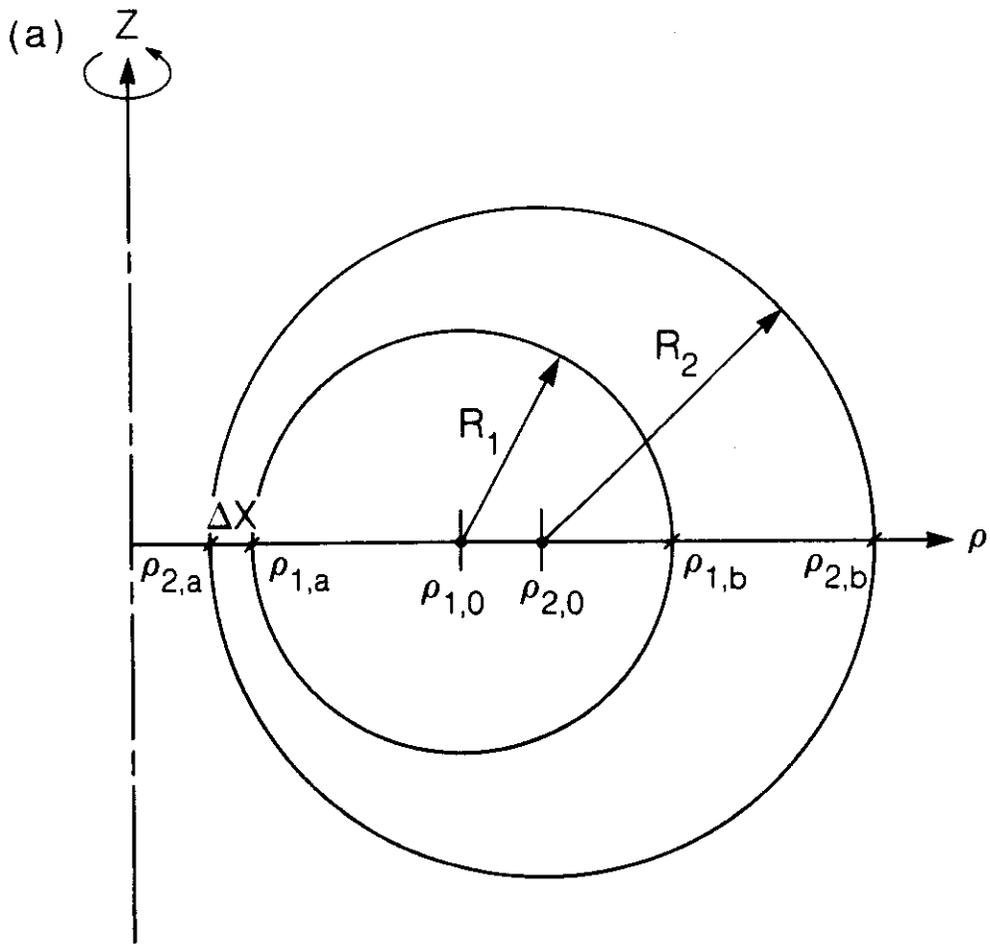
In this case, the same pair of coils was placed farther out from the axis of symmetry  $\rho_{1,0} = 25.03$  cm while maintaining the other relative dimensions. Attempts to include the axis of symmetry in the computations required a very large mesh and was therefore not used. We have, however, varied the mesh density in two cases with toroidal b.c., and compared the results with theory. Plots similar to Case A are shown in Figs. 6 and 7. Errors in  $A^*$  are  $< 0.5\%$  and variations in  $B_r$  with  $z$  are still noticeable.

#### Case C - Far

The coils are now moved to  $\rho_{1,0} = 225$  cm away from the axis of symmetry (Fig. 8). Good agreement between the toroidal case and theory is maintained ( $A^* < 0.5\%$ ).

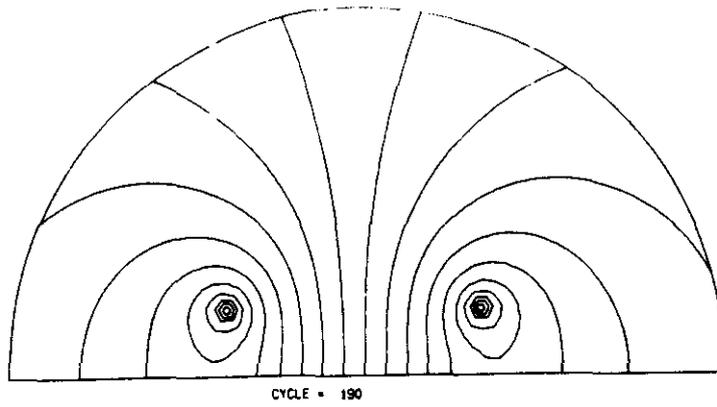
#### Case D - Very Far

Moving the coils to  $\rho_{1,0} = 100$  m maintains the accuracy of  $A^*$  at less than  $0.5\%$  (except at  $\rho = 10,000$  where  $A_{\text{theo}}^*$  is virtually zero); however, fluctuations in  $B_r$  are noticeable (Fig. 9). These fluctuations are directly related to the loss of numerical accuracy, since we iterate on  $A^*$ , which is a product of  $\rho A$  where  $\rho$  is very large.



