Computation of an Invariant for the Synchro-betatron Motion

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Part 1. Improvement on the Longitudinal Invariant in the Absence of Coupling with the Tranverse Motion for the Stationary RF-bucket.

The problem of finding 3 six-dimensional invariants for an accelerator system is an interesting mathematical problem. (Even though they do not usually exist, except in a formal power series sense).

This has also some applications in the context of the computation of the "Don Edwards" smear.

This problem requires two different approaches depending on the values of the fractional tunes ν_x, ν_y and ν_t. If these three tunes are all respectable fractions of unity, then the problem has been solved even in the presence of strong linear coupling. This has been incorporated to the code MARYLIE to 3rd order* in the phase space variables[2]. Translation codes are available at SSC for computing the same invariants from 2nd order** matrix codes such as Dimat or Transport.

Unfortunately (or fortunately), the temporal tune ν_t is often very small ($\nu_t^{ssc} = 1.4\ 10^{-3}$). Therefore an expansion in t and p_t is not the wisest choice. In fact, it is preferable to keep (t, p_t) to all order and expand in ν_t.

Our ultimate goal is to provide 3 invariants, two transverse invariants and one temporal. These invariants will be exact in (t,p_t) and expanded in ν_t, while staying exact in (ν_x, ν_y) and expanded to fourth order in (x, p_x, y, p_y).

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* quartic invariants
** cubic invariants

In this paper we will confine ourselves to a more modest goal, namely developing a well defined perturbation approach for the improvement of the Dome-Mizumachi invariant. This is an approximate invariant for the map:

$$\bar{\Psi} = \Psi + P \tag{1.1a}$$

$$\bar{P} = P - kg(\bar{\Psi}) \tag{1.1b}$$

where

$$g(\Psi + 2\pi) = g(\Psi) \tag{1.1c}$$

$$g(0) = 0$$

$$g'(0) = 1 \text{ or } 0$$

The last condition on $g'(0)$ is just a normalization of g. In this paper we will derive the invariants for $g'(0) = 1$. The non-linear case, $g'(0) = 0$, is a simple extension of our treatment. (see (4.25), (4.26), and (4.27) for a formula). In the next section, we derived the Dôme-Mizumachi invariant using a simple technique. We later develop more power tools to handle higher order invariants.

2. The Quadratic and Dôme-Mizumachi Invariants

To simplify the mathematics (and improve the invariant as we will later prove), we will rewrite (1.1) at the center of the cavity. Consider the following two maps:

$$\mathcal{M}_1: \quad \bar{\Psi} = \Psi + P \tag{2.1a}$$

$$\bar{P} = P$$

$$\mathcal{M}_2(k): \quad \bar{\Psi} = \Psi \tag{2.1b}$$

$$\bar{P} = P - k\, g(\Psi)$$

The original map (1.1) is the product $\mathcal{M}_1 \mathcal{M}_2$. (We choose the anti-matrix order when we compose maps).

Let us define a new map $\mathcal{M}$ obtained by similarity transformation for $\mathcal{M}_1 \mathcal{M}_2$:

$$\mathcal{M} = \mathcal{M}_2(\tfrac{k}{2})\ \mathcal{M}_1\ \mathcal{M}_2(k)\ \mathcal{M}_2(-\tfrac{k}{2}) \tag{2.2}$$

$$= \mathcal{M}_2(\tfrac{k}{2})\ \mathcal{M}_1\ \mathcal{M}_2(\tfrac{k}{2})\ .$$

This is the map which brings a particle from the center back to the center of the cavity.

