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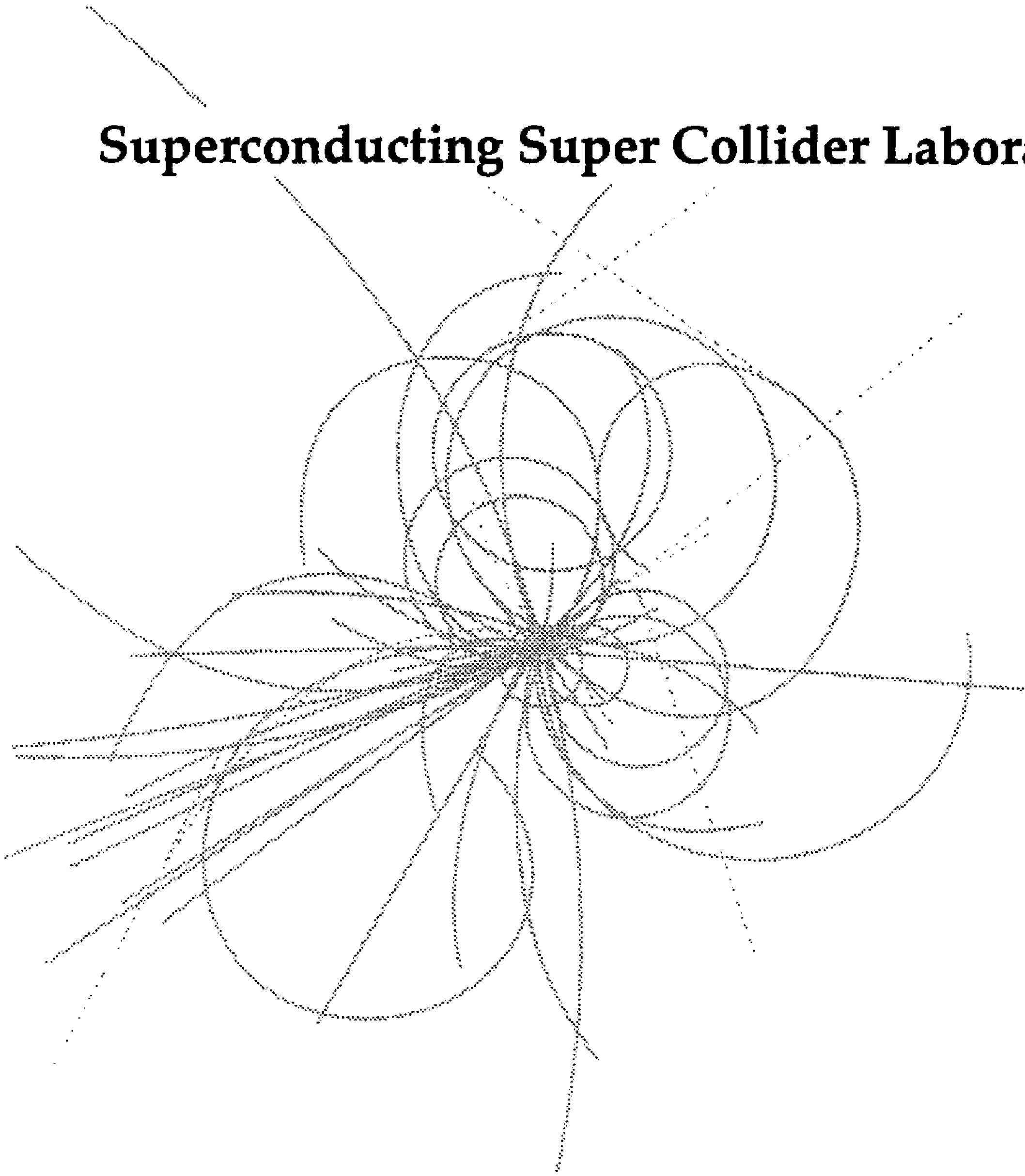

Energy Loss of Bunched Beams in RF Cavities

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ABSTRACT

Bunched, charged particle beams lose energy when they traverse cavities or other structures which can be described by resonator impedances. The calculation of this loss is extended to arbitrary quality factors by using known approximations for the sums of infinite series. For low Q-values, these expressions agree with those obtained by replacing the sums by integrals. The loss power in SSC is calculated using these expressions.