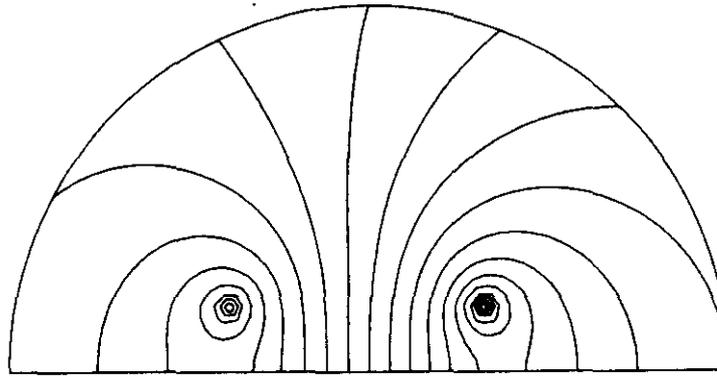
XBL 875-10166

Fig. 2. (a) The inner and outer boundary used with the toroidal coordinate system. (b) Location of the 4 current loops used in the example.



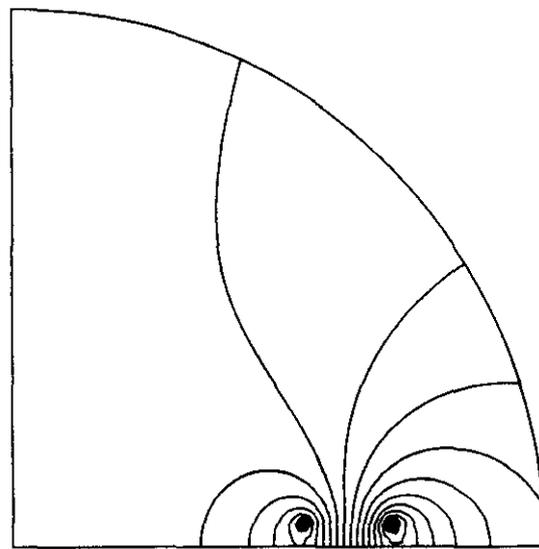
a.

XBL 875-2362



b.

XBL 875-2363

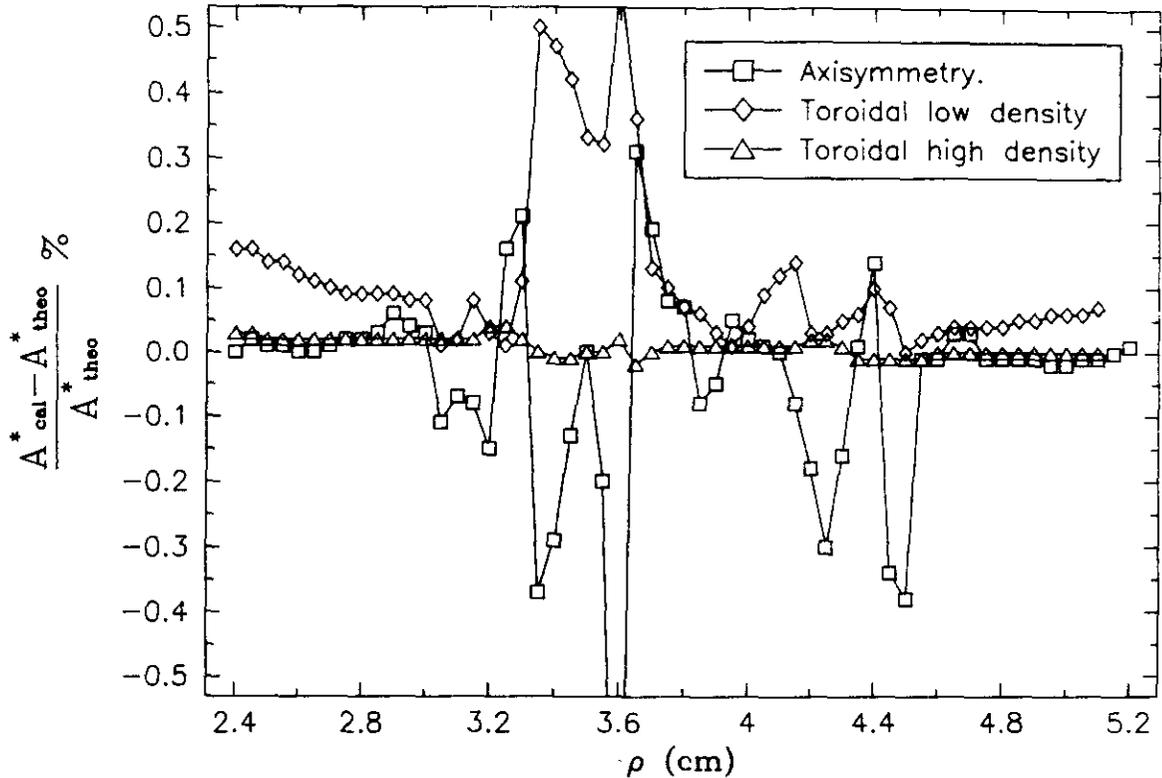


c.