First let us find the quadratic invariant of $\mathcal{M}$. To do this, we look at the linear matrix approximation of M_2 and M_1:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.3a}$$

$$M_2(\tfrac{k}{2}) = \begin{pmatrix} 1 & 0 \\ -\frac{k}{2} & 1 \end{pmatrix} \tag{2.3b}$$

$$M = M_2(\tfrac{k}{2})\ M_1\ M_2(\tfrac{k}{2})$$

$$= \begin{pmatrix} 1 - \frac{k}{2} & 1 \\ -\frac{k}{2}\,(2-\frac{k}{2}) & 1 - \frac{k}{2} \end{pmatrix} \tag{2.3c}$$

From the usual Courant-Snyder theory, we must have for the Twiss parameters:

$$\sin^2 \frac{\mu}{2} = \frac{k}{4}\ ;\ \mu = \text{tune of } M. \tag{2.4}$$

$$\beta = \frac{1}{\sin \mu}$$

$$\gamma = \sin \mu$$

$$\alpha = 0\ .$$

From (2.4), we deduce immediately the quadratic invariant of M:

$$I_2 = \Psi^2 + \frac{1}{\sin^2\mu} P^2 \tag{2.5}$$

The invariant I_2 is clearly degrading as we go away from the origin. I_2 is correct to all order in k, but is only quadratic in (Ψ, P). Since we assume a small k, we should look for an invariant which is exact in (Ψ, P) but first order in k.

This can be obtained by computing the map $\mathcal{M}$ of line (2.2):

$$\mathcal{M}\binom{\Psi}{P} = \begin{pmatrix} \Psi + P - \frac{k}{2} g(\Psi) \\ P - \frac{k}{2} g(\Psi) - \frac{k}{2} g(\Psi + P - \frac{k}{2} g(\Psi)) \end{pmatrix} \tag{2.6}$$

To help us in ordering the expansion, we look at the Twiss parameters (2.4). We notice that P is smaller than Ψ by sin μ, hence to lowest order in μ (2.6) is just:

$$\bar{\Psi} - \Psi = P - \frac{k}{2} g(\Psi) + \ldots = P + \ldots \tag{2.7a}$$

$$\bar{P} - P = -k\, g(\Psi) + \ldots$$

or

$$\bar{\Psi} - \Psi = \Delta\Psi = P - 2 \sin^2 \frac{\mu}{2}\, g(\Psi) + \ldots = P + \ldots \tag{2.7b}$$

$$\bar{P} - P = \Delta P = P = -4 \sin^2 \frac{\mu}{2}\, g(\Psi) + \ldots$$

Notice, again by using the Twiss Parameters, that $\sin^2\mu/2\ g(\Psi)$ is one order higher in μ than P. Now, we turn (2.7) into a differential equation involving a parameter ε where ε goes from 0 to 1.

$$\frac{d\Psi}{d\varepsilon} = P \tag{2.8}$$

$$\frac{dP}{d\varepsilon} = -4 \sin^2 \frac{\mu}{2}\, g(\Psi)$$

This can be written as:

$$\frac{d\Psi}{d\varepsilon} = [-H, \Psi] \tag{2.9}$$

$$\frac{dP}{d\varepsilon} = [-H, P] \quad [f, g] = \text{Poison bracket of } f \text{ and } g.$$

where,

$$H = \frac{P^2}{2} + 4 \sin^2 \frac{\mu}{2} \; \Gamma(\Psi) \tag{2.10}$$

$$\Gamma(\Psi) = \int_0^{\Psi} g(\alpha)\, d\alpha$$

The function H is the Hamiltonian which generates equation (2.8). Since it is independent of ε, it must be a constant of the motion parametrized by ε. Furthermore, we can insure that H reduces to I_2 by substituting sin μ for 2 sin $\mu/2$. This only affects third powers in μ, which are neglected anyway in this treatment.

$$H = \frac{P^2}{2} + \sin^2\mu \; \Gamma(\Psi) \tag{2.11}$$

The expression (2.11) is precisely the Dôme-Mizumachi invariant.

In the next section, we prepare the ground for a more elegant method which allows the computation of high order invariants.

3. Lie Operators

Consider two functions of phase space f and g. We can form their Poisson bracket by the usual rule:

$$[f,g] = \frac{\partial f}{\partial \Psi}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial \Psi} \tag{3.1}$$

The Poisson brackets defines a Lie product. Hence the set of functions of phase space forms a Lie algebra under the Poisson brackets.

