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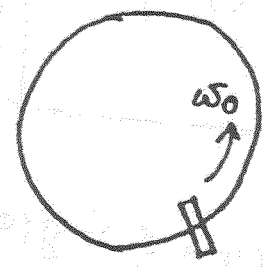
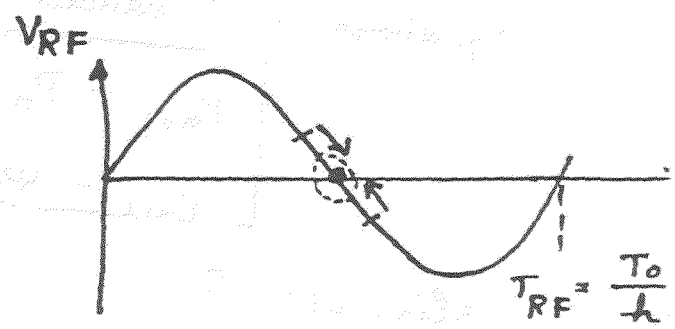
8/21/85

RF NOISE.

- RF provides longitudinal focussing in storage rings
- If there is a randomness in this focussing, the bunch undergoes noise diffusion in longitudinal phase-space
- Beam life-time may be reduced
- Colliding beam-beam luminosity is degraded.

But first <u>what</u> is an <u>RF cavity</u>?





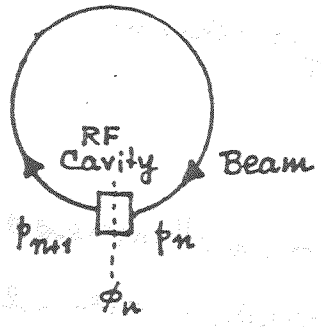
$$V = V_0 \sin(\overbrace{\omega_{rf} t + \phi_0}^{\psi})$$

$$\omega_{rf} = h\omega_0$$

<u>Noise?</u> $$V = V_0 (1 + a) \sin(\psi + \phi)$$

random small quantities

Motion without noise in stationary RF bucket.



p_s, v_s : Mom., vel. of reference synch particle

t_n : Time of nth cavity traver

$$\delta p_n = p_n - p_s$$
$$\delta \phi_n = \phi_n - \phi_s$$

Finite difference equations of motion:

$$\delta p_{n+1} - \delta p_n = \frac{eV}{v_s}\left[g(\phi_n) - g(\phi_s)\right]$$
$$\delta \phi_{n+1} - \delta \phi_n = \frac{2\pi h \eta}{p_s}\,\delta p_{n+1}$$

Define reduced variables:

$$\psi_n = \delta \phi_n$$
$$P_n = \frac{2\pi h \eta}{p_s}\,\delta p_n$$

Equations in reduced variables:

$$\boxed{\begin{aligned} P_{n+1} - P_n &= -k_0\left[g(\psi_n + \phi_s) - g(\phi_s)\right] \\ \psi_{n+1} - \psi_n &= P_{n+1} \end{aligned}} \quad k_0 = -\frac{2\pi h \eta}{p_s v}$$

Eliminate P's :

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n + k_0\left[g(\psi_n + \phi_s) - g(\phi_s)\right] = 0$$

Small oscillations or linear voltage:

$$\boxed{\psi_{n+1} + \psi_{n-1} + [K_0 - 2]\,\psi_n = 0}$$

where $K_0 = k_0\, g'(\phi_s) = -\dfrac{2\pi h \eta e V}{p_s v_s}\, g'(\phi_s)$

Solution: <u>Bounded oscillatory solution</u> <u>iff</u> $K_0 = -\frac{2\pi h \eta e V}{p_s v_s} g'(\phi_s) >$

$$\psi_n = \mathrm{Re}\,[a e^{in\mu}]$$

with $$2\cos\mu + K_0 - 2 = 0 \Rightarrow \boxed{\sin^2\frac{\mu}{2} = \left(\frac{K_0}{4}\right)}$$

<u>i.e.</u> synchrotron oscillation with complex <u>amplitude</u> 'a' and

$$\text{tune: } Q_s = \frac{\mu}{2\pi}$$

For nonlinear RF voltage, $g(\psi_n)$, analytic solution to the finite difference equations is difficult to obtain. If synchrotron oscillations are slow compared to revolution time, we can write down equivalent differential equations:

