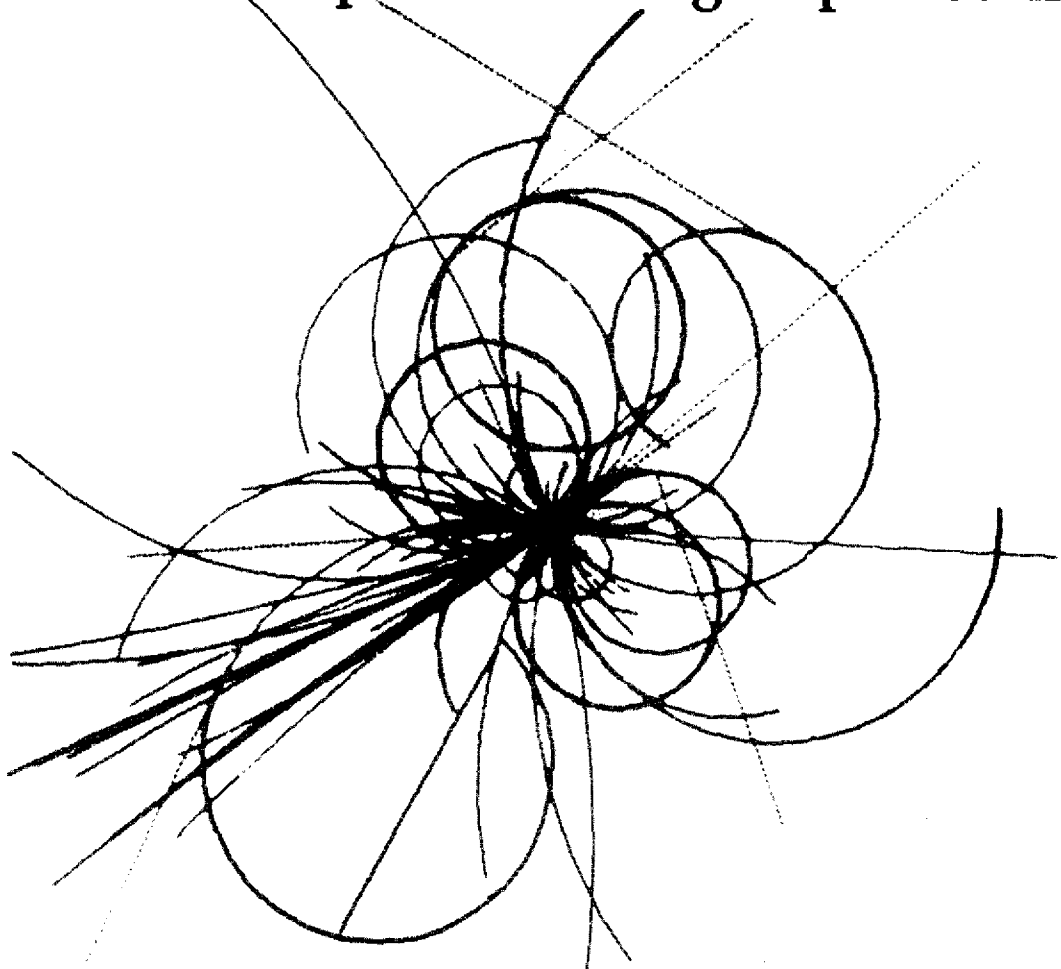
SSC-95

# The Superconducting Super Collider



# Analytical Computation of the Smear

Etienne Forest
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# ANALYTICAL COMPUTATION OF THE SMEAR

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# ANALYTICAL COMPUTATION OF THE SMEAR

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Abstract

We explain an analytic computation of the map and the "smear" for a lattice composed of cells with random sextupole and octupole errors, as well as systematic chromatic sextupoles. Results are presented for the 60° and 90° lattices.

## I. INTRODUCTION

It has often been stated that the SSC is largely dominated by errors. This implies that an "ensemble of machines" must be studied. In practice, one generates the various random components of the lattice using a certain algorithm. In the cases explored during the linear aperture studies, it amounted to generating a sequence of random numbers. Hence emerges the perennial question; do we have enough "seeds"? Are the "statistics" good enough?