Now consider a homorphism denoted by :: which promotes a function f to an operator status:

$$f \xrightarrow{\;:\;:\;} :f: \tag{3.2a}$$

$$:f:\, g = [f,g] \tag{3.2b}$$

We can now define the commutator of two Lie operators:

$$\{:f:,:g:\} = :f::g: - :g::f: \tag{3.3}$$

The commutator of Lie operators (of any operators) forms a Lie algebra. The homorphism comes from the Jacobi identity which is satisfied by all Lie products. This reads as follows:

$$\{:f:,:g:\} = :[f,g]: \tag{3.4}$$

Finally, we can also define the adjoint representation of Lie operators and their commutators. If :f: is a Lie operator we denote by #f# the adjoint of :f:. By definition, it acts on an operator :g: by taking commutators:

$$\#f\#:g: = \{:f:,:g:\} = :[f,g]:. \tag{3.5}$$

Incidentally, the operators #f# also form a Lie algebra under commutation.

$$\{\#f\#,\#g\#\} = \#\,\{:f:,:g:\}\,\# = \#:[f,g]:\#. \tag{3.6}$$

Equation (3.6) results from the two homomorphisms # # and : :. Equation (3.5) will become very useful in our perturbation treatment. In fact, consider any function F(x) defined by a power series. The following relations are true:

$$F(\#f\#):g: \;=\; \sum a_n \,\#f\#^{\,n}:g: \tag{3.7}$$

$$= \sum a_n \,:(:f:^n g):$$

$$= :\sum a_n \,:f:^n g:$$

$$= :F(:f:)\,g:.$$

After this mathematical interlude, we are now ready to proceed with the higher order theory.

4. Computation of the High Order Invariant

A. Rewriting the map $\mathcal{M}$

This process starts with a rewriting of $\mathcal{M}_1$ and $\mathcal{M}_2(k/2)$ in terms of Lie operators. It is easy to check that the following expressions are correct:

$$\mathcal{M}_1 \;=\; \exp\left(:-\frac{p^2}{2}:\right) \tag{4.1a}$$

$$\mathcal{M}_2\left(\frac{k}{2}\right) \;=\; \exp\left(:-\frac{k}{2}\;\Gamma(\Psi):\right) \tag{4.1b}$$

The map $\mathcal{M}_2$ can be rewritten:

$$\mathcal{M}_2\left(\frac{k}{2}\right) \;=\; \exp\left(:-\frac{k}{2}\,\gamma(\Psi):\right)\exp\left(:-\frac{k}{4}\;\Psi^2:\right) \tag{4.2}$$

$$\gamma(\Psi) \;=\; \Gamma(\Psi) - \frac{\Psi^2}{2}\;.$$

Expression (4.2) is exact since $\{:\gamma(\Psi):,:\Psi^2:\}$ vanishes. (Use (3.4)). The function $\gamma(\Psi)$ is of order 3 and higher in Ψ. Using (4.2) we can lump the linear parts of the map together.

$$\mathcal{M} = \exp(: -\frac{k}{2}\gamma(\Psi):)\,\mathcal{M}_L\,\exp(: -\frac{k}{2}\gamma(\Psi):) \tag{4.3a}$$

$$\mathcal{M}_L = \exp(: -\frac{k}{4}\Psi^2:)\,\exp(: -\frac{p^2}{2}:)\,\exp(: -\frac{k}{4}\Psi^2:) \tag{4.3b}$$

Each factor in $\mathcal{M}_L$ is a linear map. In fact, we have already computed the matrix representation of $\mathcal{M}_L$, the result is given by (2.3c). Can we join the exponents of $\mathcal{M}_L$? In fact, this can be done and we do have the answer[1]

$$\mathcal{M}_L = \exp(: -\frac{\mu}{2}(\sin\mu\ \Psi^2 + \frac{p^2}{\sin\mu}):) \tag{4.4a}$$

$$= \exp(: -\frac{\mu\sin\mu}{2}\ I_2:) \tag{4.4b}$$

The function I_2 is the Courant-Snyder invariant of equation (2.5). Notice that (4.4b) explicitly implies that I_2 is an invariant.