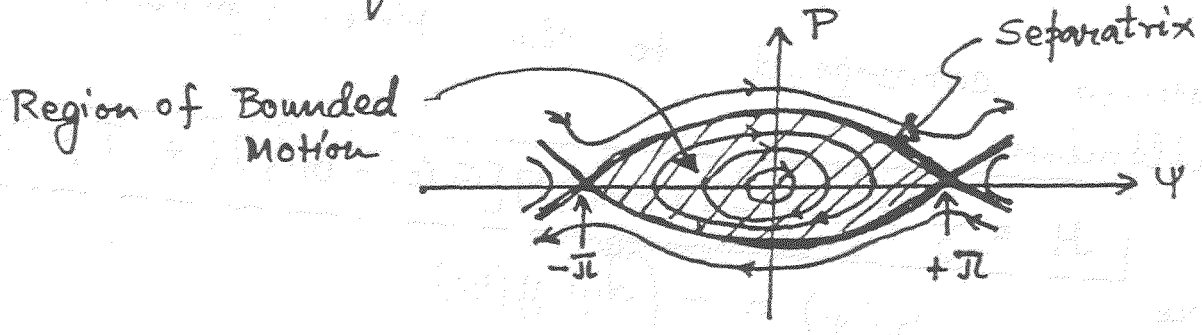
$$\left.\begin{aligned} \dot{P} &= -\frac{k_0}{T_r^2}\left[g(\psi+\phi_s) - g(\phi_s)\right] \\ \dot{\psi} &= P \end{aligned}\right\} \quad P = \frac{P_n}{T_r}$$

For a real RF cavity, $g(\psi) = \sin\psi$ and for a stationary bucket, $\phi_s = 0$. We obtain:

$$\boxed{\ddot{\psi} + \omega_s^2 \sin\psi = 0}$$

where $$\omega_s^2 = \frac{k_0}{T_r^2} = \omega_r^2\left(\frac{h\eta e V}{p_s v_s\, 2\pi}\right) = \omega_r^2 Q_s^2$$

Motion equivalent to that of a '<u>simple swing</u>'



Motion with noise in stationary RF bucket

Turn-by-turn finite difference equation for particle motion in presence of noise:

$$\boxed{\begin{aligned} P_{n+1} - P_n &= -k_0(1+a_n)\,g(\psi_n) \\ \psi_{n+1} - \psi_n &= P_{n+1} + \phi_{n+1} - \phi_n \end{aligned}}$$

where: $a_n \equiv$ Relative amplitude noise : $|a_n| \ll 1$, $\langle a_n \rangle = 0$.

$\phi_n \equiv$ Phase noise : $|\phi_n| \ll \pi$, $\langle \phi_n \rangle = 0$.

$(f_n/T_r) \equiv$ Discrete frequency noise: $f_n = [\phi_{n+1} - \phi_n]$, $\langle f_n \rangle =$

Corresponding differential equations:

$$\boxed{\begin{aligned} \dot{P} &= -\frac{k_0}{T_r^2}\,[1+a(t)]\,g(\psi) \\ \dot{\psi} &= P + \dot{\phi}(t) \end{aligned}}$$

$P = P_n / T_r$

$k_0/T_r^2 = \omega_r^2 Q_s^2 = \omega_s^2$

$f(t) = \dot{\phi}(t) =$ Smoothe frequen noise.

Eliminating P, we obtain the oscillator equation for the phase in presence of noise:

$$\ddot{\psi} + \omega_s^2\, g(\psi) = \ddot{\phi}(t) - \omega_s^2\, a(t)\, g(\psi)$$