1. Introduction

The calculation of the energy loss of a Gaussian bunch transversi resonator impedance has recently been refined in a number of SSC repo In a circular machine, the energy loss per revolution or loss power o in general is given by

$$P_{loss} = \sum_{p=-\infty}^{\infty} \mathrm{Re}Z(p\omega_o)\,|\tilde{I}\,(p\omega_o)|^2 ,$$

where $\omega_o = 2\pi/T$ is the (circular) revolution frequency, and $\tilde{I}(\omega)$ the Fo transform of the beam current

$$\tilde{I}(\omega) = \frac{1}{T} \int_{-T/2}^{+T/2} I(t) e^{-j\omega t} dt \ .$$

Since the current in a bunch must vanish outside its rf bucket, th of integration can be extended to $\pm\,\infty$.

Assuming a Gaussian bunch, the current can be written

$$I_b(t) = \hat{I} \exp(-\frac{t^2}{2\sigma^2}) \ ,$$

where $\hat{I} = I_b(0)$ is the peak current, and σ the rms bunch length in tim ($\sigma = \sigma_z/c$). The average bunch current is

$$I_b = \frac{1}{T} \int_{-T/2}^{T/2} I_b(t) dt \ = \hat{I} \frac{\sigma\sqrt{2\pi}}{T} \ .$$

The Fourier transform is obtained from Eq. (2) with extended limits

$$\tilde{I}_b(\omega) = \frac{\hat{I}}{T} \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2\sigma^2} - j\omega t)\ dt$$

$$= I_b \cdot e^{-\frac{\omega^2\sigma^2}{2}} \ ,$$

where we have introduced the average bunch current defined in Eq. (4). The loss power for a (single) Gaussian bunch thus is given by

$$P_{loss} = I_b^2 \sum_{p=-\infty}^{\infty} \mathrm{Re}Z(p\omega_o) \exp(-p^2\omega_o^2\sigma^2). \tag{6}$$

For large accelerators with short bunches, the value of $\omega_o\sigma = \sigma_z$/radius can be very small and the exponential factor then falls off only for rather high values of the summation index p. If the impedance varies slowly, one can approximate the sum by an integral, but for impedances with narrow peaks this is not permissible.

2. Multibunch Case

Most accelerators are operated with more than one bunch circulating, and for the SSC there are even some 17,000 bunches foreseen. Usually one attempts to have equally spaced identical bunches, but sometimes gaps are left in an otherwise uniform bunch train, or a bunch-to-bunch spread in population is introduced on purpose in order to damp coherent oscillations. We shall therefore treat the general case first and specialize to equally spaced identical bunches only in the last stage.

Equations (1) and (2) are correct for any current distribution. We study a current distributed over M - in general different - Gaussian bunches. Each bunch is characterized by a peak current $\hat{I}_k$, an rms bunch-length σ_k, and a position t_k

$$I(t) = \sum_{k=1}^{M} \hat{I}_k \exp\left|-\frac{(t-t_k)^2}{2\sigma_k^2}\right| . \tag{7}$$

The average current I_o is found to be the sum of the average currents in each bunch

$$I_{bk} = \hat{I}_k \cdot \frac{\sigma_k\sqrt{2\pi}}{T} \tag{8}$$

which is seen to depend only on the product of the peak current and t
width.