$$\mathcal{M}_L\, I_2 = \exp(: \frac{-\mu\sin\mu}{2}\, I_2:)\, I_2 \tag{4.5}$$

$$= I_2\ ,\ \text{since}\ [I_2, I_2] = 0\ .$$

Our final exact expression for $\mathcal{M}$ is just:

$$\mathcal{M} = \exp(: -\frac{k}{2}\gamma:)\,\exp(: -\frac{\mu\sin\mu}{2}\ I_2:)\,\exp(: -\frac{k}{2}\gamma:), \tag{4.6}$$

where $\qquad k = 4\sin^2\frac{\mu}{2}\ .$

Equation (4.6) is made of 3 maps with Lie operators of order μ^2. Unlike in $\mathcal{M}_L$, where every Lie operator was quadratic, it is not easy to check if we can combine the exponents. But, for one moment, let us assume we can do it. Then the following is true:

$$\exists\ H \text{ such that } \mathcal{M} = \exp(:H:) \tag{4.7}$$

$$\Longrightarrow\ H \text{ is an invariant}$$

In the next part of this section, we will find an expansion for H in powers of μ. This is the departure from the adiabatic expansion of Dôme-Mizumachi.

B. The General Approach to the Computation of Corrections to Adiabatic Invariants

For more generality, let us consider a map $\mathcal{M}$ written as follows:

$$\mathcal{M} = \exp(:f:)\ \exp(:g:)\ \exp(:f:) \tag{4.8}$$

It is possible to introduce a parameter σ:

$$\mathcal{M}(\sigma) = \exp(:\sigma f:)\ \exp(:\sigma g:)\ \exp(:\sigma f:). \tag{4.9}$$

The quantity σ is just a so-called "smallness" parameter. Ultimately it can be set to unity.

We assume that H can be expanded in powers of σ:

$$H(\sigma) = \sum_{n=1}^{\infty} \sigma^n H_n \tag{4.10}$$

Hence, if H exists, $\mathcal{M}$ can be written as:

$$\mathcal{M} = \exp(:H(\sigma):) \tag{4.11}$$

Using (4.8) we can derive a differential equation for $\mathcal{M}$. We differentiate $\mathcal{M}$ with respect to σ:

$$\frac{d\mathcal{M}}{d\sigma} = :f:\ \mathcal{M} + \mathcal{M}:f: + \exp(:\sigma f:)\ :g:\ \exp(:\sigma g:)\ \exp(:\sigma f:) \tag{4.12}$$

Using the properties of Lie operators mentioned in section 3, we can prove an important property: (See Appendix)

$$\forall\ (f,g)\quad \exp(:f:)\ :g:\ \exp(:-f:) = \exp(\#f\#):g: \tag{4.13}$$

$$= :\exp(:f:)\ g:$$

With the help of (4.13), we can simplify (4.12):

$$\frac{d\mathcal{M}}{d\sigma} = :f: \mathcal{M} + :\mathcal{M}f: \mathcal{M} + :\exp(:\sigma f:)\ g: \mathcal{M} \tag{4.14}$$

$$= :f + \mathcal{M}f + \exp(:\sigma f:)\ g: \mathcal{M}$$

Finally, (4.14) can be written as:

$$\frac{d\mathcal{M}}{d\sigma}\mathcal{M}^{-1} = :f + \mathcal{M}f + \exp(:\sigma f:)\ g:. \tag{4.15}$$

Here, a minor miracle occurs. Assuming the existence of $H(\sigma)$, we can rewrite $\frac{d\mathcal{M}}{d\sigma}\mathcal{M}^{-1}$ in terms of the adjoint representation of Lie operators. (See Appendix)

$$\frac{d\mathcal{M}}{d\sigma}\mathcal{M}^{-1} = \mathrm{iex}(\#H\#):\frac{dH}{d\sigma}:$$

$$= :\mathrm{iex}(:H:)\ \frac{dH}{d\sigma}:, \text{ where } \mathrm{iex}(\alpha) = \frac{\exp(\alpha) - 1}{\alpha}. \tag{4.16}$$

We now substitute (4.16) into (4.15):

$$\mathrm{iex}\ (:H:)\ \frac{dH}{d\sigma} = f + \exp(:H:)\ f + \exp(:\sigma f:)\ g + \lambda \tag{4.17}$$

where, $\lambda \in \mathbb{R}$ (λ can be set to zero).

