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EFFECT OF TEMPERATURE AND STRESS ON THE LENGTH OF A BAR WITH A TEMPERATURE-DEPENDENT YOUNG'S MODULUS

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INTRODUCTION

We consider a bar made of a material having a linear stress-strain relationship the slope -- Young's modulus -- of which is temperature dependent, and for which an equation of state is applicable -- the change in length is a function only of the initial and final stresses and temperatures, and is independent of the stress-temperature path between the two states.

We develop a relationship between the initial and final lengths, stresses, and temperatures in terms of the material properties. At the expense of working the fool thing half to death, this relationship is developed by several methods, all of which -- curiously enough -- yield the same result.

NOMENCLATURE

σ	longitudinal stress (+ = tension)
ϵ	increase in length divided by initial length resulting from combined effect of stress and temperature
E	Young's modulus, $(\partial\sigma/\partial\epsilon)_{T=const.}$
α	coefficient of linear thermal expansion, $(\partial\epsilon/\partial T)_{\sigma=0}$
T	temperature

FOR STARTERS

We express strain as

$$\epsilon = \epsilon(\sigma, T, E, \alpha)$$

We regard E as a function only of T, independent of stress (a severe limitation, perhaps, for the coil material), and α also as a function only of T. σ and T, then, are the independent variables, ~~so Eq. 1~~ this reduces to

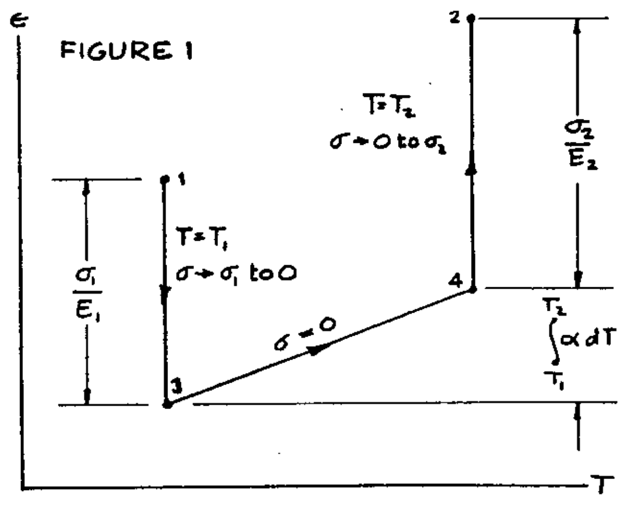
$$\epsilon = \epsilon(\sigma, T)$$

the differerential of which is

$$d\epsilon = (\partial\epsilon/\partial\sigma)_T d\sigma + (\partial\epsilon/\partial T)_\sigma dT \qquad (1)$$

SOLUTION 1

Since the process from State 1 to State 2 is path-independent, we select a convenient path, 1-3-4-2 in Fig. 1. (The order is scrambled so that 1 and 2 represent the end points of the process.)



The Process 1-3 (State 1 to State 3) occurs at temperature T_1 during which the stress is reduced from σ_1 to zero. The resulting increase in strain, by definition of E_1, is

$$\epsilon_3 - \epsilon_1 = -\sigma_1 / E_1$$

Process 3-4 occurs at zero stress as the temperature changes from T_1 to T_2. The resulting increase in strain, by definition of α, is

$$\epsilon_4 - \epsilon_3 = \int_{T_1}^{T_2} \alpha\, dT$$

Process 4-2 occurs at temperature T_2 during which the stress is increased from zero to σ_2. The resulting increase in strain, by definition of E_2, is

$$\epsilon_2 - \epsilon_4 = \sigma_2 / E_2$$

The difference in strain between State 1 and State 2 is

$$\epsilon_2 - \epsilon_1 = (\epsilon_2 - \epsilon_4) + (\epsilon_4 - \epsilon_3) + (\epsilon_3 - \epsilon_1)$$

or

$$\epsilon_2 - \epsilon_1 = \frac{\sigma_2}{E_2} - \frac{\sigma_1}{E_1} + \int_{T_1}^{T_2} \alpha \, dT \qquad (2)$$

SOLUTION 2

The term $(\partial\epsilon/\partial\sigma)_T$ in Eq. 1 is simply 1/E. The term is evaluated by a process similar to that of Solution 1 but applied to differentials and for a constant stress (Fig. 2).

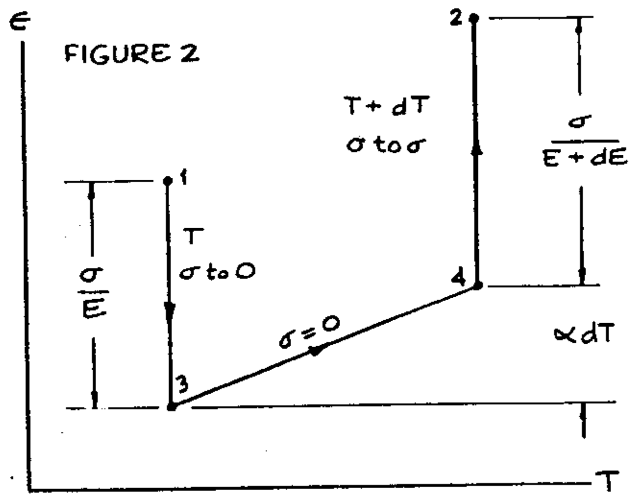


FIGURE 2

Process 1-3 Temp = T, stress → σ to zero:

$$\epsilon_3 - \epsilon_1 = \sigma/E$$

Process 3-4 Temp → T to T+dT, stress = zero:

$$\epsilon_4 - \epsilon_3 = \alpha \, dT$$

Process 4-2 Temp = T+dT, stress → zero to σ :

$$\epsilon_2 - \epsilon_4 = \frac{\sigma}{E + dE} = \frac{\sigma}{E}\left(1 - \frac{dE}{E}\right)$$