The Fourier transform of Eq. (7) is found from Eq. (2), with limi
extended to $\pm\infty$

$$\tilde{I}(\omega) = \frac{1}{T} \sum_{k=1}^{M} \hat{I}_k \int_{-\infty}^{\infty} e^{-\frac{(t-t_k)^2}{2\sigma_k^2}} e^{-j\omega t} dt$$

$$= \frac{\sqrt{2\pi}}{T} \sum_{k=1}^{M} \hat{I}_k \sigma_k \, e^{-\frac{\omega^2\sigma_k^2}{2}} e^{-j\omega t_k} .$$

Since $\hat{I}_k \sigma_k = I_{bk} \, T/\sqrt{2\pi}$, we can simplify this as

$$\tilde{I}(\omega) = \sum_{k=1}^{M} I_{bk} \exp\left(-\frac{\omega^2\sigma_k^2}{2} - j\omega t_k\right)$$

$$= \sum_{k=1}^{M} \tilde{I}_{bk}(\omega) \, e^{-j\omega t_k} ,$$

i.e., the Fourier transform of the total beam current is the sum of th
Fourier transforms of each bunch, multiplied by phase factors dependir
their positions t_k.

From now on we assume that all bunches are identical ($I_{bk} = I_b$, σ_k
and hence $\tilde{I}_{bk} = \tilde{I}_b$) and equally spaced ($t_k = kT/M$, with $k = 1,2,...M$).
Then

$$\tilde{I}(\omega) = \tilde{I}_b(\omega) \sum_{k=1}^{M} e^{-\frac{jk\omega T}{M}}$$

and in particular for $\omega_o = p\omega_o = \frac{2\pi p}{T}$

$$\tilde{I}(p\omega_o) = \tilde{I}_b(p\omega_o) \sum_{k=1}^{M} e^{-2\pi jpk/M} .$$

The sum over the p-th power of all n-th roots of unity vanishes except when p is a multiple of M, then each term is one and the sum thus simply M.

We thus have $\tilde{I}(p\omega_o) = 0$ except for $p = nM$, where n is an integer,

$$\tilde{I}(nM\omega_o) = M\tilde{I}_b(nM\omega_o) \; , \tag{13}$$

i.e., only multiples of the "bunch-frequency" $\omega_b = M\omega_o$ appear in the spectrum. The power loss of M equally spaced, identical bunches can thus be written as

$$P_{loss} = M^2 \sum_{p=-\infty}^{\infty} \mathrm{Re}Z(p\omega_b)|\tilde{I}_b(p\omega_b)|^2 \; . \tag{14}$$

However, since the sum is only over each M-th multiple of ω_o, it will be M times smaller than the sum over all lines for slowly varying impedances. We shall therefore split the factor M^2 and write the power loss as

$$P_{loss} = M \; Z_{loss} I_b^2 \; . \tag{15}$$

The effective loss impedance for Gaussian bunches is defined by

$$Z_{loss} = M \sum_{p=-\infty}^{\infty} \mathrm{Re}Z(p\omega_b)\exp[-(p\omega_b\sigma)^2] \tag{16}$$

and can be easily generalized to other distribution functions.

3. <u>Resonator Impedance</u>

The interaction of a charged particle beam with its surroundings is usually described by impedances. In accelerators or storage rings, the major contributors to the overall impedance are often the rf cavities or other unavoidable cross-section variations of the vacuum chamber. These can be approximated quite well by a number of (parallel) resonator impedances, each characterized by a resonant frequency $\omega_r/2\pi$, a shunt impedance R, and a quality factor Q (or, alternatively, by the bandwidth $\Delta\omega = \omega_r/Q$). The complex impedance (assuming a time variation $e^{j\omega t}$) can be written

$$Z(\omega) = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \quad .$$

It is often more convenient to expand this expression by partial fract

decomposition

$$Z(\omega) = \frac{R}{jS}\left(\frac{\omega_1}{\omega-\omega_1} - \frac{\omega_2}{\omega-\omega_2}\right) \; ,$$

where

$$S = \sqrt{4Q^2-1}$$

and

$$\omega_{1,2} = \frac{\omega_r}{2Q}\,(j \pm S)$$

are the (complex) poles of the impedance.

