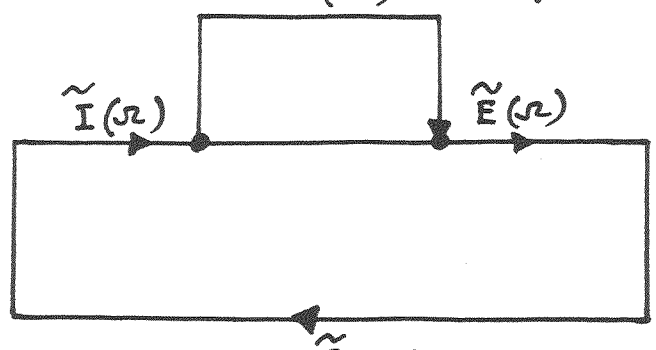
Swapan Chattopadhyay 9/11/85

Self-Consistent Collective Dynamics.

$\tilde{Z}(\Omega) \equiv$ Impedance

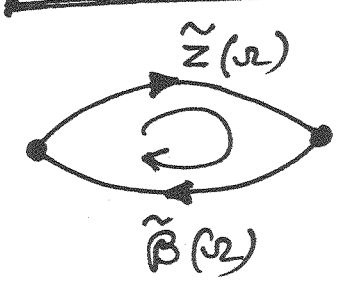


$\tilde{B}(\Omega) \equiv$ Beam. Transfer Function.

$$\left.\begin{aligned} \tilde{E}(\Omega) &= \tilde{Z}(\Omega)\,\tilde{I}(\Omega) \\ \tilde{I}(\Omega) &= \tilde{B}(\Omega)\,\tilde{E}(\Omega) \end{aligned}\right\rangle \quad \tilde{I}(\Omega)\left[1 - \tilde{B}(\Omega)\tilde{Z}(\Omega)\right] = 0$$

Self-Consistency or Dispersion Relation :

$$\boxed{1 - \tilde{B}(\Omega)\tilde{Z}(\Omega) = 0}$$



$= 1.$

$$|\tilde{B}| = |\tilde{Z}|^{-1}$$

$$\phi_B + \phi_Z = 2\pi n$$

$\tilde{B} \equiv$ Beam Transfer function $\Rightarrow$ Determined by single particle unperturbed orbits + beam distribution

$\tilde{Z} \equiv$ Determined by beam + environment.

Dispersion Relation:

$$\boxed{\hat{\tilde{B}}(\Omega)\,\hat{\tilde{Z}}(\Omega) = 1.}$$

General operator equation.

1. Could be scalar equation — lucky!!
   Generally for continuous coasting beams.

2. Could be matrix equation.
3. Could be integral equation.

can be made equivalent and sometimes transformed to scalar equation even.

General complex solution:

$$\Omega = \Omega_R + i\Omega_I$$

Ω_R: Coherent oscillation freq.

Ω_I: Growth or Damping rate

Perturbation $\xi \sim e^{-i\Omega t} = e^{-i\Omega_R t}\, \underbrace{e^{\Omega_I t}}_{\text{growth.}}$

Ω depends on beam current and $\tilde{Z}(\Omega)$.

For $I \leq I_{th}$, $\Omega_I \leq 0 \implies$ No growth
No instability

Collective Forces : Impedances.

