Fitting Errors and Chromaticity Correction

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Introduction

Correction of the chromaticity requires that the chromaticity function first be determined. Conceptually, this involves measuring the tune shift at several energy deviations, and then fitting a polynomial to the measurements. This note addresses the problem of errors introduced into the fit by measurement errors, and also by restricting the energy range over which the data points are taken. The optimum location for data points is also considered.

The technique of fitting a polynomial to an unknown function by minimizing the squared difference between the two functions is well known. Let $f(x)$ be the unknown function and $f_n(x)$ be the approximating polynomial of order n. If $f(x)$ can be determined with complete accuracy eveywhere in the interval $-1 < x < 1$, then the fitting polynomial is given by

$$f_n(x) = \sum_{m=0}^{n} a_m \, P_m(x) \; , \tag{1}$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} dx \; f(x) \; P_m(x) \; . \tag{2}$$

Here $P_m(x)$ is the Legendre polynomial. This fit gives the polynomial $f_n(x)$ which minimizes the integrated square of the fitting error $\epsilon = f(x) - f_n(x)$ over the interval $-1 < x < 1$. If it was desired to make a fit which weighted the consideration of ϵ differently at different x, then a different set of polynomials would appear instead of the Legendre polynomials. This method is applicable to a general interval by scaling the endpoints to -1 and 1.

Now consider the more physical problem where $f(x)$ is not given everwhere, but is measured at N points x_i, with measurement errors at each point. The optimum choice of the x_i are the abscissas for Gaussian integration[1]: The N samples should be taken at the N roots of the polynomial $P_N(x)$. This allows the integral of equation (2) to be computed to order $2N-1$, and hence the coefficients a_m can be calculated exactly when $f(x)$ is a polynomial of order up to $2N-1-n$. For example, using five samples to fit $f_3(x)$, one has $N = 5$ and $n = 3$ and the fit is correct if $f(x)$ is a polynomial of order six or less. The coefficients a_m are given by

$$a_m = \frac{2m+1}{2} \sum_{i=1}^{N} w_i \, f(x_i) \, P_m(x_i) \, , \tag{3}$$

where w_i is the integration weight associated with x_i. Note that a least squares fit can be made to any set of data points, but then no statement can be made about the agreement of the unknown function and the fitting function elsewhere than at the data points. Only in the case of data points selected at the abscissas x_i can the connection be made to the fitting of the function $f(x)$ over the entire interval.

Fitting of Chromaticity Measurements

The chromaticity function is $\Delta\nu = f(\delta)$, where $\Delta\nu$ is the tune shift and δ is the fractional energy deviation. In the case of the SSC, $f(\delta)$ may be fit with a cubic polynomial. We shall assume that five data points are available, including the implicit point $f(0) = 0$. The remaining four points are assumed to have an rms tune error of 10^{-4}. The fitting polynomial is required to have an rms tune error of 10^{-3} everywhere in the domain $|\delta| < 10^{-3}$[2]. Since $N = 5$ and $n = 3$, the fitting algorithm described above is valid as long as the actual chromaticity function f can be represented as a polynomial of order 6.

First consider the response of the fitted polynomial f_n to an error at one of the data points. This does not consider the accuracy of the fit itself: We do not

consider yet whether or not f is well represented by a polynomial of given order, but only how this accuracy is degraded by random measurement error. Hence, the error response is independent of the form of f. Figures 1–3 show the response of f_2, f_3, and f_4 to unit errors at the two positive x_i. The response to errors at the negative x_i are mirror image curves to these, and the point at $x_i = 0$ is assumed to be free of error. The variable x has been arbitrarily scaled from δ, so $x = \delta/\delta_0$. The case of figure 3 is illustrative of the fact that f_4 is exactly determined by five data points: The response is one at the x_i which contains the error, and zero at all the other x_i. However, the error response diverges rapidly outside of the sampling range of the data points. For $n = 3$ and $n = 2$, the error response is less, and diverges less rapidly. In general, a better fit within the fitting interval is paid for by a more rapid divergence of the fitting function outside of the fitting interval. In figure 5, the four error curves for the $n = 3$ case are summed in quadrature to find the rms error for the actual f_3 fit. The criterion of 10^{-4} measurement error and 10^{-3} fitting error at $\delta = \pm 10^{-3}$ shows that the scale factor δ_0 must be at least 0.6. Since the most extreme sampling location is at $x = 0.91$, the data must be taken over the energy range of $\pm 5.5 \times 10^{-4}$ in order to meet the specifications. For comparison, in figures 4 and 6, the error response and the sum in quadrature are shown for $N = 7$ and $n = 3$. The difference between figures 5 and 6 suggests that the rms fitting error goes down as $\sqrt{N}$, which is consistent with the improvement in error statistics as more data is taken.

Now consider the accuracy of the fit in the absence of noise. If a fit is made with f_n, then f can be thought of as the sum of a polynomial of order n and a remainder. The polynomial is fitted exactly, and the remainder can then be treated as error in the same way as measurement error. In the problem being considered here, $n = 3$, and f is assumed to be a polynomial of order 6 or less. Hence the error response to the terms δ^4, δ^5, and δ^6 can be calculated. These are shown in figures 7–9. The error term is scaled so that it is equal to one at $x = 1$. The important feature is that the fit does not over-respond to the error: Outside

of the fitting interval, the response is of the same sign and smaller in magnitude than the error. This means that any higher power error which is small enough to be ignored in itself will not interfere with the accuracy of the fit. Since the choice of $n = 3$ was based on the assumption that the terms of f of power four and higher can be neglected, one may conclude that f_3 gives a satisfactory fit to f in the absence of measurement errors, and the actual accuracy is determined by those errors.

Fitting of Chromaticity from Continuous Data

Since the restriction on the accuracy of the N point fit described above is the measurement error of the data, the remedy is to aquire as much tune data as possible. The technique envisioned here is to modulate the machine energy sinusoidally for an integer number of cycles M at a frequency sufficiently low that the beam will follow the modulation adiabatically. While this is being done, the tune data is continuously logged by an automatic procedure, and then Fourier analysed:

$$\Delta\nu(\Theta) = \sum_{m=0}^{\infty} \Delta\nu_m \cos m\Theta \ , \tag{4}$$

where $\Theta = \Omega t$, and Ω is the modulating frequency of the order of 1 $\sec^{-1}$ or less.

The time dependent data is of the form

$$\Delta\nu(\Theta) = f(\delta_0 \cos\Theta) \ , \tag{5}$$

and δ_0 is the maximum energy excursion. Now fit $\Delta\nu$ with a polynomial in δ over the range $|\delta| < \delta_0$, where $\delta = \delta_0 \cos\Theta$. Since the energy variation is sinusoidal, the data is concentrated near the extremes of the energy excursion. Hence, the fit is done with the Chebyshev polynomials $T_n(x)$, where $x = \delta/\delta_0$. These give a fit weighted by the function $W(x) = (1-x^2)^{-1/2}$ reflecting the data distribution.