XBL 875-2361

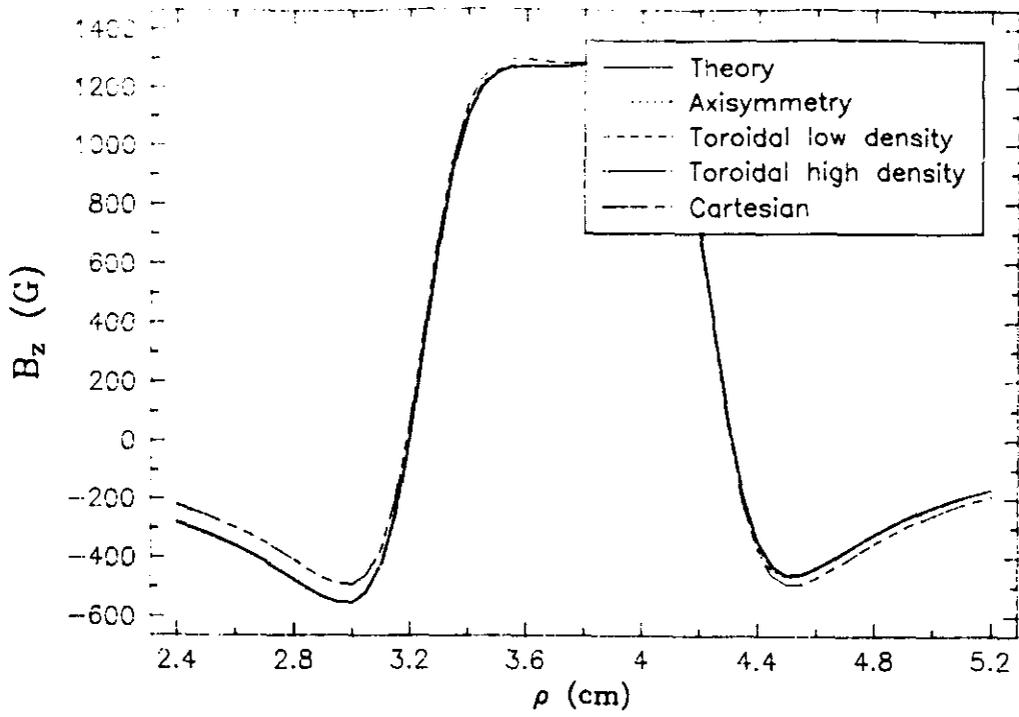
Fig. 3. Flux plot around a pair of conductors with various boundary conditions; (a) Cartesian and circular boundary (b) axisymmetry with toroidal boundary (c) axisymmetry with circular boundary (drawn to a reduced scale so as to include the axis of rotational symmetry).

*Close ,A\* distortion*  
 $\rho_{1,0} = 3.75$  (cm)

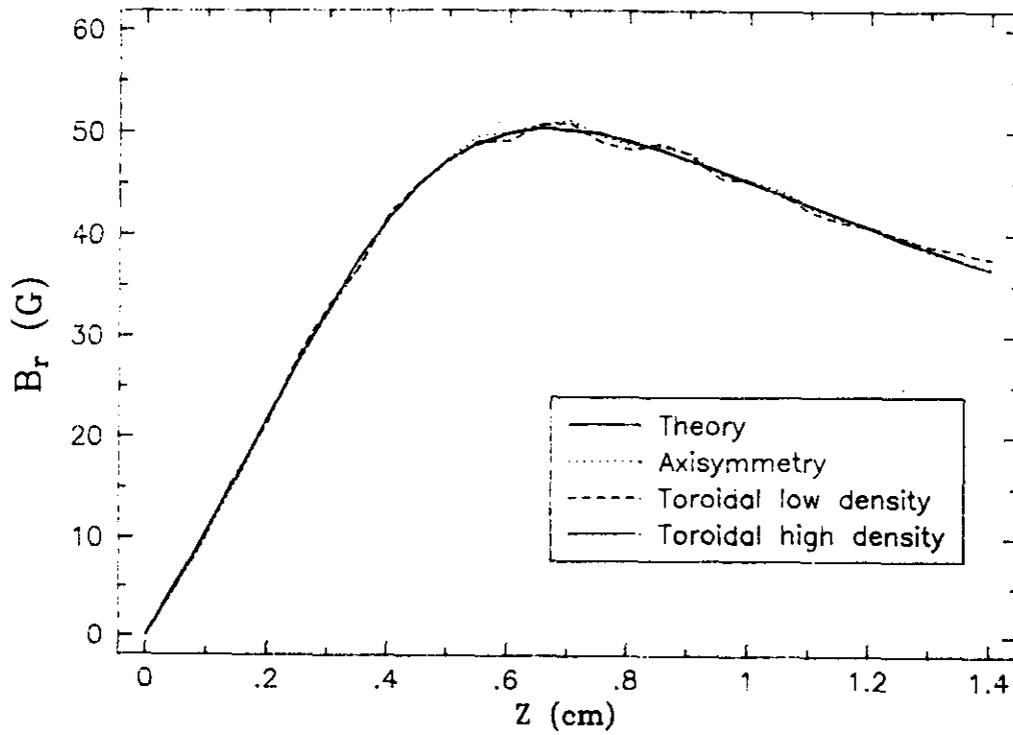


XBL 875-2374

Fig. 4. Comparison between the calculated vector potential  $A^*$  ( $= \rho A$ ) and theoretical values along the midplane of symmetry ( $z = 0$ ). The axisymmetric case includes the axis of symmetry and employs a circular boundary, whereas the toroidal case employs a circular boundary around the sources. The need for a high mesh density is evident. It is noted that numerical difficulties will arise when  $A^*$  approaches zero causing fractional errors to be large. Such difficulties are present near  $\rho_{1,0}$  for the toroidal case and exactly at  $\rho_{1,0}$  for the cartesian case. In the data presented here, no attempt was made to overcome such difficulties and large fractional errors near  $\rho_{1,0}$  accordingly do not reflect a real difference between the computed and expected values.