Equation (4.17) produces a hierarchy of equations for the various H_n. In the last part of this section, we will prove that all H_n (n even) are zero. We will then derive H_1, H_3 and H_5.

C. <u>Explicit Expression for H</u>

Consider the map $\mathcal{M}(\sigma)$. Let us first prove that $\mathcal{M}(-\sigma)$ is just the inverse of $\mathcal{M}$.

$$\mathcal{M}^{-1}(\sigma) = \exp(:\sigma f:)^{-1} \exp(:\sigma g:)^{-1} \exp(:\sigma f:)^{-1} \tag{4.18}$$

$$= \exp(:-\sigma f:) \exp(:-\sigma g:) \exp(:-\sigma f:)$$

$$= \mathcal{M}(-\sigma)$$

On the other hand, $\mathcal{M}^{-1}(\sigma)$ is given by:

$$\mathcal{M}^{-1}(\sigma) = \exp(:-H(\sigma):), \tag{4.19}$$

which implies that $-H(\sigma)$ is $H(\sigma)$. Hence, we deduce from (4.19) the absence of even H_n:

$$H(\sigma) = \sigma H_1 + \sigma^3 H_3 + \sigma^5 H_5 + \ldots \tag{4.20}$$

Using the result of (4.20), we proceed at calculating the invariant H with the help of (4.17). To first order, we recover the Dôme-Mizumachi invariant:

$$H_1 + \ldots = f + f + g = 2f + g. \tag{4.21}$$

Then, to third order one gets:

$$3\sigma^2 H_3 = \frac{\sigma^2}{2} [H_1,[H_1,f] + \frac{\sigma^2}{2} [f,[f,g]], \tag{4.22a}$$

$$H_3 = \frac{1}{6} [g,[g,f]] - \frac{1}{6} [f[f,g]] \tag{4.22b}$$

We now, find an expression for the 5th order invariant:

$$\frac{3}{2}[H_1,[H_3,H_1]]+\frac{3}{2}[H_1,[H_1,H_3]] + 5H_5 = \frac{1}{4!}:H_1:^4 f + \frac{1}{2}[H_1,[H_3,f] \tag{4.23a}$$

$$+ \frac{1}{2} [H_3,[H_1,f]] + \frac{1}{4!}:f:^4 g$$

$$H_5 = \frac{1}{5!}[H_1,[H_1,[H_1,[H_1,f]]]] + \frac{1}{10} [H_1,[H_3,f]] \tag{4.23b}$$

$$+ \frac{1}{10} [H_3,[g,f]] + \frac{1}{5!}[f,[f,[f,[f,g]]]]$$

Just as an example, we will give the functional expressions of H_1 and H_3 for the standard cavity kick:

$$\Gamma(\Psi) = 1 - \cos\Psi \tag{4.24a}$$

$$H_1 = \frac{-\mu}{2\sin\mu}\left(\Psi^2 \sin^2\mu + P^2\right) + 4\sin^2\frac{\mu}{2}\left(\cos\Psi - 1 + \frac{\Psi^2}{2}\right) \tag{4.24b}$$

$$H_3 = \frac{2}{3}\,\frac{\mu\sin^4\mu/2}{\sin\mu}\,[\sin\Psi - \Psi]^2 \tag{4.24c}$$

$$+\frac{1}{3}\,\frac{\mu^2\sin^2\mu/2}{\sin^2\mu}\,\{\sin^2\mu\,(\Psi\sin\Psi - \Psi^2) - P^2(\cos\Psi - 1)\} \tag{4.24d}$$

It is easy to check that H_1 is, to order μ^3, the same invariant as what is obtained from (2.11). In the case of a non-linear bucket, the expression in (4.24) are modified. We assume that $g'(0)$ is now zero. This implies that $\mathcal{M}_L$ is just $\exp(:-p^2/2:)$.