To calculate the power loss, we need the impedance evaluated at mu

of the bunch frequency $\omega_b = M\omega_o$

$$Z(p\omega_b) = \frac{R}{jS}\left(\frac{\nu_1}{p-\nu_1} - \frac{\nu_2}{p-\nu_2}\right)$$

with

$$\nu_{1,2} = \frac{\omega_{1,2}}{\omega_b} = \frac{p_r}{2Q}(j \pm S),$$

where

$$p_r = \frac{\omega_r}{\omega_b} = \frac{\omega_r}{M\omega_o}$$

is the ratio of resonant and bunch frequencies.

In order to evaluate Eq. (16) for the loss impedance, we need the

infinite series of the form

$$F(a,\nu) = \sum_{p=-\infty}^{\infty} \frac{\exp(-a^2p^2)}{p - \nu} \quad ,$$

where $a = \omega_b\sigma$ is real, but ν in general complex.

This series converges only very slowly for small values of a, but then it can be summed analytically to very high accuracy.[4] Indeed, the first neglected term in the derivation is of the order of $\exp(-\pi^2/a^2)$, which for $a = 1/10$ approaches $e^{-1000} \approx 10^{-400}$!

This sum is given by the expression

$$F(a,\nu) = j\pi \{w(a\nu) - e^{-a^2\nu^2}[1-j.\cot(\pi\nu)]\} , \tag{25}$$

where w(z) is the complex error function.[5] We thus obtain the general expression for the loss impedance of k equally spaced, identical Gaussian bunches traversing a resonator with any value of ω_r, R_s, Q

$$Z_{loss} = \frac{\pi R}{\omega_o \sigma S} \{\alpha_1 [w(\alpha_1) - e^{-\alpha_1^2}(1-j.\cot \pi\nu_1)]$$

$$-\alpha_2 [w(\alpha_2) - e^{-\alpha_2^2}(1-j.\cot \pi\nu_2)]\} . \tag{26}$$

In Eq. (26), we have introduced

$$\alpha_{1,2} = a\nu_{1,2} = \frac{\omega_r \sigma}{2Q}(j \pm S), \tag{27}$$

which are independent of the number of bunches, which appears only in the arguments $\pi\nu_{1,2}$ of the cotangent.

4. Broad-band Resonator

For single bunches it is often advantageous to replace the large number of (sharp) resonances of real structures by a single one with broad bandwidth. These impedances are equivalent if they have the same wake potential over the length of the bunch.

A broad bandwidth corresponds to a small quality factor. When Q compared to $p_r = \omega_r/\omega_b$, the imaginary part of the poles $\nu_{1,2}$ becomes large. Since the cotangent of complex argument can be written

$$\cot(x+jy) = -j\,\frac{1+e^{2jx}\cdot e^{-2y}}{1-e^{2jx}\cdot e^{-2y}}$$

one finds $\cot z \to -j$ for $y \gg 1$. Then the expression in square brack Eq. (25) vanishes and one gets simply

$$F(a,\nu) \approx j\pi\, w(a\nu)$$

which is exactly the same result as that obtained when the infinite s replaced by an integral.[2]

The broad-band loss-impedance becomes to a very good approximatio

$$Z^{BB}_{loss} = \frac{\pi R}{\omega_o \sigma S}\,[\alpha_1 w(\alpha_1) - \alpha_2 w(\alpha_2)]$$

and does thus not depend on the number of bunches.*

i) For $Q > 1/2$, the quantity $S = \sqrt{4Q^2-1}$ is real and hence $\nu_2 = -\nu_1^*$. Since furthermore $w(-z^*) = w^*(z)$, we find that the loss impedance is resistive

$$Z^{BB}_{loss}\ (Q > 1/2) = \frac{2\pi R}{\omega_o \sigma S}\,\mathrm{Re}[\alpha_1 w(\alpha_1)]$$

ii) For $Q < 1/2$, S is imaginary and so is $\alpha_{1,2} = j.y_{1,2}$, with

$$y_{1,2} = \frac{\omega_r \sigma}{2Q}\,(1 \pm \sqrt{1-4Q^2})$$

* This justifies including the factor M in the definition of the loss impedance Eq. (16)