To bypass this problem, we decided to go ahead with a realistic analytical computation which could be checked on selected cases by extensive tracking.[1] The checks on extreme cases allowed us to extrapolate with confidence in all the regions of interest.

In this report, we will document a technique most relevant to the study of a machine made out of cells. The technique can be adapted to include an actual insertion. However, for simplicity, we will ignore the existence of anything but the cells.

## II. THE MAP OF THE MACHINE WITH ERRORS ONLY

Consider the $i^{th}$ cell. It has a mapping $\mathscr{M}_i$. In general, it can be written as follows:

$$\mathscr{M}_i = \mathscr{A}_2^{-1} \exp(-\underline{\mu}\cdot\hat{\underline{J}}) \exp(\hat{f}_i) \mathscr{A}_2 \tag{2.1a}$$

$$\hat{g} = :g: = [g,] = -L_g \text{ ,etc...} \tag{2.1b}$$

$$\underline{J} = (J_x, J_y) \tag{2.1c}$$

Here, $J_x$ and $J_y$ are the action variables of the linear map. The operator $\hat{f}$ associated to f contains all the errors and the systematic non-linearities. The map $\mathscr{A}_2$ is a canonical transformation bringing the systematic linear part of $\mathscr{M}_i$ in Floquet variables. This is described in SSC-29,30 and 78. We will follow here the notation of SSC-78.

The full map of the machine is simply the product of the individual maps. So, assuming N cells, one gets:

$$\mathscr{M} = \prod_{i=1}^{N} \mathscr{M}_i = \mathscr{M}_1 \cdots \mathscr{M}_N$$

$$= \mathscr{A}_2^{-1}\exp(-N\underline{\mu}\cdot\hat{\underline{J}}) \exp(:\tilde{f}_1:)\ldots \exp(:\tilde{f}_N:) \mathscr{A}_2 \tag{2.2a}$$

$$\tilde{f}_K = \exp((K-1)\underline{\mu}\cdot\hat{\underline{J}}) f_K \ . \tag{2.2b}$$

Obviously, we must have $N\mu_i = 2\pi Q_i$. To proceed further, we can combine the exponents $\tilde{f}_K$ into a single grand exponent. Doing this to second order in $\tilde{f}_K$, gives the result:

$$\mathscr{M} = \mathscr{A}_2^{-1} \exp(-N\underline{\mu}\cdot\hat{\underline{J}}) \exp(:\varepsilon\tilde{W}+\varepsilon^2\tilde{V}:) \mathscr{A}_2 \tag{2.3a}$$

$$\tilde{W} = \sum_{k=1}^{N} \tilde{f}_K \tag{2.3b}$$

$$\tilde{V} = \frac{1}{2} \sum_{k<j} [\tilde{f}_K, \tilde{f}_j] \ . \tag{2.3c}$$

Equation (2.3) gives the final map $\mathcal{M}$ to second order in $\varepsilon$. Each individual polynomial $f_K$ is a function of the particular errors in the $K^{th}$ cell. Hence $f_K$ is not a constant. It would be nice, if we had an expression of the $f_K$ in terms of the errors in the cells. Such an expression would be functionally invariant cell to cell. Consider the error vector $\vec{V}_K$ = (random multipole strengths in the $K^{th}$ cell), we can rewrite $f_K$ as follows:

$$f_K = f(\vec{V}_K) = \sum_{a,b,c,d} (\Gamma^i_{abcd}\, V^i_K + \Delta^{ij}_{abcd}\, V^i_K\, V^j_K)\, |abcd\rangle + \ldots \tag{2.4a}$$

$$|abcd\rangle = J_x^{a/2} J_y^{b/2} \exp\{i(c\phi_x + d\phi_y)\} \ . \tag{2.4b}$$

The eigenvectors $|abcd\rangle$ of $\hat{J}$ are defined as in SSC-78. The matrix quantities $\Gamma$ and $\Delta$ are not error-dependent. There are the responses of the lattice to the multipole excitations vector $\vec{V}_K$.