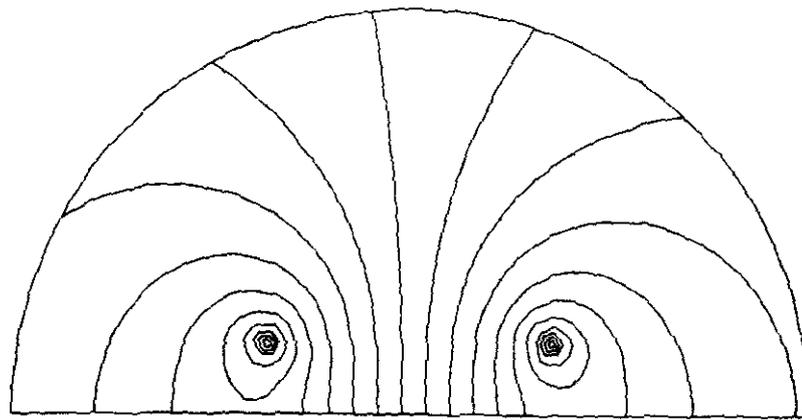


XBL 875-2372



XBL 875-2373

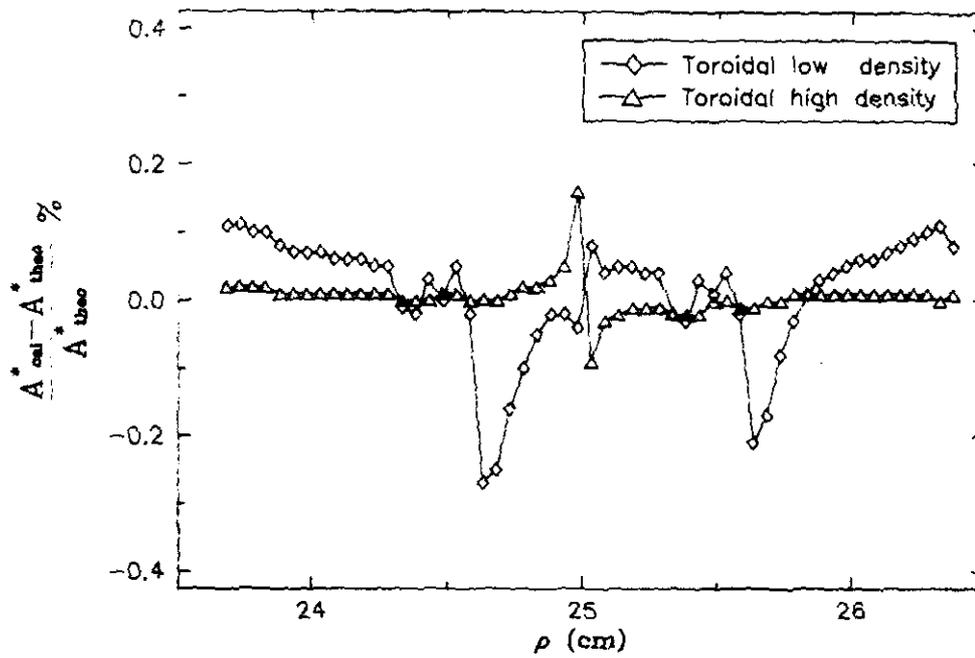
Fig. 5 The magnetic flux density in the  $z$  (top) and  $\rho$  (bottom) directions for the close case ( $\rho_{1,0} = 3.75$  cm). Case (a) is a scan at  $z = 0$  and case (b) along  $\rho = 3.75$  (Note that  $B_r$  for the Cartesian case is 0).



CYCLE = 640

XBL 875-2364

Medium,  $A^*$  distortion  
 $\rho_{1,0} = 25.03$  (cm)



XBL 875-2371

Fig. 6 Flux plot (top) and vector potential (bottom) for the toroidal case farther removed from the axis of symmetry.