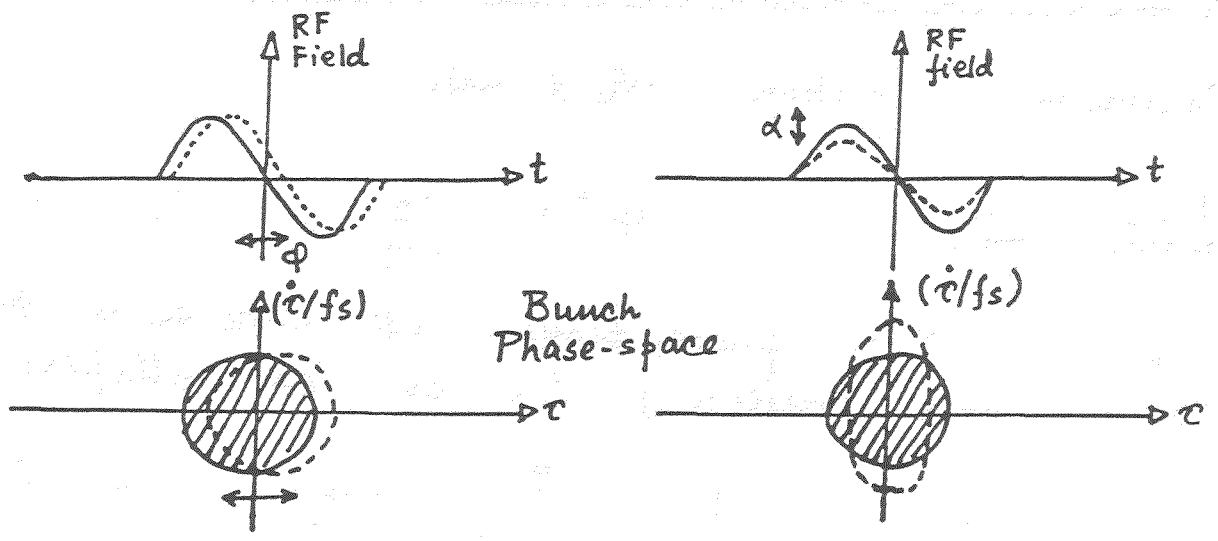
$$= \underbrace{2\pi \dot{f}(t)}_{\text{Phase or freq. Noise Term } \dot{\Omega}(t)} - \underbrace{\omega_s^2\, a(t)\, g(\psi)}_{\text{Amplitude Noise Term}}$$

Equations correspond to the time-dependent Hamiltonian:

$$\boxed{H = \frac{1}{2}P^2 + \omega_s^2\,[1+a(t)]\,[G(0) - G(\psi)] + P f(t)}$$

where

$$G(\psi) = -\int^{\psi} d\psi\, g(\psi)$$



Pure Phase Noise

$$V = V_0 \sin(\phi + \varphi)$$

Dipole-like motion

Coherently enhanced if noise power $P_\varphi(f)$ significant at $f \simeq f_s$.

Pure Amplitude Noise

$$V = V_0 (1+\alpha) \sin\phi$$

Breathing Quadrupole-like motio

Coherently enhanced if noise power $P_\alpha(f)$ significant at $f \simeq 2f_s$.

<u>Note:</u> (A) $g(\psi)$ ($\equiv \sin\psi$ for real RF) multiplies the amplitud noise term. For equal noise power levels $P_\varphi \simeq$ phase noise is more destructive than amplitude noise to the bunch phase-space for vanish small amplitudes ($\sin\psi \xrightarrow[\psi\to 0]{} \psi$). This is verified by the SPS-experience (CERN) with RF noise-dominate beam life-time and also by simple computer simulations (Y. Mizumachi).

(B) Noise entering into the equations of motion in the bunch frame is the sampled version of the actual noise at the cavity:

$$S_a(f) = \sum_{n=-\infty}^{+\infty} P_a(f+nf_r) \; ; \; S_\varphi(f) = \sum_{n=-\infty}^{+\infty} P_\varphi(f+nf_r)$$

Fokker-Planck and Diffusion Equation for bunch distribution

Constants of motion without noise :

Linear case:

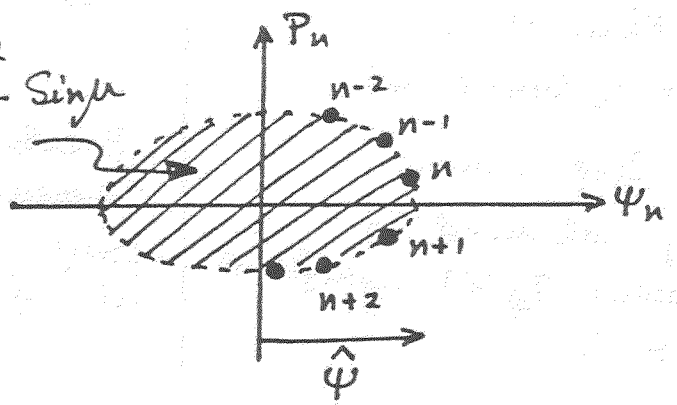
$$|a|^2 = \psi_n^2 + \left[\frac{P_n}{\sin\mu} - \tan\left(\frac{\mu}{2}\right)\cdot \psi_n \right]^2$$