The fit to order n is

$$f_n(x) = \sum_{m=0}^{n} a_m \, T_m(x) \; , \tag{6}$$

$$a_m = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_{-1}^{1} dx \, W(x) \, \Delta\nu(\Theta) \, T_m(x) \; , & m = 0 \; , \\[2ex] \dfrac{2}{\pi} \displaystyle\int_{-1}^{1} dx \, W(x) \, \Delta\nu(\Theta) \, T_m(x) \; , & m \neq 0 \; , \end{cases} \tag{7}$$

$$W(x) = \frac{1}{\sqrt{1 - x^2}} \; . \tag{7a}$$

Making the change of variable $x = \cos\Theta$ in equation (7), and using the relation $T_m(\cos\Theta) = \cos m\Theta$, one finds that

$$a_m = \Delta\nu_m \; , \tag{8}$$

where the $\Delta\nu_m$ are defined by equation (4). Hence, the coefficients of the Chebyshev fit are exactly the Fourier coefficients of the time dependent tune.

Now consider the effect of particle distributions on the measured tune. In practice, the betatron motion will have a spectrum, and the tune recorded will be some average tune. For purposes of automatic data logging, we shall assume that the measured tune is the centroid of the spectrum:

$$\bar{\nu} = \int_{-\infty}^{\infty} d\nu \; \nu \; P(\nu) \; , \tag{9}$$

where $P(\nu)$ is the normalized power spectrum of the transverse beam motion signal. Contributions to the spectrum from the amplitude dependent betatron tune spread are not of interest here, because the entire spectrum will move as

the energy shifts. However, the contribution to the spectrum from the energy spread is important because the energy spread smears the chromaticity function f. This produces an apparent chromaticity function $\bar{f}$, defined as $\Delta\bar{\nu} = \bar{f}(\delta)$. Hence, the fitting technique described above will generate a fit to $\bar{f}$ rather than f.

To evaluate $\bar{f}$, consider the particles to have an energy distribution $\rho(\delta - \bar{\delta})$, where $\bar{\delta}$ is the externally applied energy shift. When δ is mapped into $\Delta\nu$ by the chromaticity function f, the result is a distribution of betatron tunes

$$\rho_\nu(\Delta\nu) = [f'(\delta)]^{-1}\ \rho(\delta - \bar{\delta})\ . \tag{10}$$

If the particles have uncorrelated betatron motion, then the power spectrum $P(\nu)$ will be proportional to the tune distribution. Then the centroid tune shift is

$$\begin{aligned} \delta\bar{\nu} = \bar{f}(\bar{\delta}) &= \int_{-\infty}^{\infty} d\nu\ \nu\ [f'(\delta)]^{-1}\ \rho(\delta - \bar{\delta}) \\ &= \int_{-\infty}^{\infty} d\delta\ f(\delta)\ \rho(\delta - \bar{\delta})\ . \end{aligned} \tag{11}$$

This indicates that $\bar{f}$ is just the convolution of f with ρ.

When f and $\bar{f}$ are represented by polynomials, the convolution in equation (11) can be inverted easily. The polynomials are

$$\begin{aligned} f(\delta) &= \sum_{m=0}^{n} \xi_m\ \delta^m\ , \\ \bar{f}(\delta) &= \sum_{m=0}^{n} \bar{\xi}_m\ \delta^m\ , \end{aligned} \tag{12}$$

and substituting into (11), a matrix equation for the coefficients results:

$$\bar{\xi}_m = \sum_{m'=0}^{n} A_{mm'}\ \xi_{m'}\ , \tag{13}$$

$$A_{mm'} = \begin{cases} 0 \,, & m > m' \,, \\ 1 \,, & m = m' \,, \\ \binom{m'}{m} \, \langle \rho \rangle_{m'-m} \,, & m < m' \,. \end{cases} \tag{13a}$$

The $\langle \rho \rangle_m$ are the moments of ρ:

$$\langle \rho \rangle_m = \int_{-\infty}^{\infty} d\delta \; \delta^m \; \rho(\delta) \,. \tag{13b}$$

The matrix A may be considered to be well-conditioned as long as

$$\xi_m \, \langle \rho \rangle_m \ll \delta\nu_{max} \,, \tag{14}$$

where $\delta\nu_{max}$ is the maximum tune variation over the interval of interest. This assures that the contribution to $\bar{\xi}_m$ from higher power terms is relatively small compared to the contribution from ξ_m, and hence significant structure of f was not removed by the smearing. This condition is satisfied if the distribution ρ is narrow enough, or if f is smooth enough.

Analysis of Noise

The accuracy of measurement is limited by noise. The noise may be in the instruments, the actual beam motion, or in the accelerator components. The two common types of noise are white noise and $1/f$ noise. White noise has a power spectrum which is constant with frequency, and the noise power is proportional to the frequency range over which the noise is observed. Hence, white noise can be reduced by making a measurement over a narrow bandwidth: If data is averaged over a time T, then the bandwidth of the measurement is approximately $1/T$, and the noise amplitude is proportional to $1/\sqrt{T}$. Examples of white noise are thermal noise in conductors and the charge quantization noise of the particle beam. A more difficult problem is provided by $1/f$ noise. This noise has a power spectrum which is inversely proportional to frequency, and

so it is only significant when measuring quantities which are constant or slowly varying. This noise is also called 'drift', and occurs in all electronic amplifiers. The $1/f$ noise has the property that the power contained in a neighborhood of zero frequency is infinite. This is not paradoxical, since no signal has exactly zero frequency. In practice, the signal is turned off at some time while the measuring apparatus is calibrated. Therefore, the actual minimum frequency of the signal is approximately the reciprocal of the time since the apparatus was calibrated.

To calculate how noise scales with measurement time, assume that data is taken for a time T and represented as a Fourier series. The signals for white noise and $1/f$ noise, respectively, are

$$a_w(t) = \frac{1}{2}\, a_w \sqrt{\Delta\omega} \sum_{|n|<n_{max}} e^{i\phi_n}\, e^{in\Delta\omega t} \,, \tag{15}$$

$$a_d(t) = \frac{1}{2}\, a_d \sum_{|n|<n_{max}, n\neq 0} |n|^{-\frac{1}{2}}\, e^{i\phi_n}\, e^{in\Delta\omega t} \,, \tag{16}$$

where $\Delta\omega = 2\pi/T$ is the minimum bandwidth which can be resolved, and the noise bandwidth in both sums is cut off at $\omega_{max} = n_{max}\Delta\omega$ in order to make the total noise power finite. The ϕ_n are uncorrelated random phases, with $\phi_n = -\phi_{-n}$. The sums have been normalized such that $\frac{1}{2}a_w^2$ and $\frac{1}{2}a_d^2/\omega$ are the noise power per unit bandwidth. The arbitrarily large constant term of $a_d(t)$ has been omitted. This constant must be eliminated in any measurement. Consider the measurement of a dc signal s, where one half of the data interval is used for calibration with s absent, and the other half of the interval is used to measure s. Then the total signal including noise is

$$s(t) = s\ \Theta(t) + a_w(t) + a_d(t) \,, \quad -\frac{T}{2} < t < \frac{T}{2} \,, \tag{17}$$

where $\Theta(t)$ is the unit step function. The measured value of s is the difference of the average signal during the measurement interval and the average signal during

the calibration interval:

$$s_{meas} = \frac{2}{T}\left[\int_0^{\frac{T}{2}} dt\ s(t) - \int_{-\frac{T}{2}}^{0} dt\ s(t)\right] . \tag{18}$$