Since our problem has been formulated through the use of mappings, all the unnecessary information has been integrated out. The quantities $\Gamma$ and $\Delta$ can be computed with the help of a standard tracking code equipped with a map producting algorithm. (MARYLIE, RACETRACK or TEAPOT). Basically, one computes the map of a single cell (dim $\vec{V}_K$) times for the computation of $\Gamma$

$$\text{and} \quad \frac{(\dim \vec{V}_K) \times (\dim \vec{V}_K - 1)}{2} \quad \text{for } \Delta \quad .$$

The maps are fed into the analysis subroutines of MARYLIE which extract the matrices $\Gamma$ and $\Delta$.

In the next section, we compute the smear to first order in $\vec{V}$.

## III. COMPUTATION OF THE INVARIANTS TO FIRST ORDER IN $\vec{V}$

Following the results of Appendix A, we compute the canonical transformation which removes the non-secular terms in $\tilde{W}$:

$$\mathscr{N} = \mathscr{A}^{-1} \exp(-N\mu\cdot\hat{J}) \exp(\epsilon\hat{\tilde{W}})\mathscr{A} \tag{3.1a}$$

$$\mathscr{A} = \exp(\epsilon\hat{G}) \ . \tag{3.1b}$$

Splitting W in two parts,

$$W_R = \sum_{|c|+|d|\neq 0} A_{abcd} \ |abcd\rangle \tag{3.2a}$$

$$W_K = W_K(J_x, J_y) = \sum A_{ab00}|ab00\rangle \ . \tag{3.2b}$$

we get for G:

$$G = \frac{1}{1 - \exp\ (N\underline{\mu}\cdot\hat{J})} \ W_R$$

$$= \sum_{|c|+|d|\neq 0} A_{abcd} \frac{1}{1 - \exp(-i(c\mu_x+d\mu_y)N)}|abcd\rangle \ . \tag{3.3}$$

According to Appendix A, we can write the old invariants $\underline{J}$ in terms of the new invariants I:

$$\underline{J} \simeq \underline{I} + \epsilon[G(\underline{\psi},\underline{I}), \ \underline{I}] + \ldots \tag{3.4}$$

In (3.4), G is a function of $\underline{I}$ and $\underline{\psi}$ as defined by (A3) and (A4). For example, we get for $\Delta J_x(=J_x-I_x)$:

$$\Delta J_x = \epsilon \sum_{|c|+|d|\neq 0} A_{abcd} \frac{1}{1-\exp(-i(c\mu_x+d\mu_y)N)} \ |abcd\rangle \tag{3.5}$$

Now, we must compute $\Delta J_x^2$ and average of the phases $\psi_x$ and $\psi_y$:

$$\langle\Delta J_x^2\rangle = \frac{\epsilon^2}{2} \ \sum \frac{A_{abcd}A^*_{a'b'cd} \ c^2}{1 - \cos((c\mu_x+d\mu_y)N)} \ I_x^{\frac{a+a'}{2}} \ I_y^{\frac{b+b'}{2}} \ . \tag{3.6}$$

Specializing to equal emittance ($I_x = I_y = I$), we simplify (3.6):

$$\langle \Delta J_x^2 \rangle = \frac{\varepsilon^2}{2} \sum \frac{A_{abcd} A^*_{a'b'cd} c^2}{1 - \cos((c\mu_x + d\mu_y)N)} I^S \tag{3.7a}$$

$$S = 3 : \text{sextupole}$$
$$S = 4 : \text{octupole} . \tag{3.7b}$$

Finally, we must average the expression (3.7a) over the $V_K$'s. The result is in the lowest order in multipole strengths:

$$\langle A_{abcd} A^*_{a'b'cd} \rangle = \sum_{K=1}^{N} \sum_{K'=1}^{N} \Gamma^i_{abcd} \Gamma^{j*}_{a'b'cd} \langle V^i_K V^j_{K'} \rangle . \tag{3.8}$$

In the case of uncorrelated cells ("bined" or unmeasured), the seed average is given by:

$$\langle A_{abcd} A^*_{a'b'cd} \rangle = N \sum_{|c|+|d| \neq 0} \Gamma^i_{abcd} \Gamma^{i*}_{a'b'cd} \langle V_i^2 \rangle \tag{3.9}$$