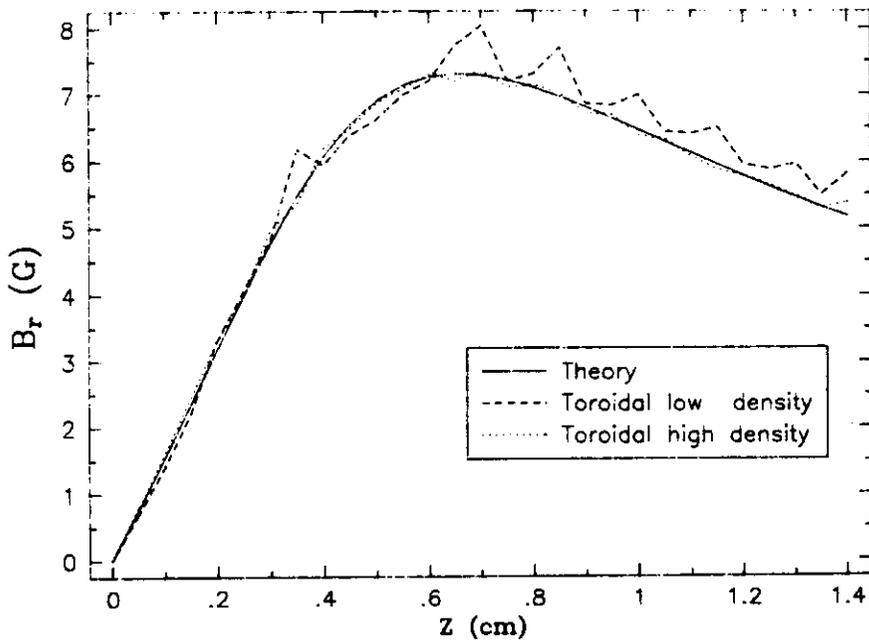
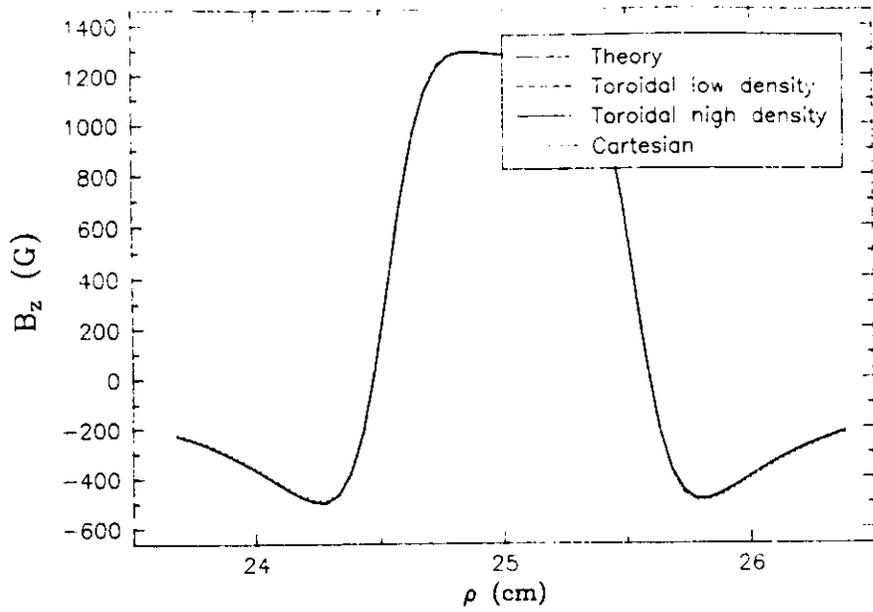
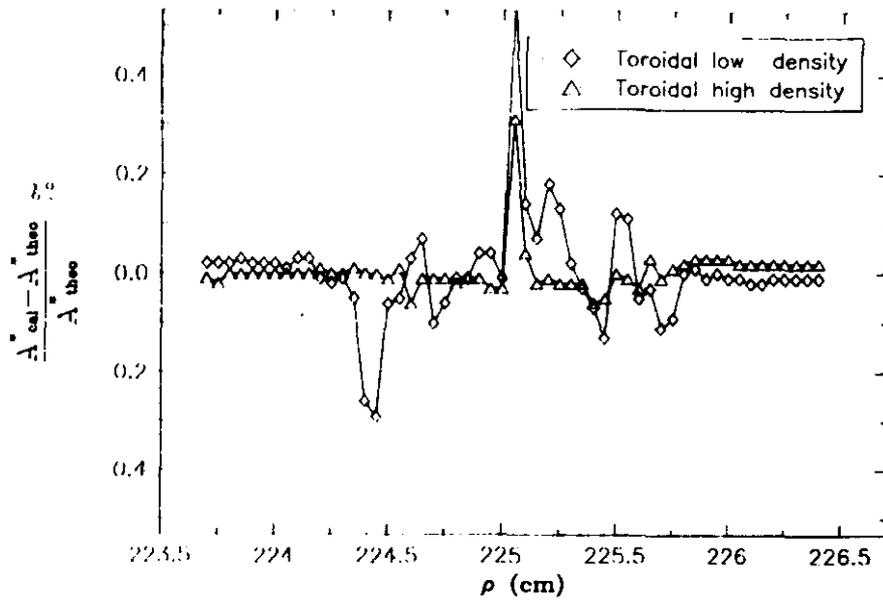
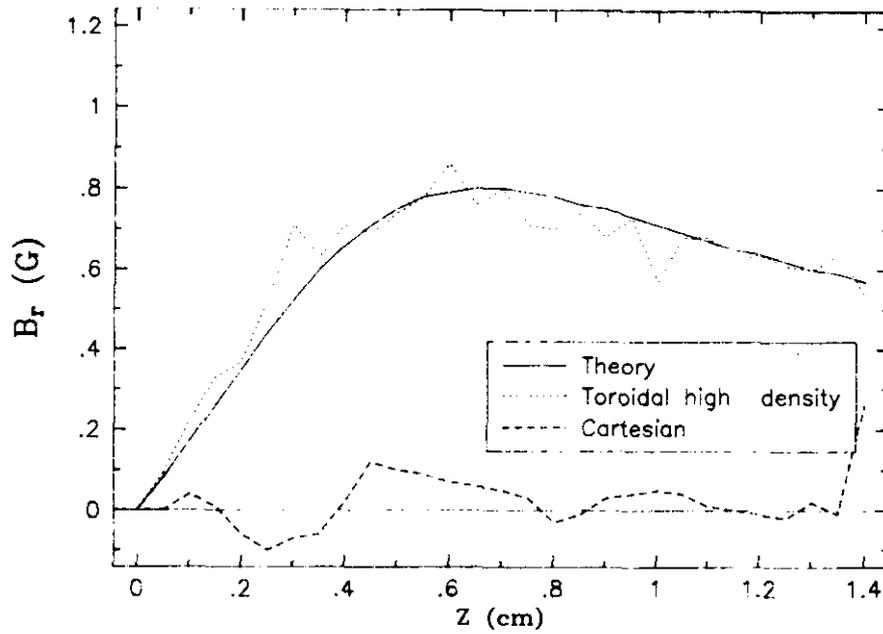


Fig. 7 The magnetic flux density in the z direction along  $z=0$  is shown on top at a distance of  $\rho_{1,0} = 25.03$  cm from the axis of symmetry (Case B -- "medium"). The difference between Cartesian and axisymmetry is barely noticeable. Flux values in the  $\rho$  direction along  $\rho = 25.03$  (bottom) are distinguishable.