In the P-ψ phase-plane, all successive points of a particle trajectory lie on an ellipse with area

$$A = 2\pi J = \frac{\pi |a|^2}{T_r} \sin\mu \; ; \quad J \equiv \text{Action variable}$$

$$2\pi J = \frac{\pi |a|^2}{T_r} \sin\mu$$

Longitudinal emittance.



Chose:

$$x = |a/2|^2 = (\hat{\psi}/2)^2 \propto J \qquad x \in [0,\infty]$$

as <u>independent variable</u>.

Nonlinear case:

Exact 'action' involves elliptic integrals. For small amplitudes, pretty well approximated by

$$J \cong \sin^2\left(\frac{\hat{\psi}}{2}\right)\cdot\left(\frac{2}{T_r}\right) \sin\left(\frac{\omega_s}{f_r}\right)$$

Chose:

$$x \approx \sin^2\left(\frac{\hat{\psi}}{2}\right) \qquad x \in [0, 1].$$

at bucket center. on the separatrix.

as independent variable.

In either case $x \propto$ longitudinal emittance of beam corresponding to a <u>matched</u> situation.

In absence of noise : $\rho(x)$ Beam distribution station

In presence of noise : $\rho(x,t)$ Time-dependent beam dis

Time evolution is of the form:

$$\frac{\partial \rho(x,t)}{\partial t} = \hat{L}\, \rho(x,t)$$

Differential operator in x, with time-dependent phase-functions as coefficie

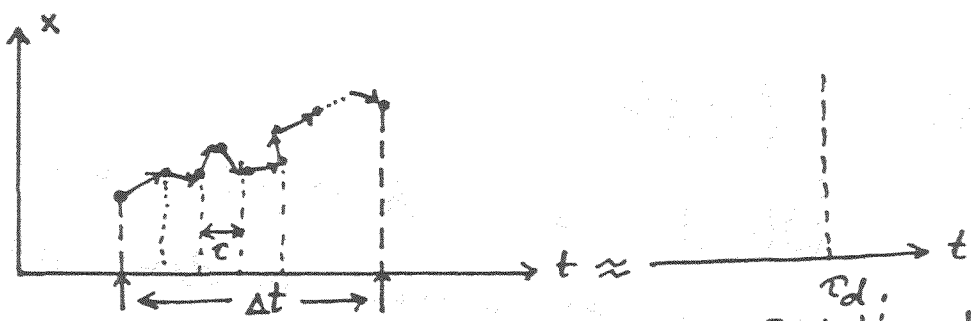
Basically follows from the Grand Continuity Equation in $6N$-dimensional phase-space.

Phase-functions are given by ensemble-aver moments like:

$$\lim_{\Delta t \to 0} \frac{\langle (\Delta x)^m \rangle}{\Delta t} = A_m(x,t), \quad m = 1, 2, 3,$$

for Markov random process.

$\tau \ll \Delta t \ll \tau_d$



Evolution time-consto

Usually $A_1(x,t) \neq 0$

$A_2(x,t) \neq 0$

but $A_m(x,t) \approx 0$ for $m > 2$

for time-scales of interest and usual random processes.

The time-evolution is then governed by the well-know Fokker-Planck equation:

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[A_1(x,t)\rho(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(x,t)\rho(x,t)\right]$$

Moreover, if noise can be included in a time-dependent Hamiltonian, one has the extra condition:

$$A_1(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x,t)\right]$$

[Consequence of Fluctuation-Dissipation Relation for the response of many-body systems].

$$\Longrightarrow \quad \boxed{\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x,t)\frac{\partial \rho(x,t)}{\partial x}\right]}$$

Nonlinear Diffusion Equation

Our problem:

What is $A_1(x,t)$ and $A_2(x,t)$ for bunch evolution under RF Noise?