Including random sextupoles and octupoles in (3.7), we now have an expression for $\langle \Delta J_x^2 \rangle$ which has two terms:

$$\langle \Delta J_x^2 \rangle = B_S^x I^3 + B_0^x I^4 . \tag{3.10}$$

The smear, which is the fluctuation of the amplitude $J_x^{1/2}$, is obtained as follows:

$$S_x = \frac{1}{2}\sqrt{B_S^x I + B_0^x I^2} = \frac{1}{2}\sqrt{\frac{B_S^x}{2\hat{\beta}} \hat{a}^2 + \frac{B_0^x}{4\hat{\beta}^2} \hat{a}^4} \tag{3.11a}$$

$$\hat{\beta} = \text{maximum } \beta . \tag{3.11b}$$

The linear aperture is obtained by the equations:

$$\begin{aligned} S_x(\hat{a}_x) &= r \quad , \; r = 6.4\% \\ S_y(\hat{a}_y) &= r \end{aligned} \tag{3.12}$$

The minimum of the couple ($\hat{a}_x$, $\hat{a}_y$) is the linear aperture. The value of r(=6.4%) was empirically determined to match the peak to peak definition used in tracking studies.

## IV. INCLUSION OF CHROMATIC SEXTUPOLES

This was done by a semi-analytical method. A ring with no-errors was produced. The chromaticity sextupole were ajusted at their proper value. Then a map was produced and analyzed in a Floquet representation. With this information, we augmented equation (3.10) as follows:

$$\langle \Delta J_x^2 \rangle = (B_s^x + B_c^x)\ I^3 + (B_0^x + B_{cc}^x)\ I^4 \ . \tag{4.1}$$

The quantity $B_c^x$ is linear in chromatic sextupole strength and $B_{cc}^x$ is quadratic. $B_{cc}^x$ turned out to be main contributor. Notice that $B_c^x$ is zero if the machine is a perfect achromat. Here also we neglected cross-terms involving a random sextupole and a chromatic sextupole.

## V. SCALING WITH CELL LENGTH

The purpose of this smear calculation was to compare cell lengths. We had 4 cases to discuss. These were the 4, 5, 6 and 7 magnets per half cell. This is a lot of tedious work especially if cross terms are to be estimated. Therefore we computed only the 4 and 7 magnet cases and used approximate scaling laws to generate any other length. In theory, we needed only one length, however the two lengths allowed us to check the validity of our crude scaling laws.

These laws are summarized:

$$\hat{\beta}' = \hat{\beta}\ \sigma \qquad \eta' = \eta\ \sigma^2 \tag{5.1a}$$

$$B_s^{x,y'} = B_s^{x,y}\ \sigma^3 \tag{5.1b}$$

$$B_0^{x,y'} = B_0^{x,y}\ \sigma^4 \tag{5.1c}$$

$$B_c^{x,y'} = B_c^{x,y}\ /\sigma^3 \tag{5.1d}$$

$$B_{cc}^{x,y'} = B_{cc}^{x,y}/\sigma^8 \tag{5.1e}$$

$$\sigma = \frac{L'}{L} \quad . \tag{5.1f}$$

The equations in (5.1) allowed us to give a continuous plot of linear aperture versus cell length. These are given in graph (5.2) and (5.3).