Var,  $A^*$  distortion  
 $\rho_{1,0}$  225 (cm)



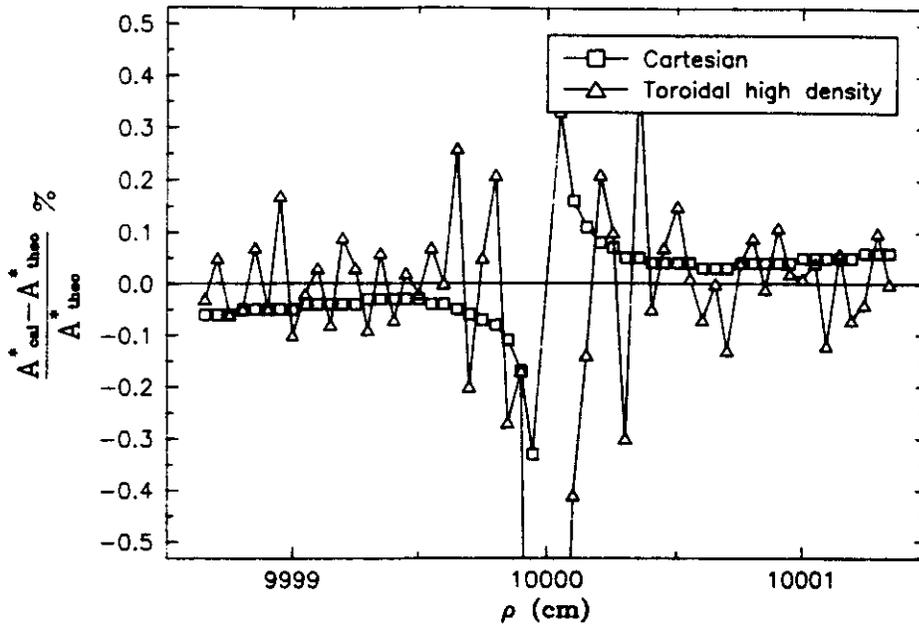
XBL 875-2368



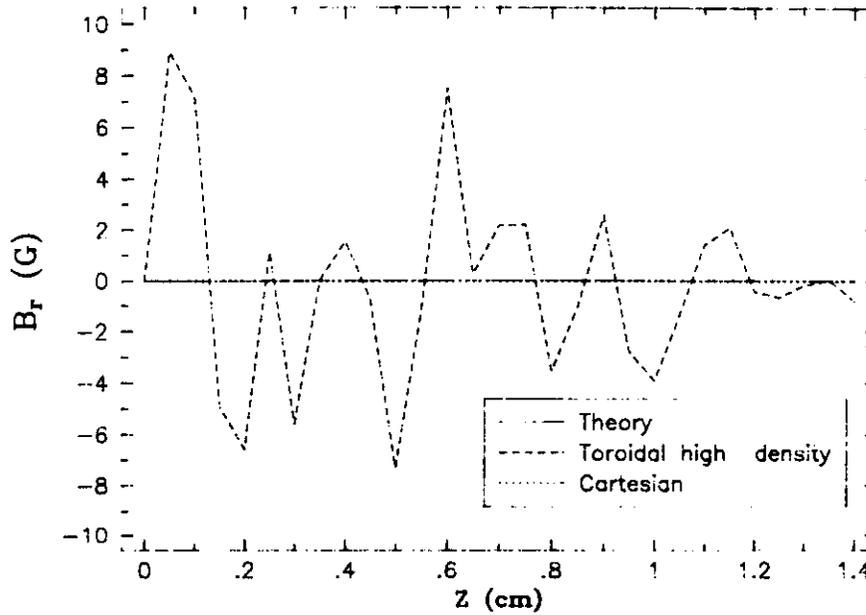
XBL 875-2369

Fig. 8. Vector potential and field values for a case where the coils have been extended to  $\rho_{1,0} = 225$  cm.

Very Far,  $A^*$  distortion  
 $\rho_{1,0} = 10000$  (cm)



XBL 875-2370



XBL 875-2365

Fig. 9. Placing the coils at  $\rho_{1,0} = 100$  m (Case D) produces the above quality of the vector potential and field. Note that  $B_r$  should be almost zero and the large fluctuations are due to loss in numerical accuracy when  $\rho$  is very large.

## APPENDIX TO SECTION VI

### Calculation of Field Components

POISSON solves for  $A$  or  $\rho A$  ( $A^*$ ), depending on the coordinate system chosen. The quantities of interest to the magnet designer or user may also include components of the field  $B$ , expressed by  $B_r$  and  $B_z$  for a problem solved with circular cylindrical symmetry. The field editor in POISSON expresses the potential in the neighborhood of a point of interest as a sum of a series of "harmonic" polynomials; the components of  $B$  are found by taking appropriate derivatives of this series. The procedure is described in section B.13.2.2 of reference \19/. The harmonic polynomials used in this series expansion involve powers of  $\rho$  that may result in exponent over- or underflow when calculated on a computer, particularly when  $\rho$  is greater than 100. In order to moderate the degradation in the field editor we employ double precision calculations, and scale the quantities  $\rho$ ,  $z$ , and  $A^*$ . The scaling is done so that  $\rho \approx 1.0$ ; one might think of it as a temporary change in units. The scaling does not affect the harmonic character of the polynomials used in the series expansion. The quantity  $A^*$  and the calculated components of  $B$  are then scaled back to appropriate units for the field edit report.

## VII. REFERENCES AND NOTES

- A. Milton Abramowitz and Irene A. Stegun (Eds.), "Handbook of Mathematical Functions" (U.S. Nat. Bu. Stds.; and Dover, N.Y.).
- B. G. Arfken, "Math. Methods for Physicists," Ed. 2 (Academic Press, N.Y.; 1970).
- C. Bateman Manuscript Project (A. Erdélyi, Ed.), v. 1, "Transcendental Functions" (McGraw-Hill, N.Y.; 1953).
- D. Henry E. Fettis and James C. Caslin, "Tables of Toroidal Harmonics," U.S.A.F. Aerospace Res. Labs. (Wright-Patterson AFB, Ohio; December 1969).
- E. T. M. MacRobert, "Spherical Harmonics," Ed. 2 (Dover, N.Y.; 1947).
- F. Math. Tables Project, "Tables of Associated Legendre Functions" (Columbia Univ. Press, N.Y.; 1945).
- G. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics" (McGraw-Hill, N.Y.; 1953).
- H. W. R. Smythe, "Static and Dynamic Electricity," Ed. 1 (McGraw-Hill, N.Y.; 1939) or Ed. 2 (1950).