The mean square measurement over many trials is found by averaging over the random phases ϕ_n:

$$\begin{aligned}\langle s^2_{meas}\rangle &= s^2 + \frac{16}{\pi}T^{-1}a_w^2\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2} + \frac{8}{\pi^2}a_d^2\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3} \\ &\approx s^2 + \frac{16}{\pi}T^{-1}a_w^2 + \frac{8}{\pi^2}\,a_d^2 .\end{aligned} \tag{19}$$

This gives rms errors from the two types of noise:

$$\epsilon_w = \frac{4}{\sqrt{\pi T}}\,a_w\ , \quad \epsilon_d = \frac{\sqrt{8}}{\pi}\,a_d\ . \tag{20}$$

The rms error due to white noise decreases in proportion to $1/\sqrt{T}$ as the measurement time increases, but the rms error due to $1/f$ noise is constant. This means that there is a minimum error which can be obtained for one measurement cycle, regardless of its duration. The error can still be reduced by using multiple measuring cycles.

The tune measurement error of 10^{-4} cited above sets an upper bound on $1/f$ noise of $a_d < 10^{-4}$ for this problem. Returning to the problem of automatic tune logging, define the rms error ϵ_m occurring in the measurement of each Fourier component $\Delta\nu_m$. Assuming that Ω is in the range where only $1/f$ noise is important, the errors are

$$\epsilon_m = \frac{1}{\sqrt{2mM}}\,a_d\ , \tag{21}$$

where M is the number of cycles used in the measurement, with the total measurement time $T = 2\pi M/\Omega$. Then let $\epsilon(x)$ represent the rms error of the Chebyshev

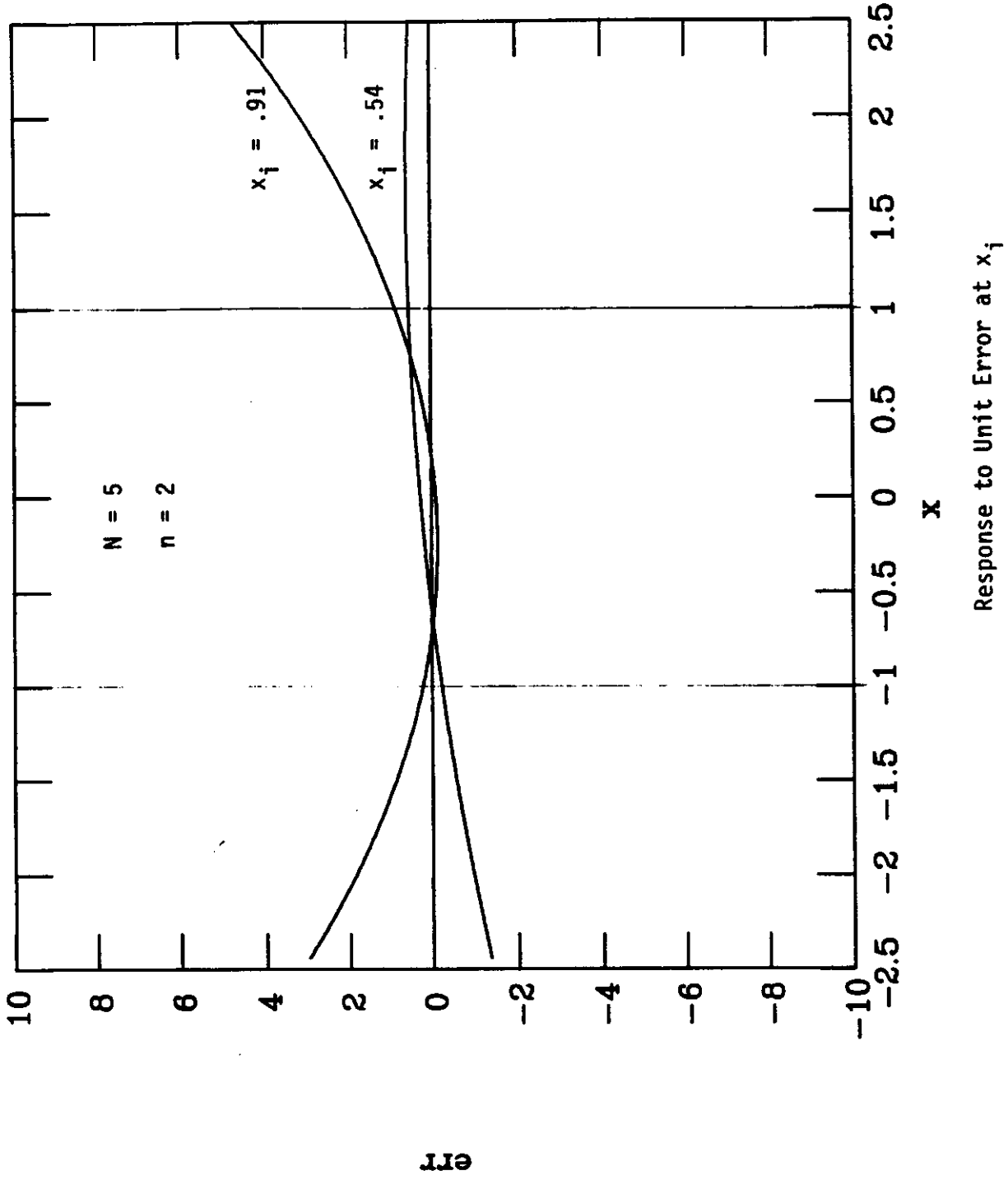
fit, recalling that $x = \delta/\delta_0$, and that δ_0 is the maximum deviation of the energy modulation. Figure 10 shows the scaled rms fitting error $\epsilon(x)\sqrt{M}/a_d$. Using again the criterion that $\epsilon < 10^{-3}$ in the range $|\delta| < 10^{-3}$, and also $a_d < 10^{-4}$ as suggested above, one can see from figure 10 that with optimum data processing, using $M = 100$ measurement cycles would make a value of $\delta_0 = 0.25$ possible, corresponding to an energy sweep of $\pm 2.5 \times 10^{-4}$. If the modulation frequency $\Omega = 1\ \mathrm{sec}^{-1}$ is practical, then this measurement would take ten minutes.