The complex error function of imaginary argument is purely real[5])

$$w(jy) = e^{y^2} \text{erfc}\,(y) \tag{33}$$

where erfc(y) = 1 - erf(y) is the "complementary error function." Then the loss impedance is again resistive

$$Z^{BB}_{loss}\ (Q < 1/2) = \frac{\pi R}{\omega_o \sigma \sqrt{1-4Q^2}} [\alpha_1 e^{\alpha_1^2} \text{erfc}(\alpha_1) - \alpha_2 e^{\alpha_2^2} \text{erfc}(\alpha_2)] \tag{34}$$

and is also independent of the number of bunches.

iii) In the limit $Q \to 1/2$, both expressions (31) and (34) become indeterminate. Taking the limit $\varepsilon \to 0$ for $Q = 1/2 \pm \varepsilon$ in Eqs. (31) or (34) yields

$$Z^{BB}_{loss}\ (Q = 1/2) = 2\pi R[(1+2\alpha^2)e^{\alpha^2} \text{erfc}(\alpha) - \frac{2\alpha}{\sqrt{\pi}}] \ , \tag{35}$$

where $\alpha = \omega_r \sigma$ for short.

5. Narrow-band Resonators

For resonances with bandwidths of the order of or smaller than the bunch frequency, it is no longer permitted to replace the sum by an integral. The resonances of (metallic) cavities usually have quality factors of some tens of thousands, and the situation $Q \geq p_r = \omega_r/M\omega_o$ is often encountered. Since for all $Q > 1/2$: $\nu_2 = -\nu_1^*$, we can rewrite Eq. (26) in any case as

$$Z_{loss} = \frac{2\pi R}{\omega_o \sigma S} \text{Re}\ \{\alpha_1[w(\alpha_1) - e^{-\alpha_1^2}(1 - j.\cot \pi\nu_1)]\}\ . \tag{36}$$

The loss impedance is again purely resistive, but now the cotangent become important. Since its argument is complex it will always rem as can be seen from another form of the cotangent of a complex vari

$$\cot(x + jy) = \frac{1}{2}\ \frac{\sin 2x - j \sinh 2y}{\sin^2 x + \sinh^2 y} \quad ,$$

where the denominator is a sum of squares which does not vanish for

However, the loss impedance will depend strongly on the resonat frequency, which is often varying with temperature and/or small def of the vacuum chamber wall. In this case it is indicated to search maximum loss impedance, which is easily done numerically by changin resonant frequency in small steps.

6. Analytic Approximations

For short bunches such that $|\alpha_{1,2}| = \omega_r\sigma \ll 1$, we can approximat complex error function by the lowest terms of its power series expan

$$w(z) \approx 1 - \frac{2jz}{\sqrt{\pi}} - z^2 + - \ldots$$

For a broad-band resonator impedance with $Q > 1/2$, one obtains t Eq. (31)

$$Z^{BB}_{loss} = \pi \frac{\omega_r}{\omega_o}\ \frac{R}{Q} \left[1 + \frac{2}{\sqrt{\pi}}\ \frac{\omega_r\sigma}{Q} + \ldots\right]$$

As can be shown from Eq. (34), this expression actually holds al $Q < 1/2$ if the stronger condition $\omega_r\sigma \ll Q$ is fulfilled.

For very long bunches, on the other hand, for which $|\alpha_{1,2}| = \omega_r\sigma$ we can use the lowest terms of the asymptotic expansion of the compl function[5]

$$w(z) \approx \frac{i}{\sqrt{\pi}\, z} \left[1 + \frac{1}{2z^2} + \frac{3}{4z^4} + ..\right] \tag{40}$$

to obtain from Eq. (31)

$$Z^{BB}_{loss} = \frac{\sqrt{\pi}}{2} \frac{\omega_r/\omega_o}{(\omega_r\sigma)^3} \frac{R}{Q^2} \left[1 + \frac{3}{(\omega_r\sigma)^2} \left(1 - \frac{1}{2Q^2}\right) + \ldots\right] . \tag{41}$$

The loss impedance in seen to decrease with the cube of the bunch length when it is larger than the (reduced) wavelength of the resonator ($\lambda/2\pi$).