The change in strain from State 1 to State 4 is

$$\begin{aligned} d\epsilon &= \epsilon_2 - \epsilon_1 = (\epsilon_2 - \epsilon_4) - (\epsilon_1 - \epsilon_3) + (\epsilon_4 - \epsilon_3) \\ &= \frac{\sigma}{E}\left(1 - \frac{dE}{E}\right) - \frac{\sigma}{E} + \alpha \, dT \\ &= -\frac{\sigma}{E^2} \, dE \quad + \alpha dT \\ &= \sigma \, d(1/E) + \alpha \, dT \end{aligned}$$

Since σ was held constant in this process,

$$(\partial \epsilon / \partial T)_\sigma = d\epsilon/dT \text{ above } = \sigma \frac{d(1/E)}{dT} + \alpha$$

Substituting that into Eq. 1 along with $(\partial \epsilon/\partial \sigma)_T = 1/E$ we obtain

$$d\epsilon = \frac{1}{E} d\sigma + \sigma d\left(\frac{1}{E}\right) + \alpha \, dT$$

which is easier to integrate when written in the form

$$d\epsilon = d\left(\frac{\sigma}{E}\right) + \alpha \, dT \qquad (3)$$

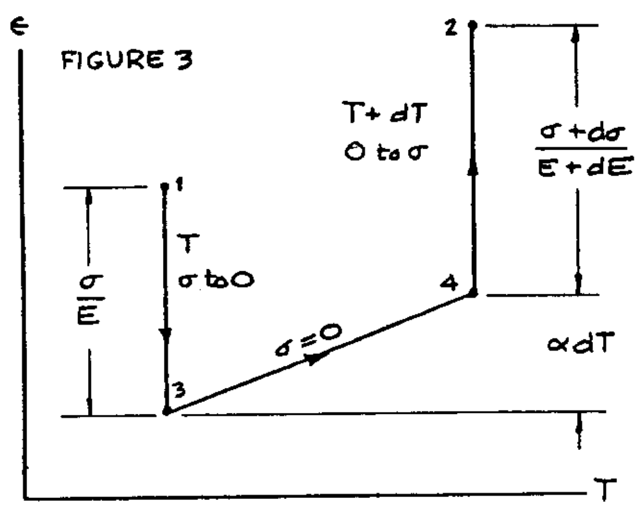
Upon integration this yields

$$\epsilon_2 - \epsilon_1 = \frac{\sigma_2}{E_2} - \frac{\sigma_1}{E_1} + \int_{T_1}^{T_2} \alpha \, dT$$

which is identical to Eq. 2 of Solution 1.

SOLUTION 3

Again we apply the procedure of Solution 1 to differentials, but we let the stress increase from σ to $\sigma + d\sigma$ as the temperature increases from T to T+dT (Fig. 3).



Processes 1-3, 3-4: same as in Solution 2

Process 4-2:

$$\epsilon_2 - \epsilon_4 = \frac{\sigma + d\sigma}{E + dE} = \frac{\sigma}{E}\left(1 + \frac{d\sigma}{\sigma}\right)\left(1 - \frac{dE}{E}\right)$$

which to first order is

$$\epsilon_2 - \epsilon_4 = \frac{\sigma}{E}\left(1 + \frac{d\sigma}{\sigma} - \frac{dE}{E}\right)$$

The change in strain from State 1 to State 4 is

$$d\epsilon = \epsilon_2 - \epsilon_1 = (\epsilon_2 - \sigma_4) + (\epsilon_4 - \epsilon_3) - (\epsilon_1 - \epsilon_3)$$

$$d\epsilon = \frac{\sigma}{E}\left(1 + \frac{d\sigma}{\sigma} - \frac{dE}{E}\right) - \frac{\sigma}{E} + \alpha T = \cdots = d\left(\frac{\sigma}{E}\right) + \alpha dT$$

which is identical to Eq.3 of Solution 2.

SOLUTION 4

Equation 1 is a perfect differential, and so the following applies:

$$\frac{\partial}{\partial T}\left(\frac{\partial \epsilon}{\partial \sigma}\right) = \frac{\partial}{\partial \sigma}\left(\frac{\partial \epsilon}{\partial T}\right) \qquad (4)$$

Since E is a function only of T,

$$\frac{\partial}{\partial T}\left(\frac{\partial \epsilon}{\partial \sigma}\right)_T = \frac{\partial}{\partial T}\left(\frac{1}{E}\right) = \frac{d}{dT}\left(\frac{1}{E}\right)$$

and so .

$$\frac{d}{dT}\left(\frac{1}{E}\right) = \frac{\partial}{\partial \sigma}\left(\frac{\partial \epsilon}{\partial T}\right)$$

The expression for $\partial\epsilon/\partial T$ that satisfies this equation and also gives $(\partial\epsilon/\partial T)_{\sigma=0} = \alpha$ is

$$\frac{\partial \epsilon}{\partial T} = \sigma\frac{d(1/E)}{dT} + \alpha$$

and so Eq. 1 becomes

$$d\epsilon = \frac{1}{E}d\sigma + \sigma d(1/E) + \alpha dT = d(\sigma/E) + \alpha dT$$

which is identical to Eq. 3 of Solution 2