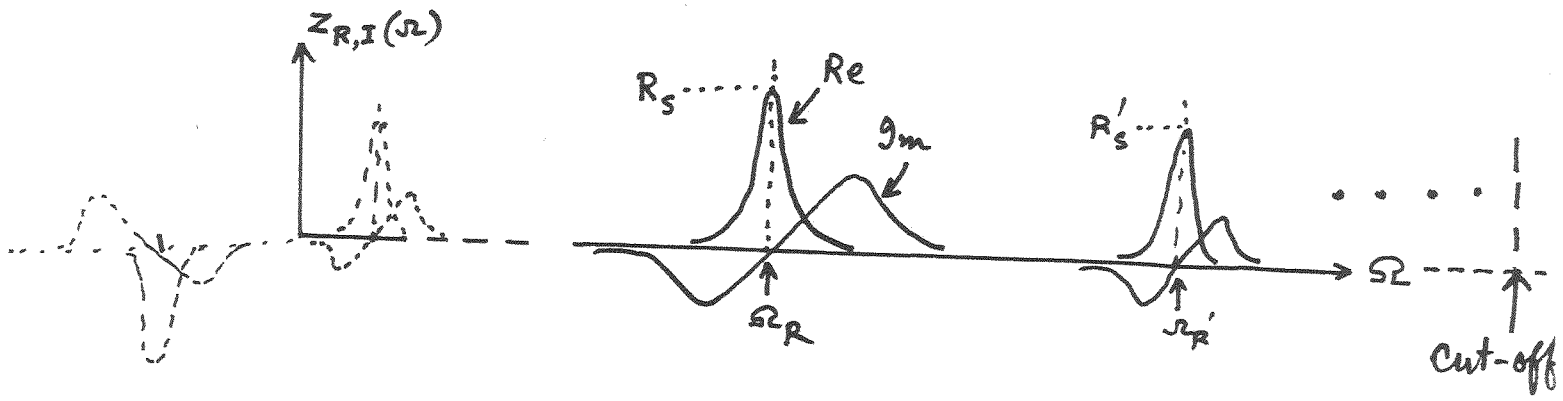
$$\tilde{E}_{\parallel}(\Omega) \sim \tilde{Z}_{\parallel}(\Omega) \cdot \tilde{I}_{Beam}(\Omega)$$

$$\tilde{F}_{\perp}(\Omega) \sim \tilde{Z}_{\perp}(\Omega) \cdot \tilde{d}_{Beam}(\Omega)$$

A. Parasitic (higher-order) modes of RF cavities.

Long:

$$Z_{\parallel}(\Omega) = \frac{R_s}{1 - iQ\left[\frac{\Omega}{\Omega_R} - \frac{\Omega_R}{\Omega}\right]}$$



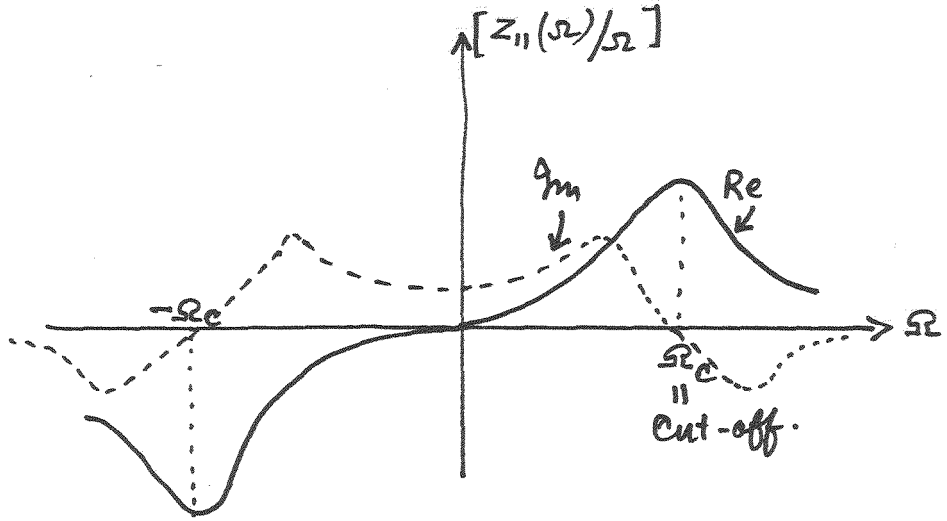
$R_s \rightarrow$ Shunt impedance
$\Omega_R \rightarrow$ Resonant frequency
$Q \rightarrow$ Quality factor.

Trans:

$$Z_{\perp}(\Omega) = \left(\frac{\Omega_R}{\Omega}\right) \frac{R_T}{1 - iQ\left[\frac{\Omega}{\Omega_R} - \frac{\Omega_R}{\Omega}\right]}$$

B. Broad-band

$Q = 1$ Resonator:



For $\sigma_\ell < b$, "Spear scaling law":

$$(Z/n)_{eff.} = (Z/n)_0 \left(\frac{\sigma_\ell}{b}\right)^{1.68}$$

b = chamber radius

σ_ℓ = Bunch length.