Already for $\omega_r\sigma = 1$, the long bunch expression is a factor $2\sqrt{\pi}Q$ smaller than the short bunch limit: The power loss of long bunches is thus seen to be considerably less than for short ones, and excessive power loss in accelerators can be alleviated by lengthening the bunches.

7. Energy Loss of SSC Bunches

According to present plans for the SSC, almost every sixth rf bucket will be filled with bunches from the HEB booster. There will be 15 batches of 1130 bunches each, separated by 14 small gaps of 10 empty locations and a longer one of 190 empty locations for the abort kicker. A total of 16950 bunches will thus occupy most of the 17280 possible places. The spectrum will therefore contain mainly multiples of the bunch frequency (62.5 MHz), while the revolution frequency harmonics will stay below the 1% level (see Appendix for the evolution of the bunch spectrum during injection).

At top energy (20 TeV), the bunch length is expected to be 7.3 cm rms for the rf voltage of 20 MV at 375 MHz. During acceleration, the bunches are slightly shorter but should remain above 6 cm rms.

The impedances are less well known. In a circular accelerator, tl contributors to energy loss are usually the rf cavities. There are a 40 cells foreseen for SSC, with a design to be scaled down from the 3! PEP cavities.[6] The higher mode frequencies should scale approximatel proportion to the fundamental frequency, while the R/Q values remain t if the geometric proportions are not changed. The Q-values should cha the square root of the frequencies, which is close enough to unity to neglected.

However, the exact frequencies of the higher modes will be spread certain range due to construction tolerances and temperature differenc the 40 cells. Assuming a variation of 5.10^{-4}, the frequency spread oc bandwidth corresponding to a resonance with a quality factor of 2000. unloaded quality factor of the fundamental resonance is about 40,000, Q-values of the higher modes will be reduced by the same factor 20.

This reduction does not apply to the fundamental resonance, howeve must be kept fixed by tuners. The power loss into the fundamental mod however, is taken into account as "beam loading" in the design of the system, and thus will be excluded here.

The power loss of the strongest higher modes of the scaled rf cavi shown in Figs. 1-6 as a function of detuning over a full bunch-frequen interval. Figure 7 shows the sum over all 18 higher (longitudinal) mo trapped in the cavity (assuming a cut-off frequency corresponding to a proportionally scaled beam pipe). As can be seen from the figures, the loss is extremely low near the center frequencies and becomes large onl detuning of nearly 30 MHz. At the peak of the second harmonic, the loss may increase as much as 4 orders of magnitude. This has to be taken into account in the detailed design of the rf cavities which should avoid higher modes at multiples of the bunch frequency.

During injection, only part of the circumference of the accelerator is filled with bunches, and the revolution frequency harmonics will be stronger. This is compensated partially by the reduced current, but because of the narrow spacing of lines there will be a much higher probability of falling exactly onto one or more resonances. The power loss during injection could thus be larger than at full current, but would be limited to a period less than the injection time of some 20 minutes. The evolution of the loss during injection is discussed in the Appendix.

8. Conclusions

The energy loss of bunches in the SSC traversing the rf cavities is strongly reduced by the fact that the spectrum of a train of evenly spaced, identical bunches contains only multiples of the bunch frequency. Due to the large number of bunches in the SSC, the bunch frequency is very high (62.5 MHz). Then all higher modes in the rf cavities could be sufficiently far from integer multiples of the bunch frequency that the energy loss is strongly reduced over that calculated by a simple broad-band model. However, this must be taken into account in the design of the rf cavities, as the resonant loss at the shortest higher modes may be larger by up to 4 orders of magnitude, surpassing by far the loss calculated with the broad-band model.