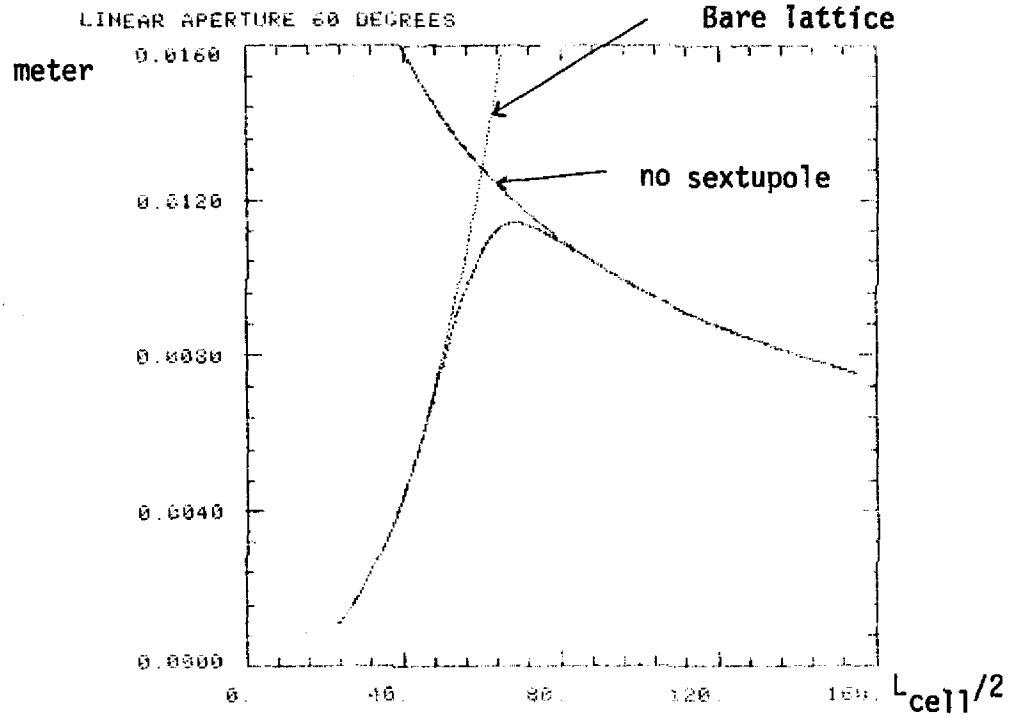


Fig.(5.2)

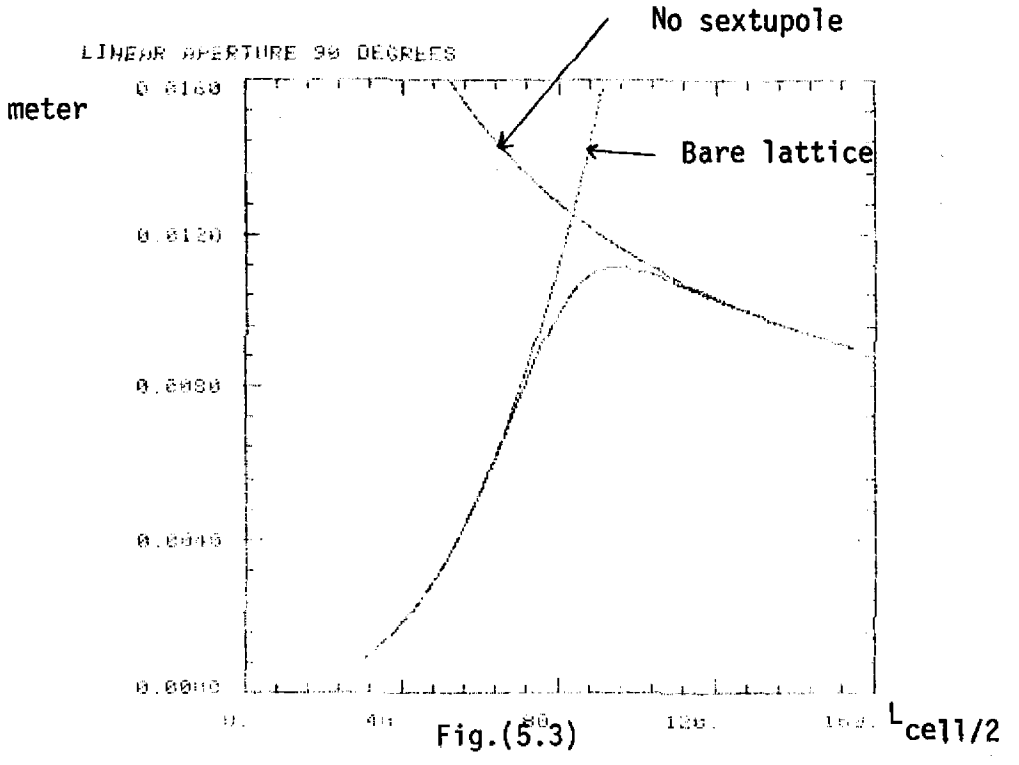


Fig.(5.3)

## VI. OFF-MOMENTUM LINEAR APERTURE

In this calculation, we do not claim exactness. All that was done, was to compute the feed-down effect on ($a_2$, $b_2$, $a_3$, $b_3$) on the basis of an average dispersion number. Using (5.1a), we can scale $\eta$ for different cell length. The feed-down on the rms-values obeys that following formulas:

$$\langle a_2^2 \rangle = \sum_{i=2}^{\infty} \left[\frac{i(i-1)}{2} \quad a_i \quad (\eta\delta)^{i-1}\right]^2 \tag{6.1a}$$

$$\langle a_3^2 \rangle = \sum_{i=3}^{\infty} \left[\frac{i(i-1)(i-2)}{6} \quad a_i \quad (\eta\delta)^{i-2}\right]^2 \tag{6.1b}$$

Similar expressions apply for the erect multipoles.