C. Free-Space Impedance

For frequencies far above cut-off, relevant for extremely small bunches:

$$\left(\frac{Z}{n}\right)_{\substack{free\\space}} = 300\left(\frac{b}{R}\right) \text{ ohms.}$$

$$\text{or } 300\left(\frac{b}{R}\right)\cdot\frac{\Delta\theta}{2\pi}$$

$\Delta\theta$ = Effective Bend angle in wigglers etc.

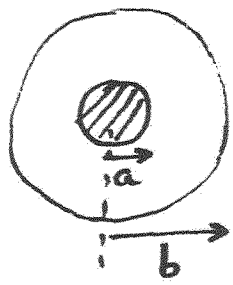
D. Space-charge

Long. $$\frac{Z_{\parallel,n}}{n} = i\,\frac{Z_0}{2\beta\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)$$

Trans. $$Z_\perp = i R Z_0 \frac{1}{\beta^2\gamma^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

R = Av. Radius of ring.

Z_0 = Free Space Impedance $= 119.92\,\pi\ \Omega s.$



Round Beam in round pipe.

For elliptical beam in round chamber:

$$Z_\perp^x = i R Z_0 \frac{1}{\beta^2\gamma^2}\left[\left\langle\frac{1}{\sigma_x(\sigma_x+\sigma_y)}\right\rangle - \frac{1}{b^2}\right]$$

$$Z_\perp^y = i R Z_0 \frac{1}{\beta^2\gamma^2}\left[\left\langle\frac{1}{\sigma_y(\sigma_x+\sigma_y)}\right\rangle - \frac{1}{b^2}\right]$$

E. Resistive Wall

Long. $$\frac{Z_{\parallel,n}}{n} = \frac{(1-i)}{\sqrt{2}\,b}\sqrt{\frac{R Z_0 \beta}{n\sigma\mu}}$$

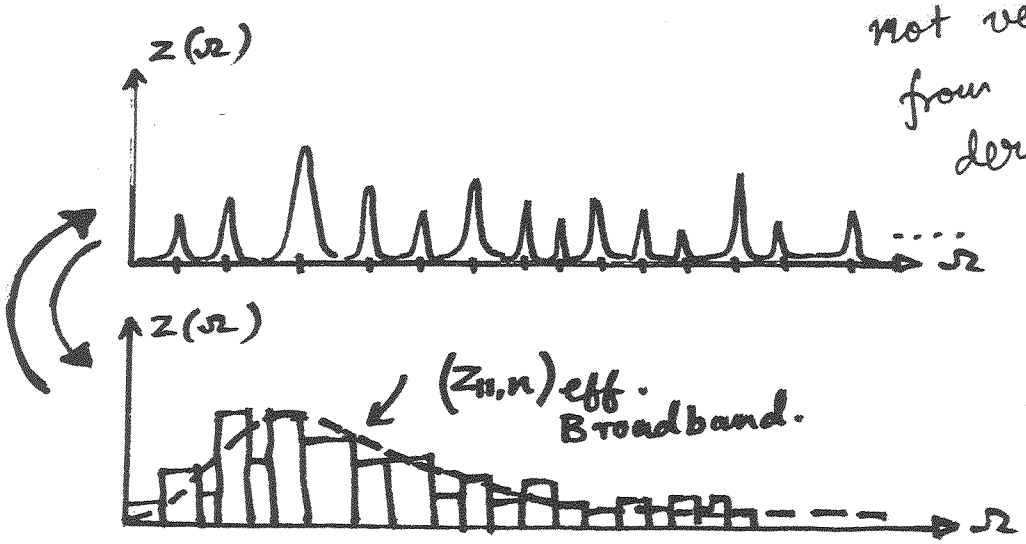
σ = Conductivity

μ = Permeability

Trans. $$Z_\perp = R Z_0 (1-i)\frac{\delta(\Omega)}{b^3}$$

$$\delta(\Omega) = \sqrt{\frac{2}{\Omega\sigma\mu}} \qquad \Omega > 0; \quad \delta(-\Omega) = -i\,\delta(|\Omega|).$$

F. Broadband component of RF.



not very dissimilar from Wien's law derived from Planc's Distribution

Three ways to determine:

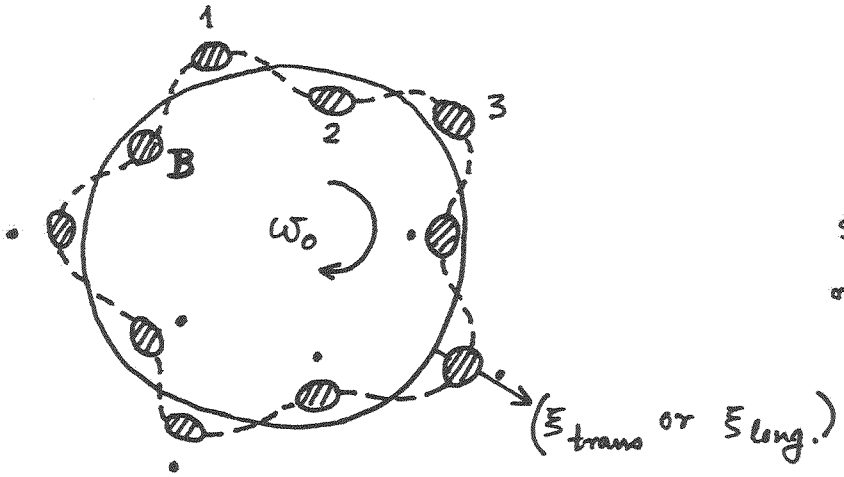
(a) calculated as above

(b) Calculating exactly what the resonators do to a single bunch and comparing to an equivalent broadband impedance

(c) Comparing real frequency-shifts induced in a long bunch by all the resonators & a single $Q = 1$ resonator.

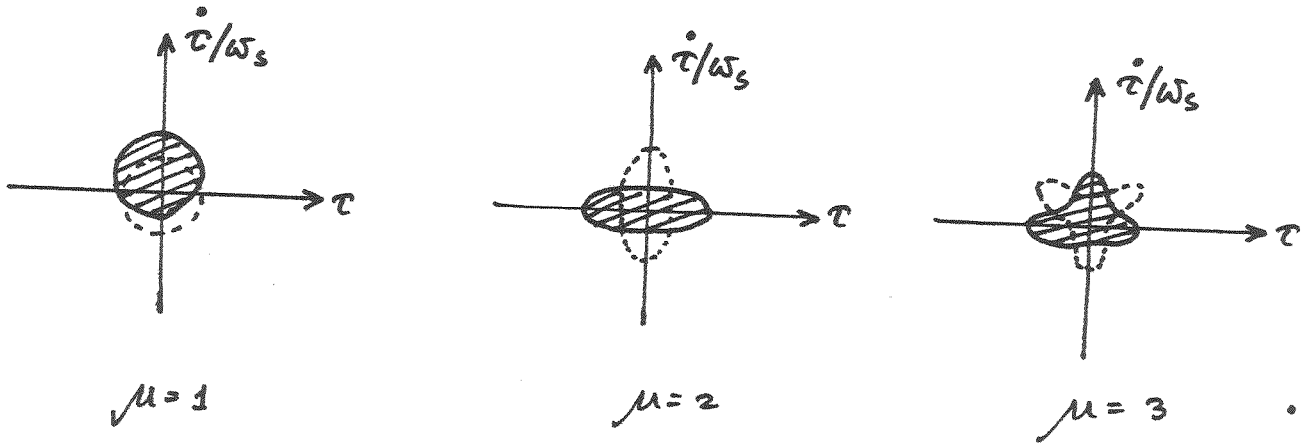
$$\left(Z_{11,n}/n\right)^{eff.}_{reactive} = \sum_{i \equiv \text{resonators}} \left(\frac{Z_{11,n}}{n}\right)^{i}$$

Coupled Bunch Modes.



$s = \{0, \dots, (B-1)\}$

or $\equiv \{1, \dots, B\}$



$\mu = 1$ $\mu = 2$ $\mu = 3$ . . .