Linear case: Finite difference equation with noise can be handled exactly.

$$A_1(x) = \frac{1}{T_r^2} \tan^2\left(\frac{Q_s}{2}\right) \left[S_\phi(\omega_s) + 2x\, S_a(2\omega_s) \right]$$

$$\frac{1}{2} A_2(x) = \frac{1}{T_r^2} \tan^2\left(\frac{Q_s}{2}\right) \left[x\, S_\phi(\omega_s) + x^2 S_a(2\omega_s) \right]$$

Neglect amplitude noise and assume $Q_s \ll 1 \Rightarrow \tan\left(\frac{Q_s}{2}\right) \simeq (Q_s/2)$.

Then:

$$A_1 = \frac{\omega_s^2}{4} S_\phi(\omega_s)$$

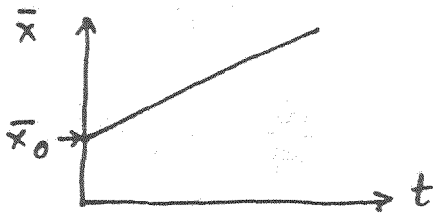
$$A_2 = \frac{\omega_s^2}{4} x\, S_\phi(\omega_s).$$

Define:

$$\bar{x} = \int_0^\infty dx \cdot x \cdot \rho(x,t)$$

$$\Rightarrow \quad \boxed{\frac{d\bar{x}}{dt} = \frac{\omega_s^2}{4} S_\phi(\omega_s) = D}$$

Sol^n $\Rightarrow$ $\boxed{\bar{x} = \bar{x}_0 + Dt}$



Linear growth of beam emittance.

Note: on validity:

Autocorrelation time of RF noise $\ll$ Bunch length in time

so that noise samples are indep. for each particle (Markov assumpt

Otherwise, no diffusion but coherent dipole jitter, for linear RF. Nonlinear RF $\rightarrow$ filamentation $\rightarrow$ diffusion (phase mixing)

(Ensemble Average) $\equiv$ (Av. over particles) $\rightarrow$ loss of coherenc

Nonlinear case:

Same $A_1(x)$ and $A_2(x)$ provided we replace:

$$S_\phi(\omega_s) \longrightarrow \frac{\sum_{\mu=1,3,\ldots}^{\infty} S_\phi(\mu\omega_s)\left[\frac{\mu^4}{ch^2(\mu v)}\right]}{\sum_{\mu=1,3,\ldots}^{\infty}\left[\frac{\mu^4}{ch^2(\mu v)}\right]}$$

$$(\omega_s^2/4) \longrightarrow \frac{\omega_{s0}^2}{4}\,\frac{2K(r^2)}{\pi}\left[1-r^2\alpha(r^2)\right]$$

$$r = \sin(\hat{\phi}/2), \quad 0 \le r^2 \le 1, \quad 8/15 \le \alpha(r^2) \le 1$$

$$v = (\pi/2)\left[K'(r^2)/K(r^2)\right]$$

K = Complete Elliptic integral of the first kind.

We have amplitude-dependent diffusion.

For small amplitudes x and a matched beam let x correspond to the frontier trajectory that bounds the beam in phase-space. The

$$x \propto \text{long. emittance}$$

and

$$\frac{dx}{dt} \simeq \frac{\omega_s^2}{4} S_\phi(\omega_s) = D(x).$$

$$\Rightarrow \quad x = x_0 + \underbrace{D(x_0)\,t}_{\text{initial growth.}} + \cdots\cdots$$

Bunch area doubling time (initially):

$$\boxed{\tau_d \simeq \frac{x_0}{D(x_0)} = \frac{4x_0}{\omega_s^2 S_\phi(\omega_s)}}$$

Preliminary Naive Application to the SSC:

$$f_s = \omega_s/2\pi = 1.65 \text{ Hz}.$$

$$S_\phi(f_s) = 10^{-3} (\text{rad})^2/(\text{sec}) \quad (-60 \text{ dBC/Hz})$$

for a typical rela stable oscillator.