Notes:

\1/ Previous applications of a boundary condition similar to that introduced here have been described, for other coordinate systems, in the following reports of the Lawrence Berkeley Laboratory:

ESCAR-28 (1975); L. Jackson Laslett  
LBID-172 (1980); L. Jackson Laslett, with Victor O. Brady  
SSC-MAG-5/LBL-17064 (1984); Laslett, Caspi, and Helm  
SSC-MAG-12/LBL-18063 (1984); do.  
SSC-MAG-28/LBL-18798 (1984); do.  
SSC-MAG-31/LBL-19050 (1985); do.  
SSC-MAG-41/LBL-19483 (1985); do.  
SSC-MAG-51/LBL-19172 (1985); do.  
SSC-MAG-68/LBL-20893 (1985); do.

\2/ Ref. B, Sect. 2.13.

\3/ Ref. H, Fig. 4.13 (p. 75).

\4/ Ref. E, Chapt. XII, Sect. 5, pp. 228-230.

\5/ Limiting forms for the functions  $P_{\nu}^1(z = \cosh \eta)$  and  $Q_{\nu}^1(z = \cosh \eta)$ , in the limit of small argument, are given in Sect. 3.9.2 of Ref. C, p. 163, and the behavior of such functions also is illustrated in Ref. F. Specifically, the cited formulas indicate that the functions  $P_{\nu}^1$  approach proportionality to  $(z - 1)^{1/2}$ , or to  $\eta$ , as  $\eta \rightarrow 0$  and  $z \rightarrow 1$ , while the functions  $Q_{\nu}^1$  approach proportionality to  $(z - 1)^{-1/2}$  or to  $\frac{1}{\eta}$  in this limit.

The formula

$$P_{\nu=n-1/2}^1(z = \text{Cosh } \eta) \rightarrow \frac{\nu(\nu+1)}{2} \eta = \frac{(n-1/2)(n+1/2)}{2} \eta$$

for the limiting form approached by the functions of the first kind as  $n \rightarrow \infty$  may be independently derived directly from formulas (cited elsewhere in this report) that relate such functions to complete elliptic integrals - - thus:

$$P_{-1/2}^1 = \frac{1}{\pi} \frac{E_A - K_A}{\text{Sinh } \frac{\eta}{2}} \rightarrow -\frac{\eta}{8}$$

$$P_{1/2}^1 = \frac{1}{\pi} \frac{(\text{Cosh } \eta) E_A - K_A}{\text{Sinh } \frac{\eta}{2}} \rightarrow +\frac{3\eta}{8} \quad (\text{for } \eta \rightarrow 0)$$

with parameter  $m_A = (\text{Tanh } \frac{\eta}{2})^2 \cong \frac{\eta^2}{4}$ ,

$$\text{Cosh } \eta \cong 1 + \frac{\eta^2}{2}, \quad E_A \cong \frac{\pi}{2} \left[ 1 - \frac{m_A}{4} \right], \quad \& \quad K_A \cong \frac{\pi}{4} \left[ 1 + \frac{m_A}{4} \right];$$

and (by induction) one then can extend the evaluations to higher degree through use of the recursion relation for Legendre functions of varying degree. This same small- $\eta$  result for  $P_{\nu=n-1/2}^1(z = \text{Cosh } \eta)$  may also be obtained by development of the formula

$$P_{\nu=n-1/2}^m(\text{Cosh } \eta) = \frac{\Gamma(n+m+1/2)(\text{Sinh } \eta)^m}{\Gamma(n-m+1/2) 2^m \sqrt{\pi} \Gamma(m+1/2)}$$

$$\int_0^\pi \frac{(\sin \phi)^{2m} d\phi}{(\text{Cosh } \eta + \cos \phi \text{ Sinh } \eta)^{n+m+1/2}}$$

cited by Irene A. Stegun as Eqn. (8.11.2) in Chapt. 8 of Ref. A., p. 336.

\6/ The vector potential  $A_\phi$  of a single centered current-carrying loop of radius  $a$  has been shown by Smythe [Ed. 2; Ref. H, Sect. 7.10] to be given (in rationalized MKS units) by

$$\frac{\pi}{\mu I} A_\phi = \left( \frac{1}{m_B} \frac{a}{\rho} \right)^{1/2} \left[ \left( 1 - \frac{m_B}{2} \right) K_B - E_B \right],$$

wherein we have elected to employ (in place of Smythe's "modulus"  $k$ ) the "parameter"

$$m_B = k^2 = \frac{4a\rho}{(a + \rho)^2 + z^2}.$$

In terms of toroidal coordinates (for a coordinate system in which the characteristic dimension "a" is identical to the radius of the current-carrying loop), this elliptic-integral parameter may be written

$$m_B = \frac{2 \operatorname{Sinh} \eta}{\operatorname{Cosh} \eta + \operatorname{Sinh} \eta} = \frac{2\sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}}$$

where  $z = \operatorname{Cosh} \eta$ .