Long : $$\Omega = \underbrace{(nB + s + \mu Q_s)}_{\nu_L}\omega_0 + \Omega'$$

Trans : $$\Omega = \underbrace{(nB + Q + s + \mu Q_s)}_{\nu_T}\omega_0 + \Omega'$$

Example

Complex Coherent Frequency Shift

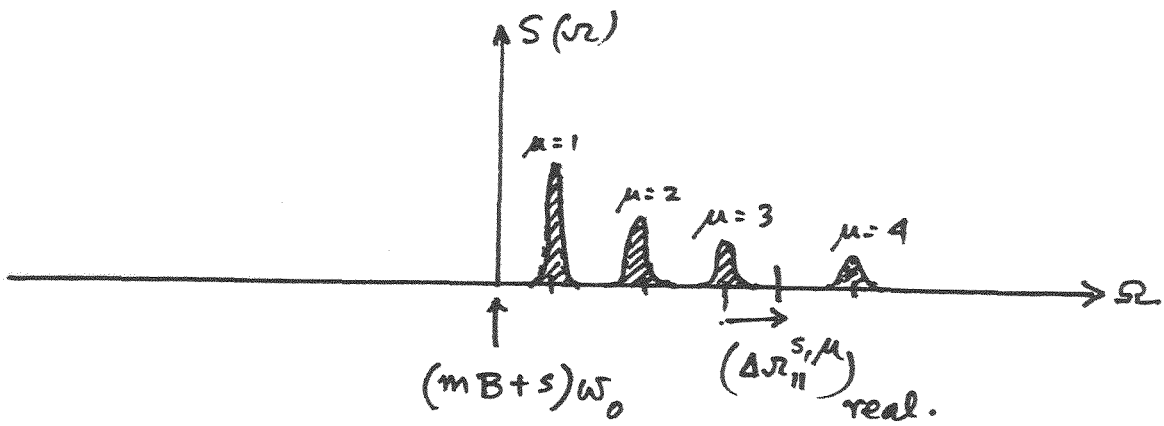
Small Gaussian Bunches

No Synchr. Freq. Spread.

$$\Delta\Omega_{\parallel}^{s,\mu} = \Omega - \mu\omega_s = i\,\frac{e\eta I_0 B\omega_0^2}{2\pi E\omega_s}\cdot\frac{L^{2(\mu-1)}}{2^{\mu}(\mu-1)!}\cdot Z_{\parallel,\,eff}.$$

$$Z_{\parallel,\,eff.} = \sum_{n=-\infty}^{+\infty} (\nu_L^n)^{2\mu-1}\, e^{-(\nu_L^n)^2 L^2/2}\, Z_{\parallel}(\nu_L^n\omega_0)$$

$$\nu_L^n = (nB + s + \mu Q_s)$$



Landau Damping with Freq. Spread.

Dispersion Relation:

Small bunches: $\omega_s(r) = \omega_{s0}\left(1 - \frac{h^2}{16} r^2\right)$

Gaussian Bunches:

$$1 = i \frac{e\eta I_0 B}{\pi p \beta c} \cdot \frac{1}{h^2} \cdot \left(\frac{\omega_0}{\omega_{s0}}\right)^2 2^{(2-\mu)} (\mu!)^{-2} L^{(2\mu-4)} Z_{\parallel,\mathrm{eff}} \left[G_\mu(\Omega)\right]^{-1}$$