References

1. Abramowitz and Stegun, Handbook of Mathematical Functions, Section 25.4

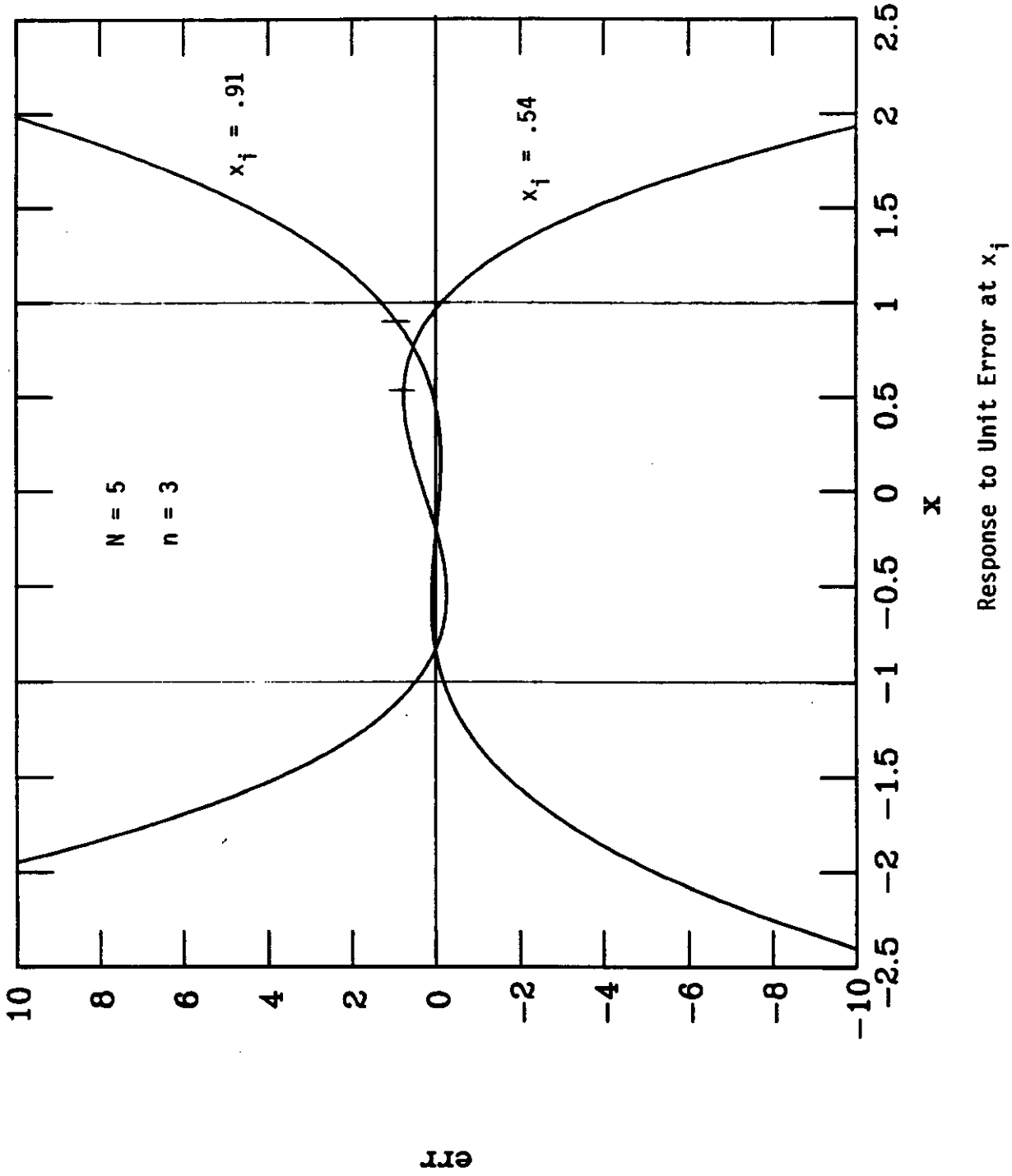
2. The values of n and N and the error criteria were suggested by J. Peterson.

Figure 1.



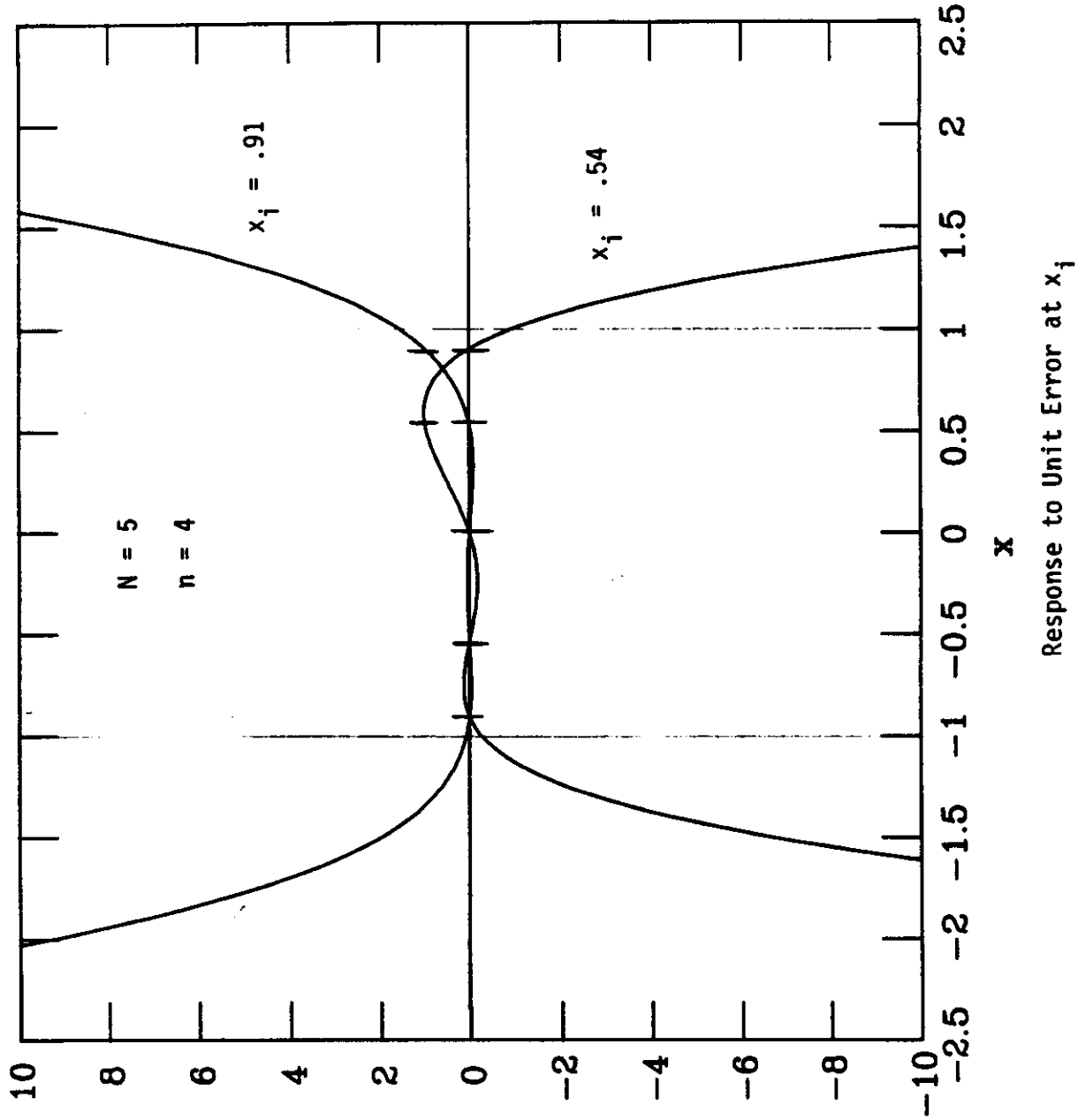
Response to Unit Error at x_i

Figure 2.