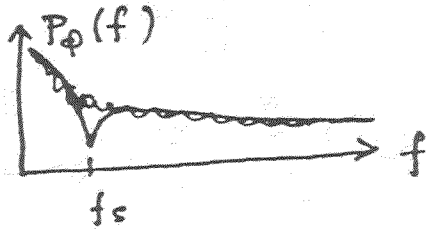
Take: $x_0 \simeq 0.27 \Rightarrow \hat{\phi} \simeq \pi/3$

Initial bunch area doubling time $\tau_d \simeq 60$ second

Ways out:

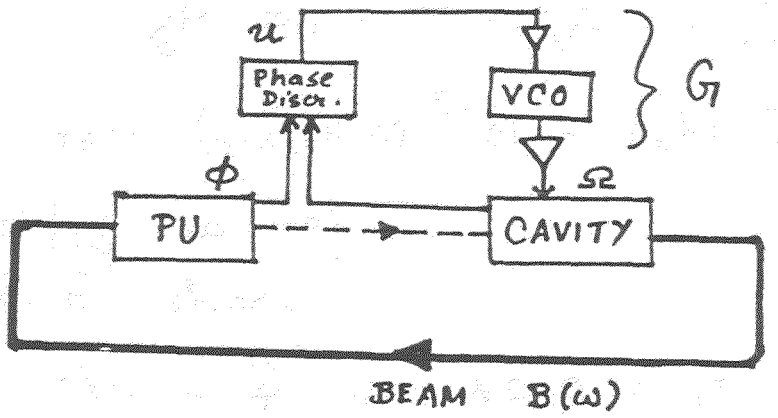
1. Use oscillators with better noise prope
2. Suppress noise density at synchrotron frequency by careful design of RF f
3. Design RF system <u>without oscillators</u> at all

With feedback, it is the closed-noise at multiples of synchrotron frequency that determines the bunch diffusion.



Let us look at RF & beam feedb effects.

ingle Bunch Feedback on RF



$$\phi(\omega) = B(\omega)\,\Omega$$

$$\Omega(\omega) = \frac{G(\omega)\,u(\omega)}{[1 + G(\omega)\,B(\omega)]}$$

$$P_\Omega(\omega) = \frac{|G|^2\,P_u(\omega)}{|1 + G\,B(\omega)|^2} \simeq \frac{P_u(\omega)}{|B(\omega)|^2}$$

for $G \gg 1$.

What is $B(\omega)$??

Oscillator response:

$$\ddot{\phi}_i + (\omega_s^i)^2 \phi_i = \dot{\Omega}(t) \qquad i = 1, 2, \ldots, N$$

Define $\tilde{\phi}_B = \frac{1}{N} \sum_{i=1}^{N} \tilde{\phi}_i$

$$\tilde{\phi}_B = \left[\frac{(-i)}{N} \sum_{i=1}^{N} \frac{\omega}{(\omega_s^i)^2 - \omega^2} \right] \tilde{\Omega}$$

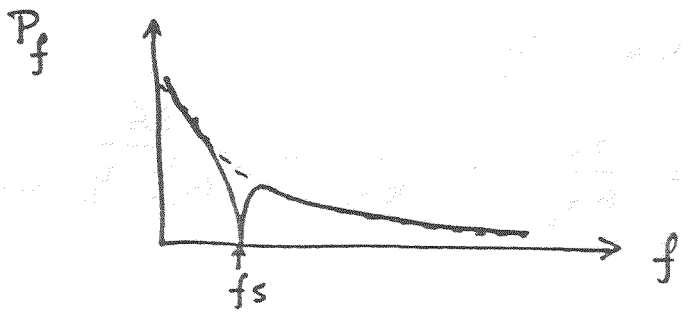
$$B(\omega) = \frac{-i}{N} \sum_{i=1}^{N} \frac{\omega}{(\omega_s^i)^2 - \omega^2}$$

If no synchrotron freq. spread,

$$\omega_s^i = \omega_s \quad \text{same for all } i.$$

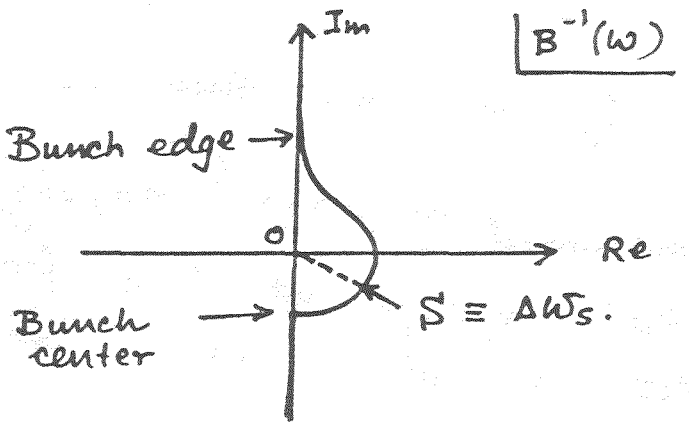
$$B(\omega) \xrightarrow[\omega \to \omega_s]{} \infty, \qquad P_\Omega(\omega) \xrightarrow[\omega \to \omega_s]{} 0.$$