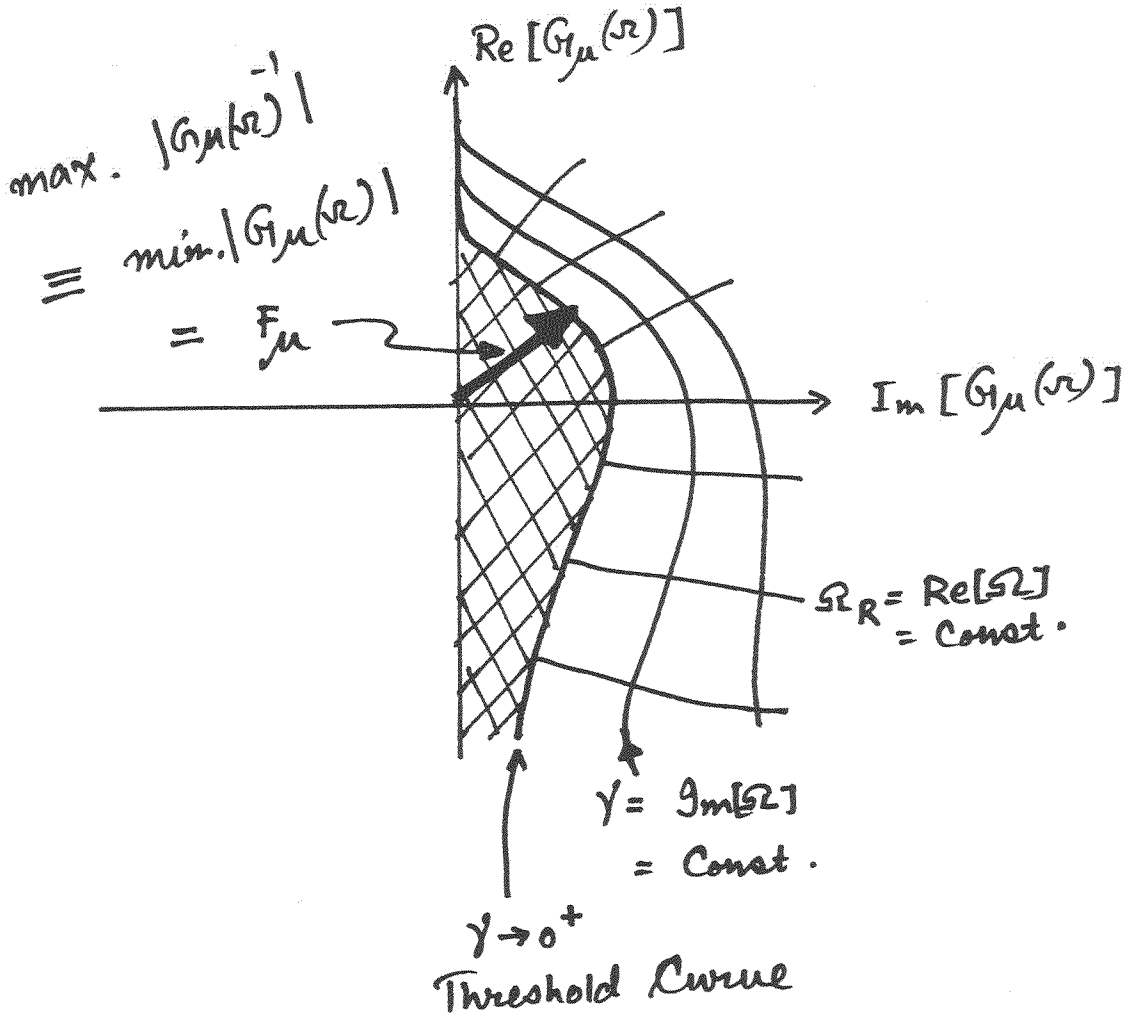
$$\left[G_\mu(\Omega)\right]^{-1} = \int_0^\infty dx \frac{x^\mu e^{-x}}{(x-y)}$$

$$y = \frac{8}{\mu \omega_{s0} h^2 L^2} (\mu \omega_{s0} - \Omega)$$

Self-consistently solved by mapping Technique:

$$\Omega = \Omega_R + i\gamma$$

$\uparrow$ growth rate.



μ	$F_\mu' = F_\mu \cdot (\mu!)$
1	0.777
2	1.12
3	1.37
4	1.59
5	1.78
6	1.95
7	2.11
8	2.25
9	2.39
10	2.52

Single Bunch Micro-Wave Thresholds.

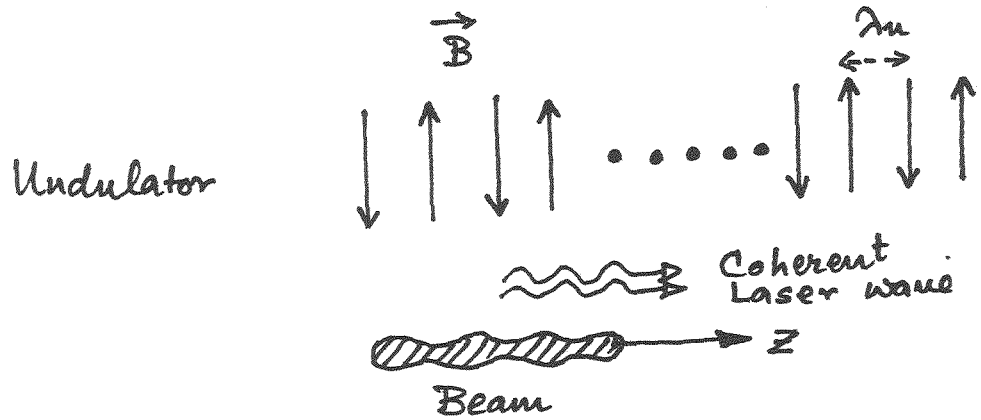
$$\hat{I}_L = \frac{2\pi |\eta| \sigma_p^2 (E/e)}{(Z/n)_{eff}}.$$ Longitudinal

$$\hat{I}_T = \frac{4\sqrt{2\pi}\,(E/e)\,\nu_s}{Z_T\,\bar{\beta}}$$ Transverse.