Response to Unit Error at x_i

Figure 3.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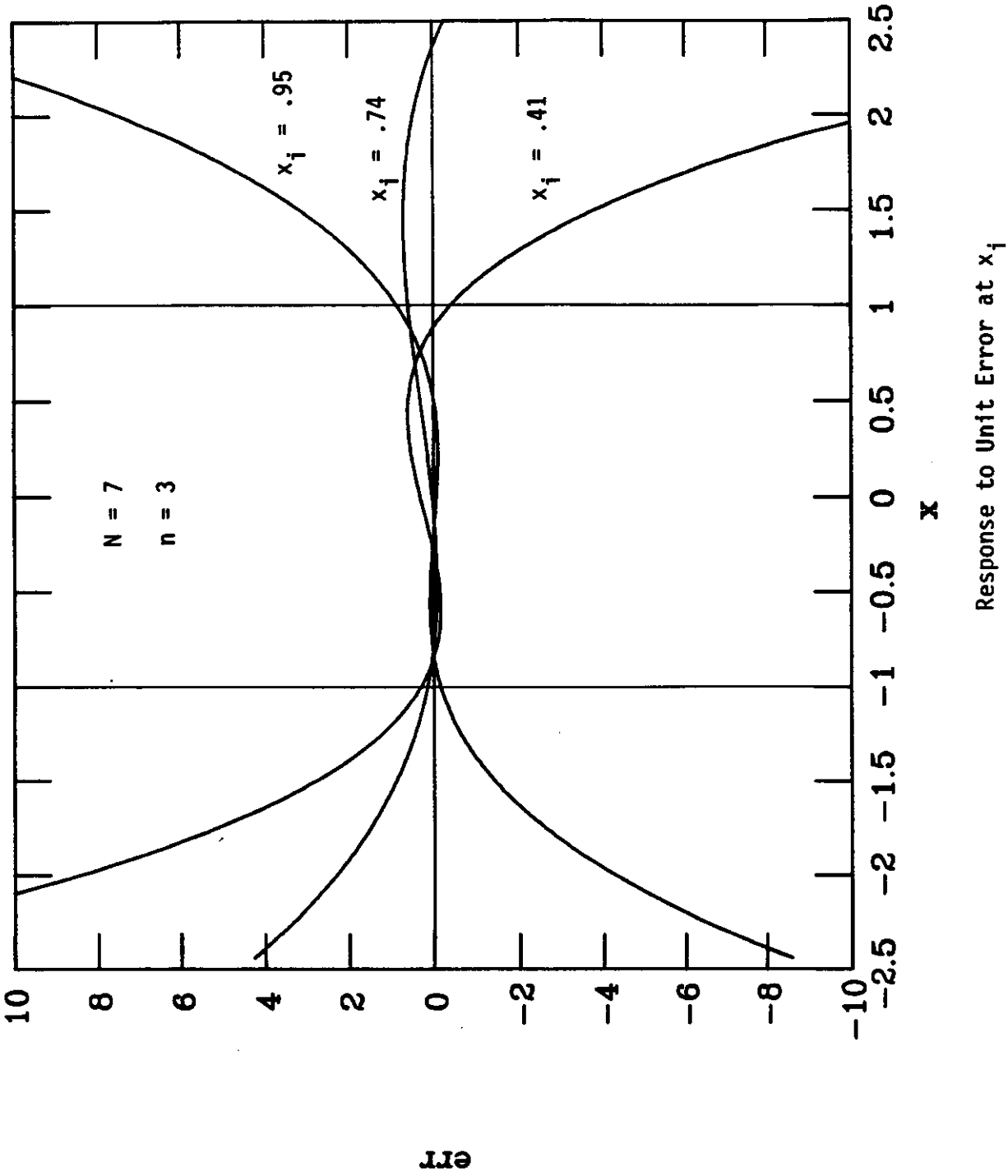
n = 4
x_i = .91
x_i = .54
err
x
Response to Unit Error at x_i
10
8
6
4
2
0
−2
−4
−6
−8
−10
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5

Figure 4.



Response to Unit Error at x_i

Figure 5.