For the example of a Fourier series representation of $g(\Psi)$; we obtain:

$$g(\Psi) = \sum_{n=1}^{\infty} A_n \sin(n\Psi) \tag{4.25a}$$

$$\text{with,} \quad \sum_{n=0}^{\infty} A_n\, n = 0 \tag{4.25b}$$

The invariant H_1 is just:

$$H_1 = \frac{-p^2}{2} + 4\sin^2\frac{\mu}{2}\left\{\sum_{n=1}^{\infty}\frac{A_n}{n}(\cos(n\Psi) - 1) + \frac{\Psi^2}{2}\right\} \tag{4.26}$$

where $\quad 4\sin^2\frac{\mu}{2} = k.$

Finally, H_3 is given by:

$$H_3 = \frac{2}{3} \sin^4 \frac{\mu}{2} \left[\sum_{n=1}^{\infty} A_n \sin(n\,\Psi) - \Psi \right]^2 \tag{4.27}$$

$$-\frac{1}{3} \sin^2 \frac{\mu}{2}\; P^2 \sum_{n=1}^{\infty} nA_n \left(\cos(n\,\Psi) - 1\right) .$$

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APPENDIX

The purpose of this appendix is to prove properties (4.13) and (4.16).

To prove (4.13), we intoduce a parameter ε as follows:

$$A(\varepsilon) = \exp(:\varepsilon f:)\ :g:\ \exp(:-\varepsilon f:) \tag{A1}$$

We now obtain a differential equation for A. Let us differentiate $A(\varepsilon)$ with respect to ε.

$$\frac{dA}{d\varepsilon} = :f:\ A - A\ :f: = \{\ :f:,A\ \} = \#f\#A \tag{A2}$$

It is easy to check that $\exp(\#\varepsilon f\#)A(0)$ is a solution of (A2). Furthermore $A(0)$ is just $:g:$. This proves the result:

$$\exp(:f:):g:\ \exp(:-f:) = \exp(\#f\#):g:. \tag{A3}$$

Equation (4.16) is proven in the same manner. We introduce a parameter ε as follows:

$$\mathcal{M}(\sigma;\varepsilon) = \exp(:\varepsilon H(\sigma):) \tag{A4}$$

Now, consider an operator $A(\varepsilon)$ defined by the equation:

$$A(\varepsilon) = \left[\frac{d}{d\sigma}\mathcal{M}(\varepsilon)\right]\mathcal{M}^{-1}(\varepsilon) \tag{A5}$$

Differentiating with respect to ε;

$$\frac{dA}{d\varepsilon} = \frac{d}{d\sigma}(:H:\mathcal{M})\mathcal{M}^{-1} - (\frac{d}{d\sigma}\mathcal{M})\mathcal{M}^{-1}:H: \tag{A6}$$

$$= :\frac{dH}{d\sigma}: + :H:A - A:H:$$

$$= :\dot{H}: + \#H\#A$$

Equation (A6) is very similar to (A2), except that it has a driving term. The homogeneous part has been solved on line (A2). We will try a solution using the variation of parameter method:

$$A = \exp(\#\varepsilon H\#)\mathcal{F}(\varepsilon) \tag{A7}$$

Substituting A into (A6) gives the result:

$$\frac{d\mathcal{F}(\varepsilon)}{d\varepsilon} = \exp(\#-\varepsilon H\#):\dot{H}: \tag{A8}$$

Notice that A(0) is zero. Hence F(0) must be zero also.

$$F(\varepsilon) = \left[\int_0^{\varepsilon} \exp(\# - xH\#)dx\right]:\dot{H}: \tag{A9}$$

$$= \frac{-\exp(\#-\varepsilon H\#)+ I}{\#H\#} \quad :\dot{H}:$$

Finally, we get for A(1):

$$A(1) = \frac{d\mathcal{F}}{d\sigma}\mathcal{F}^{-1} = \frac{\exp(\#H\#)- I}{\#H\#} \quad :\dot{H}: = i\exp(\#H\#):\dot{H}: \tag{A10}$$

References

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p.261 for a discussion of the Courant-Snyder invariants and their relation to the linear map.

p.185 for a definition of a Lie algebra.

p.214 for some comments about the adjoint representation.

2. Etienne Forest, Normal Form Algorithm on Non-linear Symplectic Maps, SSC-N-29.

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