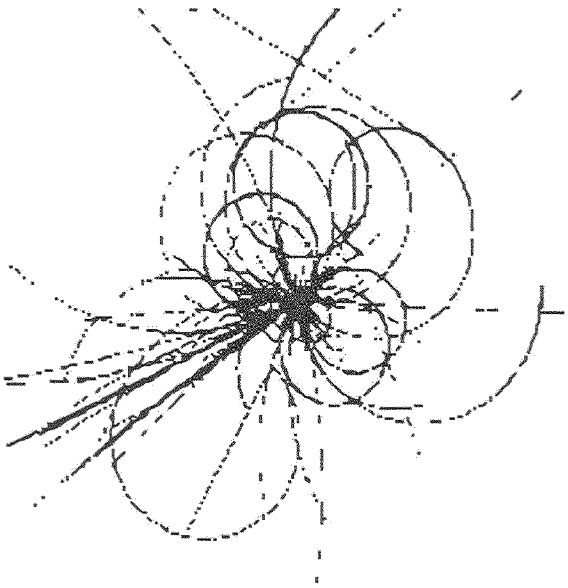
THE SUPERCONDUCTING SUPER COLLIDER LABORATORY



SSC-N-179

NOTES

SSC-N-179

GENERALIZED FORMULA FOR POWER LOSS OF A GAUSSIAN BUNCH IN RESONATORS

Huan Lee

SSC Central Design Group

May 1986

ABSTRACT

Recently Furman derived a practical formula for computing the power lost by a gaussian bunch in periodic orbit traversing a resonator with quality factor $Q > 1/2$. We give a generalized expression valid for all Q values.

A practical formula to compute the power loss by a short gaussian bunch traversing a resonator was derived by Furman[1] recently. His formula is valid for resonator quality factor $Q > 1/2$. With a slightly different formulation, we obtain a general expression valid for all physical Q values. We shall take Eq. (10) in Furman's note as the starting point, which states that, for a gaussian bunch of total charge $N_B e$ and longitudinal width σ_z, moving in a periodic orbit of length $2\pi R$ with a frequency $f_o = \omega_o/2\pi$, the power loss is given by

$$P = R_S(N_B e f_o)^2(R/\sigma_z) \int_{-\infty}^{\infty} \frac{e^{-s^2} ds}{1 + Q^2(s/\alpha - \alpha/s)^2} \tag{1}$$

where $\alpha = (\omega_R/\omega_o)(\sigma_z/R)$, and R_s, Q, and ω_R are the shunt impedance, quality factor, and resonance frequency of the resonator, repectively. The conditions which justify the integral representation Eq. (1) are $\omega_R/\omega_o >> Q$ and $(\sigma_z/R)^2 >> 1$. The discussion of these conditions is given in Ref. 1.

Consider the identity

$$\frac{1}{1 + iQ(x-1/x)} = -\frac{ix}{Q}\left[\frac{1}{(x^2-1)-ix/Q}\right]$$

$$= -\left(\frac{ix}{Q}\right)\frac{1}{(x-x_1)(x-x_2)}$$

$$= -\left(\frac{i}{Q}\right)\frac{1}{(x_1-x_2)}\left[\frac{x_1}{x-x_1} - \frac{x_2}{x-x_2}\right], \tag{2}$$

where

$$x_{1,2} = \frac{1}{2Q}\left[i \pm \sqrt{4Q^2-1}\right] \tag{3}$$

Replacing Q by -Q in (2), we get

$$\frac{1}{1-iQ(x-1/x)} = \left(\frac{i}{Q}\right)\frac{1}{(x_1 - x_2)}\left[\frac{x_1}{x + x_1} - \frac{x_2}{x + x_2}\right] \tag{4}$$

It follows that

$$\frac{1}{1+Q^2(x-1/x)^2} = \frac{1}{2}\left[\frac{1}{1+iQ(x-1/x)} + \frac{1}{1-iQ(x-1/x)}\right]$$

$$= \frac{i}{Q}\frac{1}{(x_1-x_2)}\left[\frac{x_1^2}{(x_1^2-x^2)} - \frac{x_2^2}{(x_2^2-x^2)}\right] \qquad (5)$$

Let $x = \alpha X$, and $s_{1,2} = \alpha X_{1,2}$, the integral in Eq. (1) can be written as

$$I(Q) \equiv \int_{-\infty}^{\infty} \frac{e^{-s^2}ds}{1+Q^2(x-1/x)^2}$$

$$= \frac{i}{Q}\frac{1}{(x_1-x_2)}\int_{-\infty}^{\infty} dse^{-s^2}\left[\frac{s_1^2}{s_1^2-s^2} - \frac{s_2^2}{s_2^2-s^2}\right]$$

$$= \frac{\alpha\pi}{Q}[s_1w(s_1) - s_2w(s_2)]/(s_1-s_2) \qquad (6)$$

where $w(z)$ is the complex error function[2] defined by

$$w(z) = \frac{iz}{\pi}\int_{-\infty}^{\infty}\frac{e^{-s^2}ds}{z^2-s^2} \qquad (\mathrm{Im}\ z > 0)$$

From (3) $\mathrm{Im}\ x_{1,2} > 0$ for $Q > 0$, clearly Eq. (6) is well-defined for all nonzero Q values. In particular, for $Q > 1/2$ we can write

$$I(Q) = \frac{2\pi}{\sqrt{4Q^2-1}}\mathrm{Re}[s_1w(s_1)] \qquad (7)$$

by using the facts that $s_2 = -s_1^*$ and $w(-s_1^*) = [w(s_1)]^*$. Expression (7) is just Furman's result.

The function $I(Q)$ given by (6) is obviously continuous at $Q = 1/2$, we can easily evaluate it as

$$I(\tfrac{1}{2}) = 2\alpha\pi\frac{d}{ds}[sw(s)]\Big|_{s=i\alpha}$$

Using the identity[2]

$$\frac{dw(s)}{ds} = \frac{2i}{\sqrt{\pi}} - 2sw(s) ,$$

we get $I(\frac{1}{2}) = 2\pi\alpha[(1+2\alpha^2)w(i\alpha) - \frac{2\alpha}{\sqrt{\pi}}]$. The positivity of I(1/2) is insured by the fact $w(i\alpha) \geq \frac{1}{\sqrt{\pi}}$.

REFERENCES

1. M. Furman, SSC-N-142, May 1986.
2. Handbook of Mathematical Functions, ed. by M. Abramowitz and I.A. Stegun, Dover (1965), p.297-298.

1673S