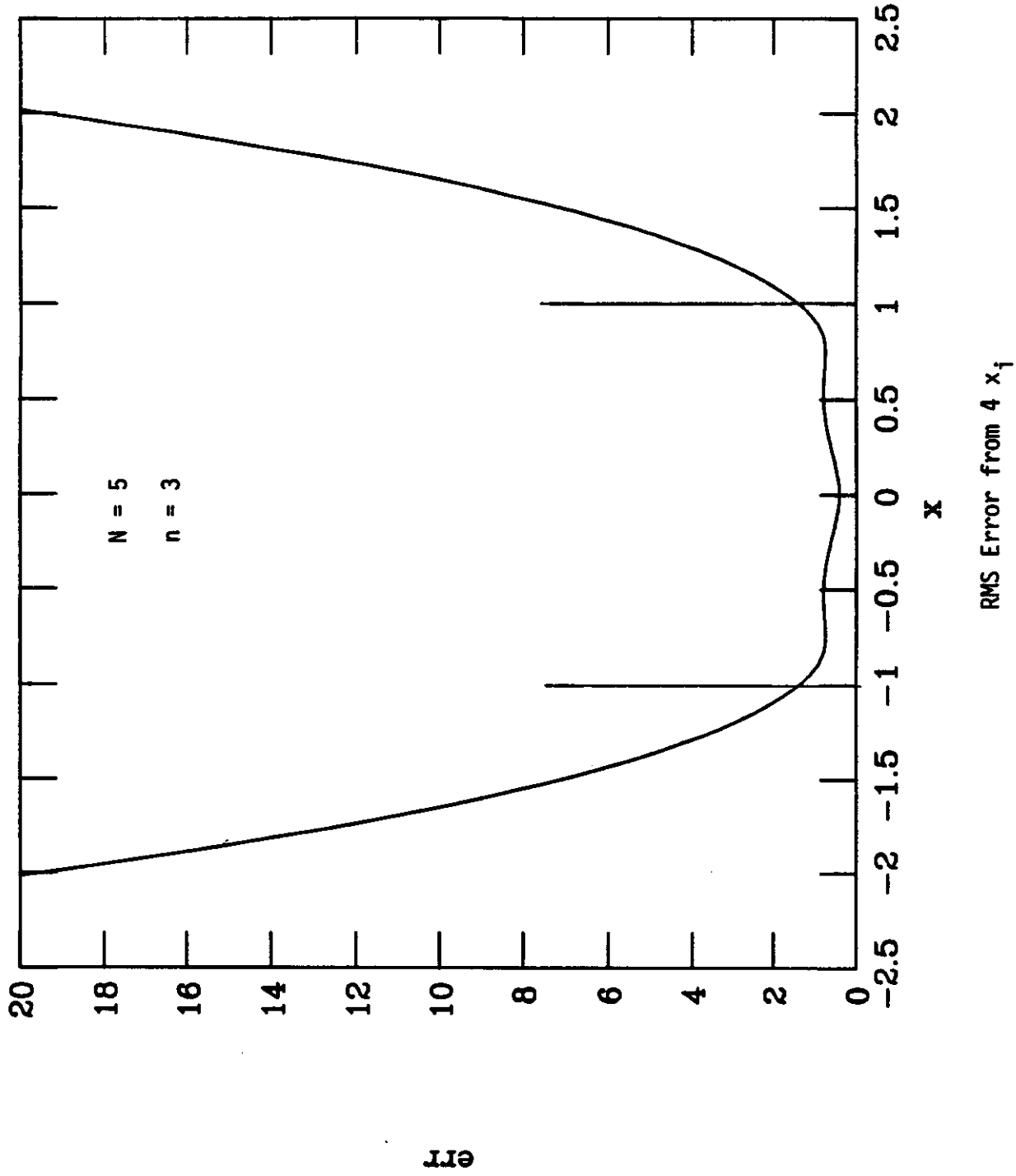

N = 5
n = 3
20
18
16
14
12
10
8
6
4
2
0
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
x
err
RMS Error from 4 x_i

Figure 6.

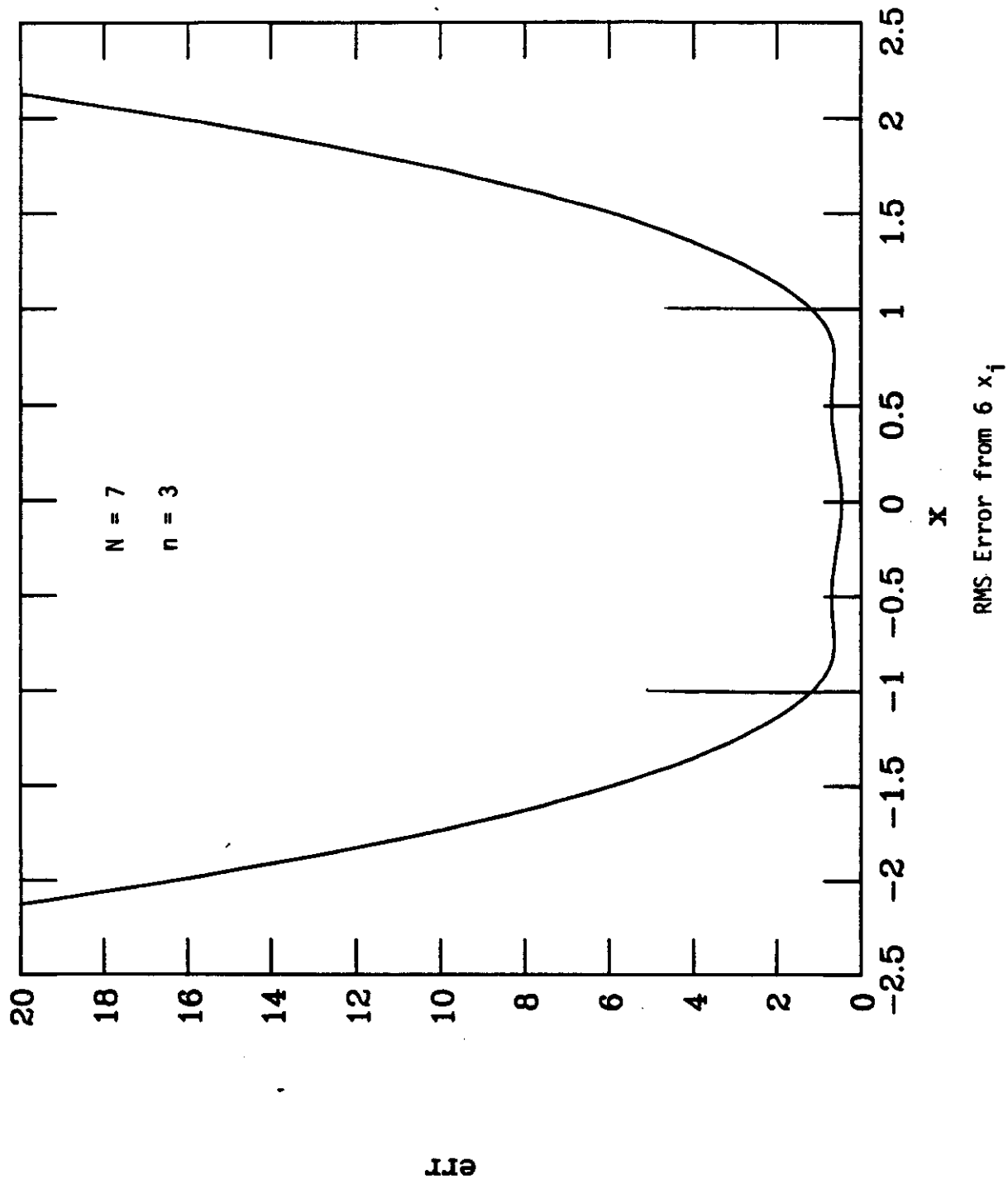
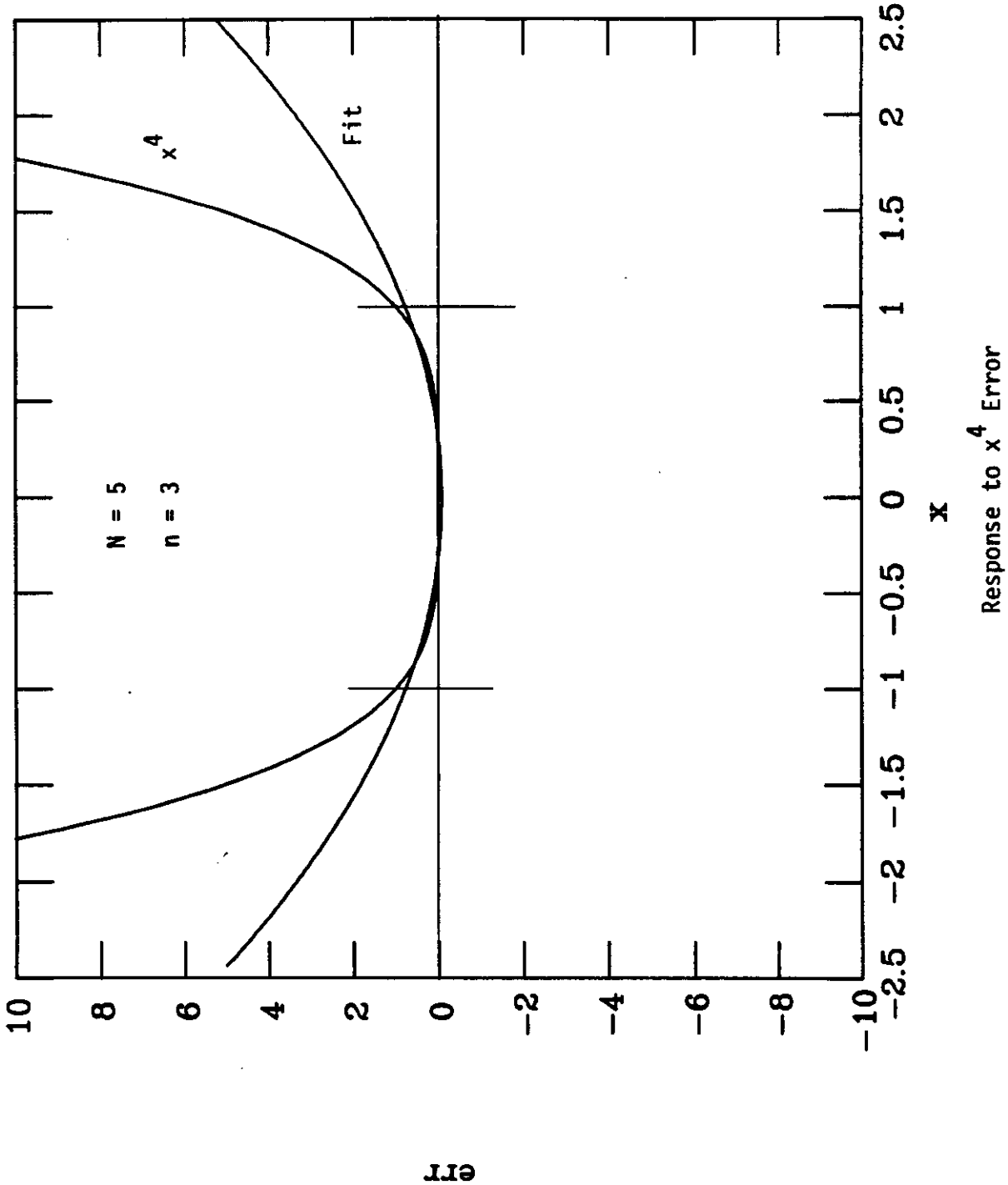