$$\sigma_\ell = \frac{\eta R}{\nu_s}\sigma_p.$$

FEL Calculations.

High Gain FEL



$$P = P_0 \, e^{gz}$$

$$g = 4\pi\sqrt{3}\,(\rho/\lambda_u)$$

$$\rho^3 = \frac{K^2 [JJ] r_e n_b \lambda_u^2}{32\pi\gamma^3} = \frac{1}{16\pi}\cdot\frac{r_e}{ec}\cdot\frac{K^3[JJ]}{2(1+K^2/2)}\cdot\frac{\lambda}{\gamma^2}\cdot\frac{\hat{I}}{\sqrt{\varepsilon_x\varepsilon_y}}$$

$$K = 0.934 \;\; \lambda_{u\,[cm]} \cdot B_{[Tesla]}$$

$$[JJ] = [J_0(\xi) - J_1(\xi)]^2$$

$$\xi = \left[K^2/4\left(1+\frac{K^2}{2}\right)\right]$$

$$P_{sat.} = \rho\, P_{beam} = \rho\, \hat{I}\, E/e \,.$$

Effect of Energy-Spread

$$g \rightarrow g' = f(\sigma_p, \rho)\, g$$
$$P_{sat} \rightarrow P'_{sat} = h(\sigma_p, \rho)\, P_{sat}. \quad \Big\} \; \sigma_p = \left(\frac{\Delta p}{p}\right)_{rms}.$$

'f' and 'h' obtained by solving a Dispersion Relation which is a 'cubic equation' for a 'rectangular' or a 'Lorentzian' longitudinal momentum distribution.

Cubic Equations: Example:

$$\lambda^3 - i\left(2\frac{\sigma_p}{\rho}\right)\lambda^2 + \left(\rho - \frac{\sigma_p^2}{\rho^2}\right) + 1 = 0$$

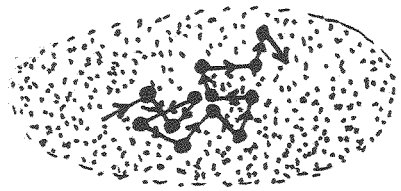
for Lorentzian distribution.

Sol^{n}: $\lambda = \lambda_R - i\lambda_I$

$$\rho_{eff.} = \lambda_I \rho$$
$$l_{eff.} = l/\lambda_I.$$

For Gaussian, we will have to solve a Dispersion Relation in the form of an integral as before.

Intrabeam Scattering



$$\frac{1}{\tau_i} = \frac{\pi^2 \alpha^2 M N (\log)}{\gamma \tilde{\Gamma}} \left\langle \int_0^\infty d\lambda \frac{\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}} \left\{ \mathrm{Tr}\, L^i \, \mathrm{Tr}\,(L+\lambda I)^{-1} - 3\, \mathrm{Tr}\, L^i (L+\lambda I)^{-1} \right\} \right\rangle_{\text{lattice Average}}$$

$$L = \sum_{i=1,2,3} L^i = L^{(h)} + L^{(\ell)} + L^{(v)}$$

$$L^{(h)} = \frac{\beta_x}{\epsilon_x} \begin{pmatrix} 1 & -\gamma\phi & 0 \\ -\gamma\phi & \frac{\gamma^2\eta^2}{\beta_x^2} + \gamma^2\phi^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^{(\ell)} = B \frac{\gamma^2}{\sigma_\eta^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

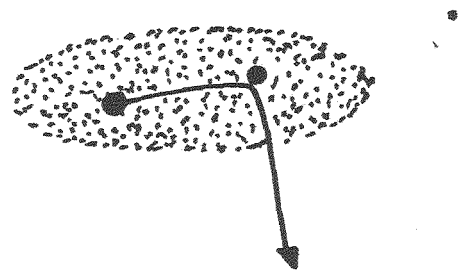
$B = 1$ unbunched
$= 2$ bunched

$$L^{(v)} = \frac{\beta_z}{\epsilon_z} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi = \eta' - \frac{\beta_x' \eta}{2\beta_x}$$

Touschek Scattering.