$$\frac{dx}{dt} = \frac{\omega_s^2}{4} P_\phi(\omega_s) = \frac{1}{4} P_\Omega(\omega_s) = 0.$$



Noise totally suppressed at $f = f_s$. No bun
diffusion.

What if there is a spread in ω_s^i
What does $B(\omega)$ look like?



(Same
the bu
beam
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integra

$$|B(\omega)| \sim (1/S)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{4} P_\Omega(\omega_s) = \frac{P_u(\omega_s)}{4} S^2$$

For small amplitudes:

$$\omega_s^2 = \omega_{s0}^2\left(1 - \frac{\hat{\phi}^2}{8}\right) \quad \text{and} \quad S = \Delta\omega_s \simeq \frac{\omega_{s0}}{16}\hat{\phi}^2 = \frac{\omega_{s0}}{4}x^2$$

$$\frac{dx}{dt} = \omega_{s0}^2 \frac{P_u(\omega_s)}{16} x^2 = k x^2$$

If $x = x_0$ at $t=0$, initial bunch area doubling time of:

$$\tau_d = \frac{x_0}{k x_0^2} = \frac{1}{k x_0} = \frac{16}{\omega_{s0}^2 P_u(\omega_s) x_0}$$

<u>Application to SSC</u>:

$f_{s0} = 1.65$ kHz

$\hat{\phi} = \pi/3 \Rightarrow x_0 = 0.27$

$u \sim 10^{-3}$ rad/$\sqrt{\text{Hz}}$ (high!!)

$\Rightarrow \tau_d \sim$ few hundred hours!!

$u_{measured} \sim 2\times10^{-6}$ rad/$\sqrt{\text{Hz}}$

Flat except at low frequencies of a few Hz, where it is unknown.

$$\left.\begin{aligned} \hat{\phi} &= \omega_{RF}\cdot(\Delta t) \\ \omega_s\hat{\phi} &= \eta\left(\frac{\Delta E}{E}\right)\cdot\omega_{RF} \end{aligned}\right\} \text{Initial Slope} = D = k x_0^2 = \eta^2(\omega_{RF})^4 \frac{\varepsilon_s^2}{E^2}\frac{P_u(\omega_s)}{(16)(16)}$$

$\varepsilon_s = (\Delta E)(\Delta t) =$ Emittance

For long times, one has to solve the full-blown diffusion equation.

The eigen-solution corresponding to the slowest decay, with the boundary condition

$$f(x=1,t)=0 \quad \text{on the separatrix}$$

determines the beam life-time, correspond to (1/e) reduction in beam intensity und steady state. Approximately:

$$\tau_{eq} \simeq \frac{20 \cdot 96}{\omega_{so}^2 \, P_u(\omega_s)}$$

for flat P_u and reasonable $\omega_s(x)$.

<u>Comparison:</u>
<u>with SPS:</u>

$$\left.\begin{array}{ll} f_{so} \sim 200 \text{ Hz} & \text{SPS} \\ f_{so} \sim 1.65 \text{ Hz} & \text{SSC} \end{array}\right\}$$

$$(\tau_{eq})_{SSC} \sim 10^4 \, (\tau_{eq})_{SPS} \quad \text{for same}$$

$$(\tau_{eq})_{SPS} \sim 200 \text{ hours.}$$

Looks like we are saved!

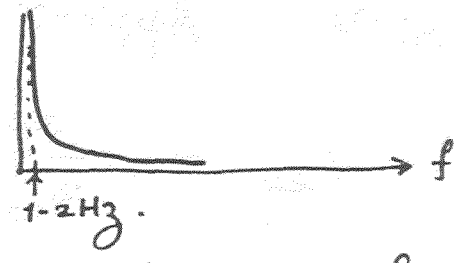
But, ...