N = 7
n = 3
20
18
16
14
12
10
8
6
4
2
0
err
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
x
RMS Error from 6 x_i

Figure 7.



Response to x^4 Error

Figure 8.

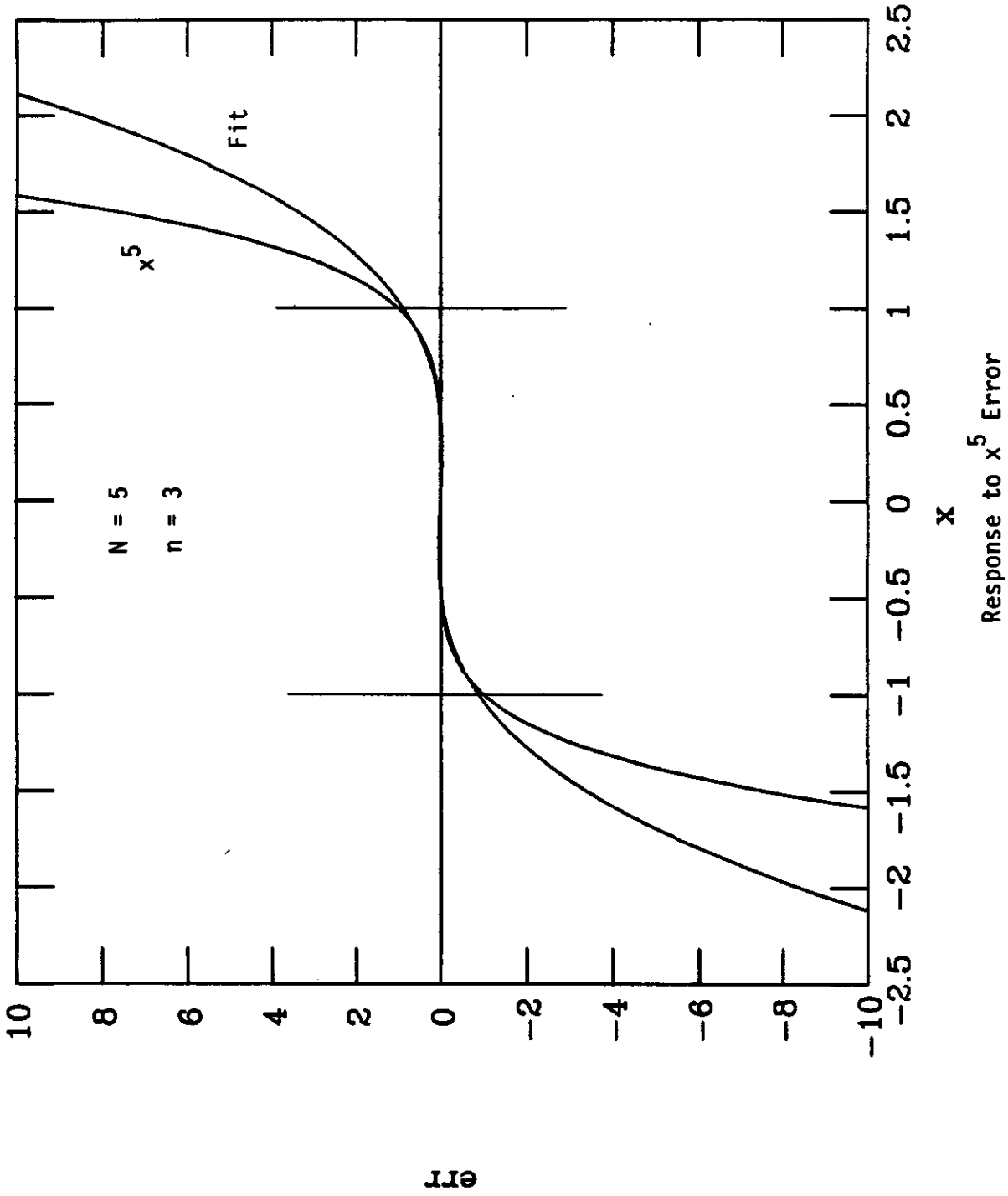
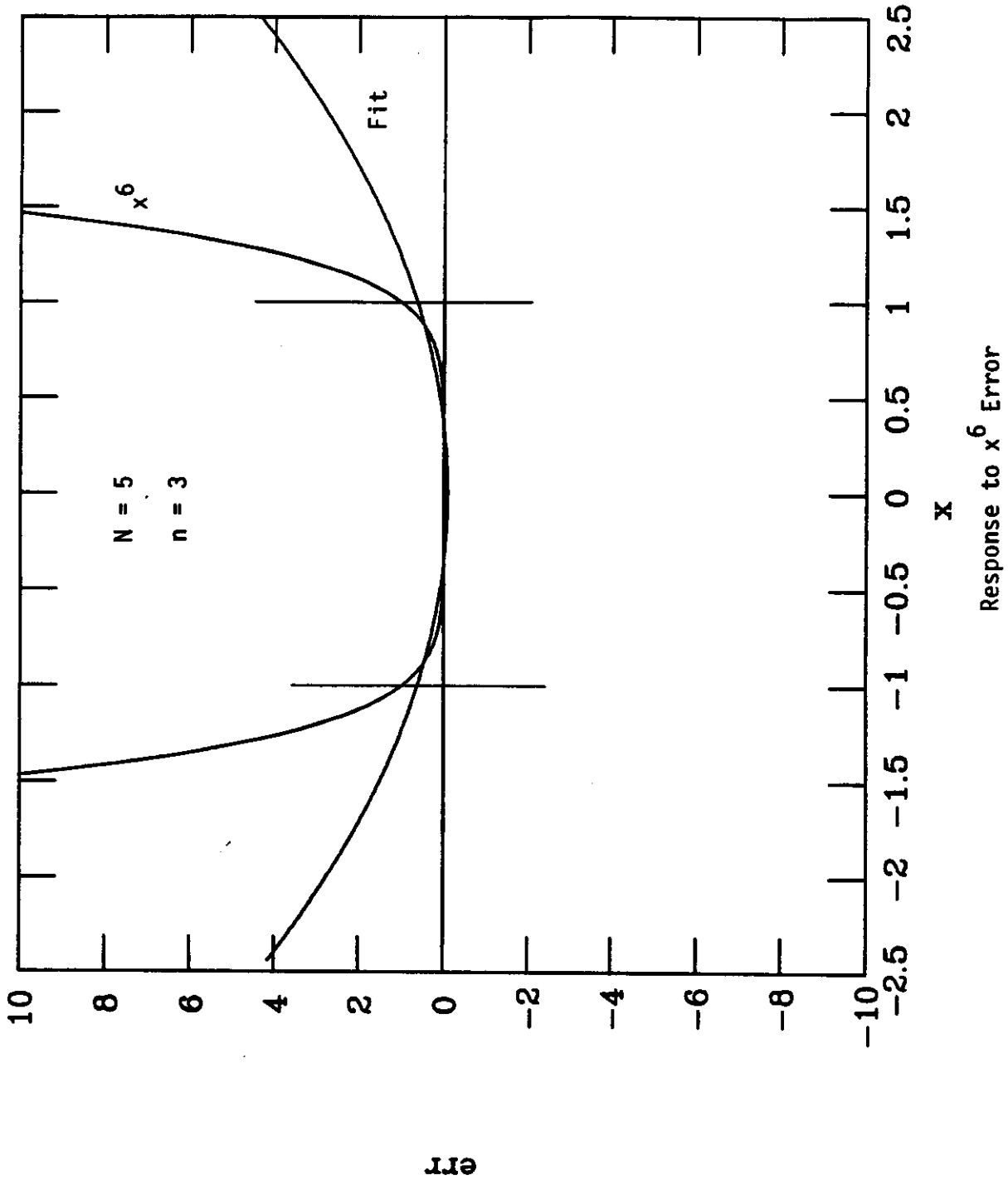