Large-angle scattering between particles within the beam.



$$\frac{1}{\tau_T} = \frac{\sqrt{\pi}\, r_e^2\, c\, N_b\, F(x)}{\gamma^3 \sigma_{x'} \left(\frac{\Delta p}{p}\right)^2_{BKT} V_B}$$

$$F(x) = \int_1^\infty \left[2u - \ln(u) - 2\right] \frac{e^{-ux}}{2u^2}\, du$$

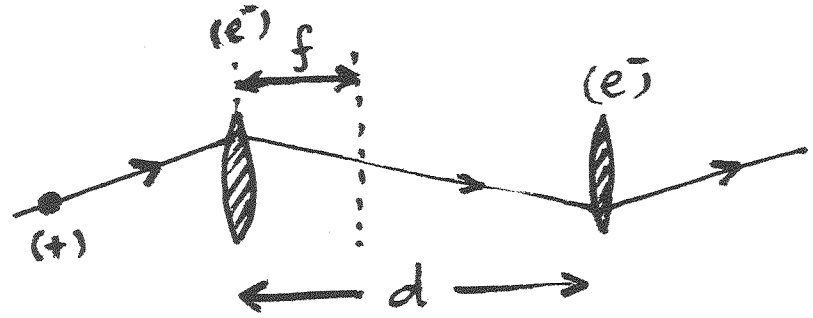
$$x = \left[\frac{1}{\gamma \sigma_{x'}} \left(\frac{\Delta p}{p}\right)_{BKT}\right]^2$$

$\left(\Delta p / p\right)_{BKT}$ ~ Min {Bucket Acceptence, Dynamic Aperture, Physical Aperture}.

$\sigma_{x'}$ → includes effect of lattice dispersion on beam dimensions.

Ion Trapping

Thin lens model:



Stability:

$$d \leq 4f$$

$$\Rightarrow |Tr(M)| \leq 2$$

Critical Mass for Stability:

$$A \geq A_c$$

$$A_c = 2\left(\frac{\pi R}{n}\right)^2 \left(\frac{m_e}{m_p}\right)\left(\frac{I}{I_A}\right) \frac{1}{\sigma_{y,x}(\sigma_x + \sigma_y)}$$

$$I_A = 17{,}045 \text{ Amp.}$$

Ion Accumulation Limit:

$$\bar{d}_i = \frac{N}{2\pi R} \cdot \frac{\beta}{2\pi \sigma_x \sigma_y} = \frac{\beta}{B} \cdot d_e$$

$B =$ Bunching factor $= 2\pi R / n l_B$.

$\eta =$ Neutralization factor $= \dfrac{\bar{d}_i}{d_e} = \beta / B$.

Time Shifts:

$$\Delta Q_{y,x} = r_e \cdot \frac{m_e c^2}{E} \int_0^{2\pi R} \frac{\bar{d}_i}{\left(1 + \frac{\sigma_{y,x}}{\sigma_{x,y}}\right)} \beta_{y,x}(s)\, ds.$$

$$= r_e \cdot \frac{m_e c^2}{E} \cdot \bar{d}_i \cdot 2\pi R \left\langle \frac{\beta_{y,x}}{1 + \frac{\sigma_{y,x}}{\sigma_{x,y}}} \right\rangle$$

$$\frac{\Delta Q_y}{\Delta Q_x} \propto \left(\frac{\sigma_x}{\sigma_y}\right)$$

$$(\Delta Q_{x,y})_{s.c} = r_e \frac{m_e c^2}{E} \int \frac{de}{\left(1 + \frac{\sigma_{x,y}}{\sigma_{y,x}}\right)} \frac{\beta_{x,y}(s)}{\gamma^2}\, ds$$

$$\Delta Q_{x,y}\,(\text{ions}) = \eta \gamma^2\, \Delta Q_{x,y}\,(\text{space-charge}).$$