Outstanding Problems:

Hardware issues:

(1) We really do not know the level of around extremely low frequencies $\omega_s \sim 10$

(2) We do not know the VCO oscillator no few Hz off-set from the carrier.



It is expected to be very high. How h

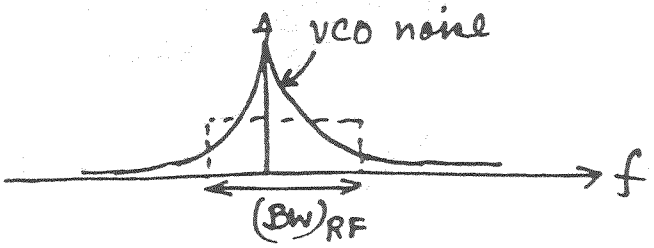
(3) SPS has few bunches.
SSC has many bunches.

High f_s, small no. of bunches → RF f
experi
demonc
by SP

Is RF feedback on many bunches indi
feasible?

Main dipole damping loop does
provide correction of Ω at $(nf_r + f_s)$ (n
lines, seen by the other bunches. Thes
noise components are important, since
is usually broadband.

(4) We need to correct nf_r lines only within the cavity bandwidth < VCO bandwidt

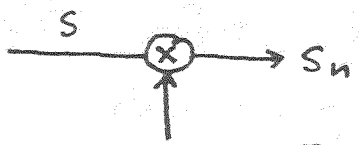


Strings of bunches within $\Delta t = 1/(BW)_{RF}$ see the same noise & can be corrected for by correcting nf_r-lines within a cavity bandwidth, which still may be large!

(5) At low fs, many of the other noise sources may be dominant e.g. frequency loop noise, etc. (magnetic noise).

(6) There will inevitably be coherent mode damping loops. Residual uncorrected components e.g. remnant quantization noise in a digitally processed dipole feedback loop will act as incoherent noise in the RF phase-space.

What is the minimum no. of bits in a coherent feedback system necessary to ensure a comfortabl RF lifetime?



e_n = Quantization error in measured RF phase

RMS $\quad \langle e_n \rangle = 2^{-B}/\sqrt{3}$ $\qquad$ B = no. of bits.

If white noise, $\quad BW = \Delta f = f_0/2$ $\quad$ (since e_n is sampled)

$$\Rightarrow \quad P_u = \langle u^2 \rangle / 2\Delta f = \langle e_n \rangle^2 / 2\Delta f = 2^{-2B}/3f_0 .$$

$$\tau_{eq} \simeq \frac{20}{\omega_s^2 P_u} = 60 f_0 \times 2^{2B}/\omega_s^2$$

If $\quad V =$ RF voltage

$\Delta V =$ Maximum voltage delivered by feedback,

Peak Frequency modulation $\quad \Omega = -i\omega \dfrac{\Delta V}{V}$.

But $\quad \Omega = \dfrac{Gu}{1+GB} \simeq \dfrac{u}{B}$.

$$u_{max} = 2^{-B} \quad ; \quad B = -i\omega/\omega_s^2 - \omega^2$$

$$\Rightarrow \quad \frac{\Delta V}{V} = 2^{-B} \frac{\omega_s^2 - \omega^2}{\omega^2} \lesssim 2^{-B} \quad \text{for } \omega \to \omega_s .$$

Theoretical Issues:

1. Rigorous formulation and practical solution of the RF noise diffusion problem <u>including RF feedback</u>

2. Simulation (and tracking) of noise diffusion <u>including RF feedback</u> for various nonlinear RF buckets & noise sources (e.g. independent vs. constant random picks)

3. Simulation vs. theory.

4. Conflicting requirements on freq. spread

$\Rightarrow$ RF noise prefers $\Delta\omega_s \downarrow$ so that $B(\omega_s)\uparrow$ and $P_\phi(\omega_s)\downarrow$

$\Rightarrow$ Stability against coherent modes prefers $\Delta\omega_s \uparrow$ Landau damping.

Experimentally for SPS, a crude compromise for the SPS case is

$$\left(\frac{\text{Bucket Area}}{\text{Bunch Area}}\right) \gtrsim 3 .$$

Can we establish such a limit theoretically for the SSC ??

5. Effect of distorted $B(\omega)$ due to ring impedance on RF ~~cavity~~ noise.