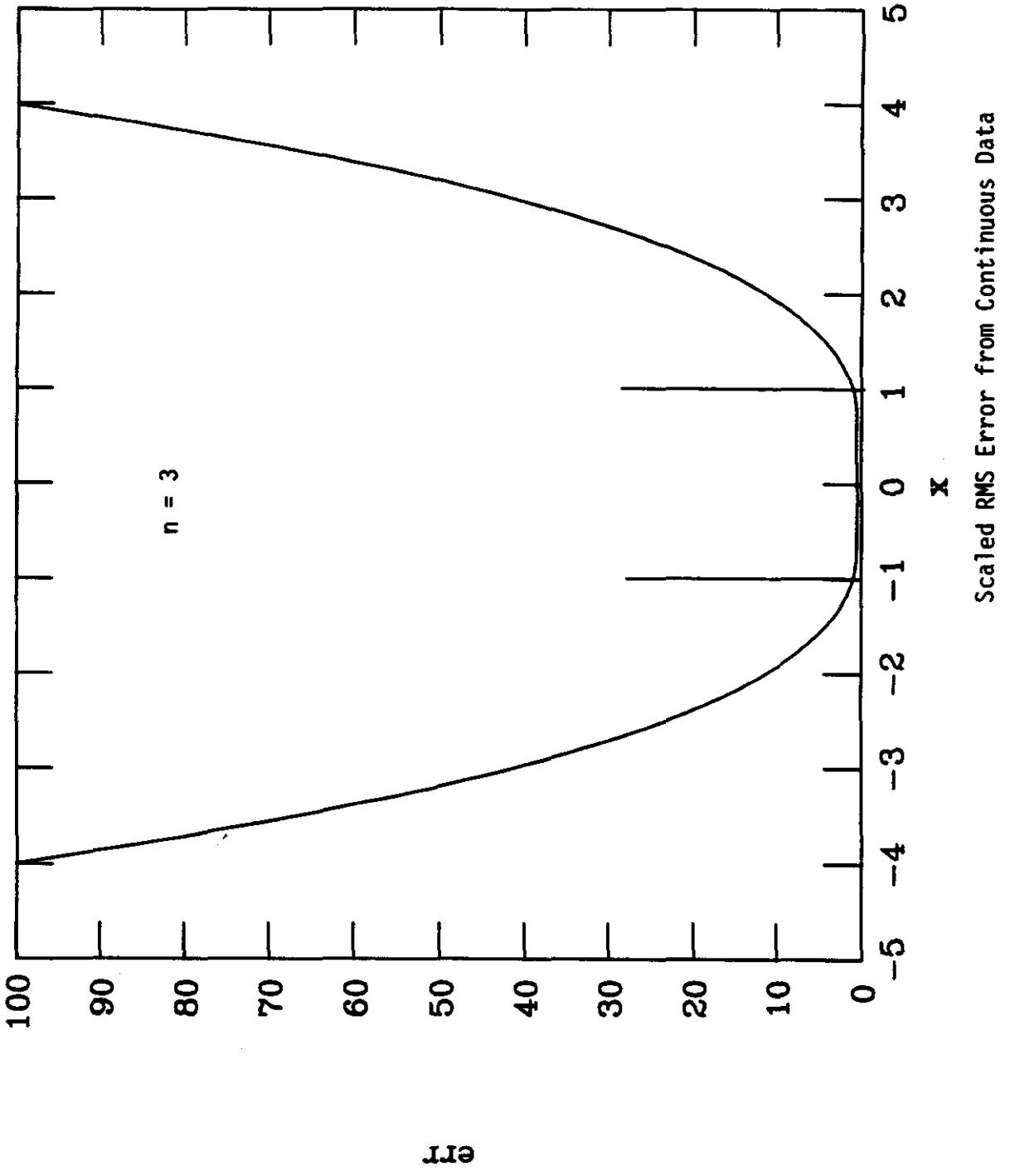

N = 5
n = 3
x^5
Fit
10
8
6
4
2
0
−2
−4
−6
−8
−10
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
x
Response to x^5 Error
err

Figure 9.



Response to x^6 Error

Figure 10.



Scaled RMS Error from Continuous Data