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[1] For example, Beat Leemann found the following results by tracking:

| Magnets/ Half Cell | Phase Advance | Coil Diameter (cm) | $A_{TRACKING}$(mm) | $A_{THEORY}$(mm) |
|---|---|---|---|---|
| 7 | 60° | 4 | 8.06 ± 0.4 | 8.4 |
| 4 | 90° | 4 | 10.7 ± 0.4 | 10.5 |
| 6 | 90° | 4 | 10.65 ± 0.6 | 10.5 |
| 6 | 90° | 6 | 17.5 ± 0.75 | 17.1 |

## APPENDIX A. PROOF OF THE EQUIVALENCE OF TIME AND PHASE AVERAGE TO ALL ORDER IN THE PERTURBATION

Consider a map $\mathscr{M}$. We assume that $\mathscr{M}$ is integrable. That is:

$$\exists \mathscr{A} \text{ such that } \mathscr{A}\mathscr{M}\mathscr{A}^{-1} = \exp[-\mu.\hat{J}+\hat{F}(J)] \tag{1}$$

where $[J_i, \phi_j] = \delta_{ij}$.

$$\hat{J} = :J: = -L_J = [J,] \tag{2}$$

Using (1), we can construct n-invariants of the map:

$$I_i = \mathscr{A}^{-1}J_i \quad \text{or} \quad I_i = J_i \tag{3}$$

Canonically conjugate to $I_i$, we can define new phases $\psi_i$:

$$\psi_i = \mathscr{A}^{-1}\phi_i \tag{4}$$

The "smear" is by definition the average oscillation of $J_i$ under repeated application of $\mathscr{M}$:

$$S_i = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} (\mathscr{M}^m J_i)^2 - \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \mathscr{M}^m J_i{}^2 \tag{5}$$

We first compute $\mathscr{M}^m J_i$.

$$\begin{aligned}
\mathscr{M}^m J_i &= \mathscr{A}^{-1}\exp[-\mu.\hat{J}+\hat{F}(J)]^m \mathscr{A} J^i \\
&= \mathscr{A}^{-1}\exp[-m\{\mu.\hat{J}-\hat{F}(J)\}]\mathscr{A}\mathscr{A} I_i \\
&= \exp[-m\{\mu.\hat{I}-\hat{F}(I)\}][\underbrace{I_i+G_i(I,\psi)}_{\mathscr{A} I_i}] \qquad (6) \\
&= I_i + G_i[I,\psi+m(\mu-\frac{\partial F}{\partial I})] \\
&= I_i + G_i[I,\psi+m\omega(I)]
\end{aligned}$$

If $\omega$ (I) has irrational and incommensurate components:

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \Leftrightarrow \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} d\psi_1 \ldots d\psi_n \quad . \tag{7}$$

Hence the phase average using the $\mathscr{A}$-script (Analytical) and the numerical average over turns must be equivalent.

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Appendix B: Cell used in the study. (provided by D. Johnson)

qf/2 oo os sf $N_b$*( b o) qd oo sd os $N_b$*( b o) qf/2

Drifts : o = .8 m

os= 2.0 m

oo= 3.98m

Dipole: $L_b$=16.54m B=6.5997 T Bρ=66712.8 Tm

Quadrupole: $L_q$= variable B'= 212 T/m

Sextupole : $L_s$= 0 B"L/Bρ= variable*

| $N_b$ | | 4 | 7 |
|---|---|---|---|
| $60^o$ | $L_q/2$ | 2.02m | 1.22m |
| | $L_{cell}$ | 158.76m | 259.61m |
| $90^o$ | $L_q/2$ | 2.82m | 1.72m |
| | $L_{cell}$ | 161.96m | 261.60m |

*Provided by Beat Leemann numerical studies.