Resonances in the SSC

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Recent studies of instabilities in the CERN $Sp\bar{p}S$[1], the CERN Antiproton Accumulator[2] and the Fermilab Tevatron[3] have identified high order nonlinear resonances as a source of instability in particle beams which are stored for long time periods. The resonances alone do not generate the long-time instability, but tune modulation provides synchrobetatron sidebands to the resonances. Overlap of these sidebands causes stochastic (chaotic) motion which leads to diffusional beam blow-up. In this section we study constraints on resonance size and consequently on magnetic field quality placed by this instability in the SSC.[4]

The nonlinear beam-beam interaction also can cause beam blow-up through the overlap of synchrobetatron sidebands, the resulting constraints on SSC design are described elsewhere.

A resonance is defined by a condition upon the betatron tunes ν_x, ν_y of the form

$$N\nu_x + M\nu_y = P \tag{1}$$

where N, M, P are integers. The order of the resonance is given by the sum $|N| + |M|$. Accelerator tunes are chosen to be far from low order resonances, but it is impossible to avoid all resonances.

In the resonance approximation the dynamics is assumed to be dominated by a single nearby resonance.[5] To simplify discussion we first consider only one-dimensional motion ($y = 0$, $y' = 0$), so that the resonance condition is $N\nu_x = P$, and extrapolate the results to include the effects of coupled resonances ($M \neq 0$) of the same order. In this resonance approximation the Hamiltonian of 1-D particle motion can be written as:

$$H = I_x \left(\nu_x - \frac{p}{N}\right) + A(I_x) + B_N(I_x) \cos(n\Psi) \tag{2}$$

where I_x is the amplitude of motion, Ψ is the resonant phase, and the independent variable is $\Theta = s/R$.

$A(I_x)$ gives the variation of tune with amplitude and includes the zero-harmonic contributions of all multipoles and nonlinear fields in the ring, including the beam-beam interaction. If we place all linear focusing in the first term of the Hamiltonian, include in the nonlinear fields only multipole content of the dipoles, and ignore closed orbit and momentum offset, we find

$$A(I) = \sum_{\substack{N=4 \\ (N\ \text{even})}}^{\infty} \frac{I^{\frac{N}{2}}\,(N-1)!}{2^N \pi\,[(\frac{N}{2})!]^2} \int \frac{B_o b_{N-1}}{B_p}\, \beta_x^{\frac{N}{2}}\, ds \tag{3}$$

Assuming the ring is composed of m independent dipoles with random multipoles of magnitude b_{N-1}, the contribution of a particular multipole may be estimated by:

$$A_N(I) \cong \frac{I^{\frac{N}{2}}\,(N-1)\,!}{2^{N-1}[(\frac{N}{2})!]^2}\ \frac{\langle\beta_x^{\frac{N}{2}}\rangle\, b_{N-1}}{\sqrt{m}} \tag{4}$$

The nonlinear detuning is is found from the derivative of A (I)

$$\Delta Q_{NL} = \frac{dA_N(I)}{dI} \cong \frac{N}{2}\,\frac{A_N}{I}\,(I) \tag{5}$$

The contributions of the multipoles to ΔQ_{NL} are included in Table 1. These are evaluated at $I = 100 \times 10^{-9}$ m-R (10σ radius at injection).

B_N gives ΔQ_N, the stopband width of the resonance of order N, from

$$\Delta Q_N = \frac{N}{2} \frac{B_N}{I}$$

where (6)

$$B_N = \sqrt{a_p^2 + b_p^2}$$

$$a_p,\ b_p = \frac{I^{\frac{N}{2}}}{2^N \pi N} \int ds\ \frac{B_o}{B_\rho}\ \beta^{\frac{N}{2}}\ b_{N-1} \cos(P\Theta)\ \sin(\rho\Theta)$$

$$\left|B_N\right| = \frac{\langle b_{N-1}\rangle \quad I^{\frac{N}{2}}\ \langle\beta^{\frac{N}{2}}\rangle}{2^{N-1} \quad \sqrt{m} \quad\quad N}$$

Only the lowest order contribution to the resonance $v = P/N$ is included in this formula.

Synchrotron oscillations and power supply ripple can cause tune modulation of the form

$$v = v + \Delta \cos(v_s\ \Theta)$$

where Δ is the modulation amplitude and v_s is the modulation frequency in turns. The single resonance Hamiltonian becomes

$$H = I\ [(v - \frac{P}{N}) + \Delta \cos(v_s\Theta)] \qquad (7)$$

$$+ A(I) + B_N(I) \cos(n\Psi)$$

The time dependence can be removed from the first term in equation 7 by a change in variables

$$\bar{\Psi} = \Psi - \frac{\Delta}{v_s} \sin (v_s\Theta)$$

which changes our single resonance Hamiltonian into one containing an infinite number of subresonances.

$$H = I(\nu-\frac{P}{N}) + A(I) + B_N(I) \sum J_k(\frac{\Delta}{\nu_s}) \cos(N\bar{\Psi}+k\nu_s\Theta) \tag{8}$$

The subresonances are spaced ν_s/N part in tune, with their central amplitudes, found from the solution of

$$\nu - \frac{P}{N} + k\nu_s + NA'(I) = 0$$

spaced in amplitude by

$$\delta I = \frac{\nu_s}{A''N}$$

The resonance full width in amplitude is found after expanding equation 8 to second order in $\Delta p/p$

$$\Delta I = 4\sqrt{\frac{B_N(I)\ J_k(\frac{N\Delta}{\nu_s})}{A''(I)}} \tag{9}$$

If the resonance width is greater than the resonance spacing then the Chirikov overlap criterion is satisfied and we may expect stochastic motion with particle trajectories that travel from resonance to resonance. This threshold occurs when:

$$\frac{4N}{\nu_s}\sqrt{A''(I)\ B_N(I)\ J_k(\frac{N\Delta}{V_s})} > 1 \tag{10}$$

This condition sets a lower limit on ν_s for each resonance order. The limits on ν_s under the pessimistic assumptions of using $\langle x^2\rangle = (10\sigma)^2$ at injection, $\Delta Q_{NL} \cong 0.01$, and replacing the Bessel function by its maximum value (1.0) are displayed in Table 1. The results indicate that it is necessary to avoid all resonances lower than seventh order within the tune spread when $\upsilon_s \cong 0.001$, as is planned for the SSC. The constraint is somewhat less restrictive than that placed by the beam-beam interaction. In that case, avoidance of resonances through ~ tenth order is desired.

Table 1

Resonance widths due to random multipoles in the dipoles.

Resonance Order N	$\sqrt{\langle b^2_{N-1} \rangle}$ $(cm^{-(N-1)})$	ΔQ_N Stop Band Half Width	ΔQ_{NL} Nonlinear Detuning	v_s Stochasticity Threshold
3	2.15×10^{-4}	0.0076	0	0.12
4	0.35×10^{-4}	0.00028	8.3×10^{-4}	0.027
5	0.59×10^{-4}	0.00010	0	0.018
6	0.059×10^{-4}	2.3×10^{-6}	2.3×10^{-5}	0.003
7	0.076×10^{-4}	6.7×10^{-7}	0	0.0017
8	0.016×10^{-4}	3.2×10^{-8}	1.1×10^{-6}	0.0004
9	0.021×10^{-4}	9.3×10^{-9}	0	0.00023
10	0.003×10^{-4}	3.0×10^{-10}	4×10^{-8}	0.000044
11	0.007×10^{-4}	1.6×10^{-10}	0	0.000034

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