\7/ An equivalent elliptic-integral expression for  $A_\phi$  can be written in terms of complete elliptic integrals of parameter

$$m_A = \left( \frac{1 - \sqrt{1 - m_B}}{1 + \sqrt{1 - m_B}} \right)^2,$$

which then for the present problem becomes

$$m_A = \left( \operatorname{Tanh} \frac{\eta}{2} \right)^2 = \frac{z - 1}{z + 1}.$$

The elliptic-integral expression introduced above in Note \6/ for the vector potential  $A_\phi$  then may be transformed [through use of formulas cited by

L. M. Milne-Thomson as Eqns. (17.3.29-30) of Chapt. 17 in Ref. A, p. 591]

as follows:

$$K_B = \frac{2}{1 + \sqrt{1 - m_B}} K_A$$

$$E_B = (1 + \sqrt{1 - m_B}) E_A - 2 \frac{\sqrt{1 - m_B}}{1 + \sqrt{1 - m_B}} K_A$$

$$\begin{aligned} \left[ \left( 1 - \frac{m_B}{2} \right) K_B - E_B \right] &= (1 + \sqrt{1 - m_B}) (K_A - E_A) \\ &= \frac{2}{1 + \sqrt{m_A}} (K_A - E_A) \end{aligned}$$

so that the expression for  $A_\phi$  may be written

$$\begin{aligned} \frac{\pi}{\mu I} A_\phi &= \sqrt{\frac{(\cosh \eta + \sinh \eta)(\cosh \eta - \cos \xi)}{2 \sinh^2 \eta}} \frac{2}{1 + \tanh \frac{\eta}{2}} [K_A - E_A] \\ &= (\cosh \eta - \cos \xi)^{\frac{1}{2}} \frac{\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2}}{\sqrt{2} \sinh \frac{\eta}{2} \cosh \frac{\eta}{2}} \frac{1}{1 + \frac{\sinh \eta/2}{\cosh \eta/2}} [K_A - E_A] \\ &= (\cosh \eta - \cos \xi)^{\frac{1}{2}} \frac{K_A - E_A}{\sqrt{2} \sinh \frac{\eta}{2}} \end{aligned}$$

The associated Legendre function  $P_{-\frac{1}{2}}^1(z = \cosh \eta)$  can, moreover, be written in terms of complete elliptic integrals of parameter  $m_A$  as

$$P_{-\frac{1}{2}}^1(z = \cosh \eta) = \frac{1}{\pi} \frac{E_A - K_A}{\sinh \frac{\eta}{2}} \quad (\text{see Appendix to Sect. III of this report}),$$

so that the vector potential in this particular example is seen to be given by a numerical constant times the single term

$$(\cosh \eta - \cos \xi)^{\frac{1}{2}} P_{-\frac{1}{2}}^1(z = \cosh \eta) .$$

\8/ Irene A. Stegun, in Chapt. 8 of Ref. A. In writing (p. 337) Eqns. 8.13.1 or 8.13.2 for  $P_{-\frac{1}{2}}(z = \text{Cosh } \eta)$  and Eqns. 8.13.5 or 8.13.6 for  $P_{\frac{1}{2}}(z = \text{Cosh } \eta)$ , Ms. Stegun has elected to express the elliptic integrals explicitly as functions of their modulus. Elliptic integral parameters, such as are cited in the body of the present report, are the square of the respective values of the moduli.

\9/ The relations cited by L. M. Milne-Thomson as Eqns. 17.3.29 and 17.3.30 in Chapt. 17 of Ref. A (P. 591) are for elliptic integrals whose parameters are related (as here) by

$$m_B = \frac{4 \sqrt{m_A}}{(1 + \sqrt{m_A})^2} \quad \text{or} \quad m_A = \left( \frac{1 - \sqrt{1 - m_B}}{1 + \sqrt{1 - m_B}} \right)^2$$

\10/ See Irene A. Stegun, Eqn. 8.6.6 in Chapt. 8 of Ref. A, p. 334.

\11/ See Irene A. Stegun, Eqn. 8.5.3 in Chapt. 8 of Ref. A, p. 334.

\12/ Ref. C, Sect. 3.9.2, formula (19), p. 164.

\13/ Ref. F and Ref. D. The values of Legendre functions of the second kind, as tabulated in Ref. D, may be related to values of functions of the first kind through use of the equality

$$P_{\mu-\frac{1}{2}}^{\nu}(z) = \frac{(s^2 - 1)^{\frac{1}{4}}}{\sqrt{\pi/2}} e^{-i\mu\pi} \frac{Q_{\nu-\frac{1}{2}}^{\mu}(s)}{\Gamma(\mu - \nu + \frac{1}{2})}$$

where  $s = z/\sqrt{z^2 - 1}$ , or specifically (with  $\nu$  replaced by 1 and  $\mu$  replaced by an integer  $n$ )

$$P_{n-\frac{1}{2}}^1(z) = \frac{[4(s^2 - 1)]^{\frac{1}{4}} 2^{n-1}}{\pi} \frac{(-1)^n Q_{\frac{1}{2}}^n(s)}{1.3.5 \dots (2n - 3)} \quad \text{for } n \geq 2 ;$$

$$P_{\frac{1}{2}}^1(z) = \frac{[4(s^2 - 1)]^{\frac{1}{4}}}{\pi} [-Q_{\frac{1}{2}}^1(s)] \quad \text{for } n = 1 ; \text{ and}$$

$$P_{-\frac{1}{2}}^1(z) = \frac{[4(s^2 - 1)]^{\frac{1}{4}}}{\pi} \frac{1}{2} [-Q_{\frac{1}{2}}^{(0)}(s)] \quad \text{for } n = 0 .$$

[We note that the associated Legendre functions  $Q$ , as tabulated in Ref. D, intentionally permit the order to exceed the degree.]

\14/ See Sect. 3 of the Appendix to Sect. III.

\15/ See Sect. 1 of the Appendix to Sect. III.

\16/ See Sect. II.

\17/ L. M. Milne-Thomson, in Ref. A, Chapt. 17, pp. 591-592 (esp. Eqns. 17.3.34 and 17.33.36).

\18/ The present VAX version of Program SPQLE is based on a similar CDC-6600 Program prepared for use in the Electron-Ring-Accelerator project of this Laboratory (see ERAN-151; 1971).

\19/ "Reference Manual for the POISSON/SUPERFISH Group of Codes", Los Alamos Accelerator Code Group, LA-UR-87-126 (Jan 1987).