During injection, the circumference of the machine is only fill partially, and the revolution frequency harmonics are excited more : The energy loss may thus be increased over the injection period, bu limited as also the current is only building up to its full value.

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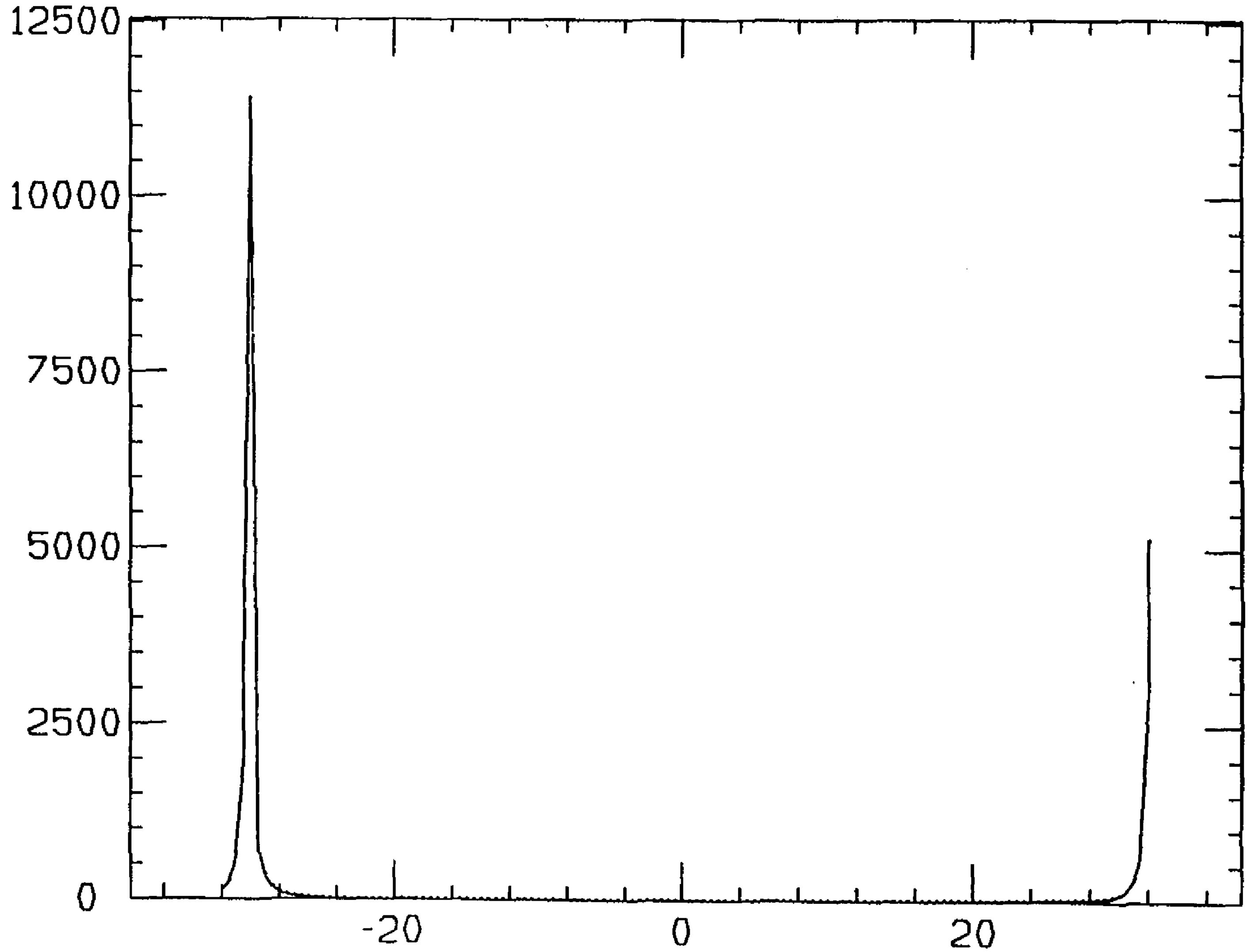
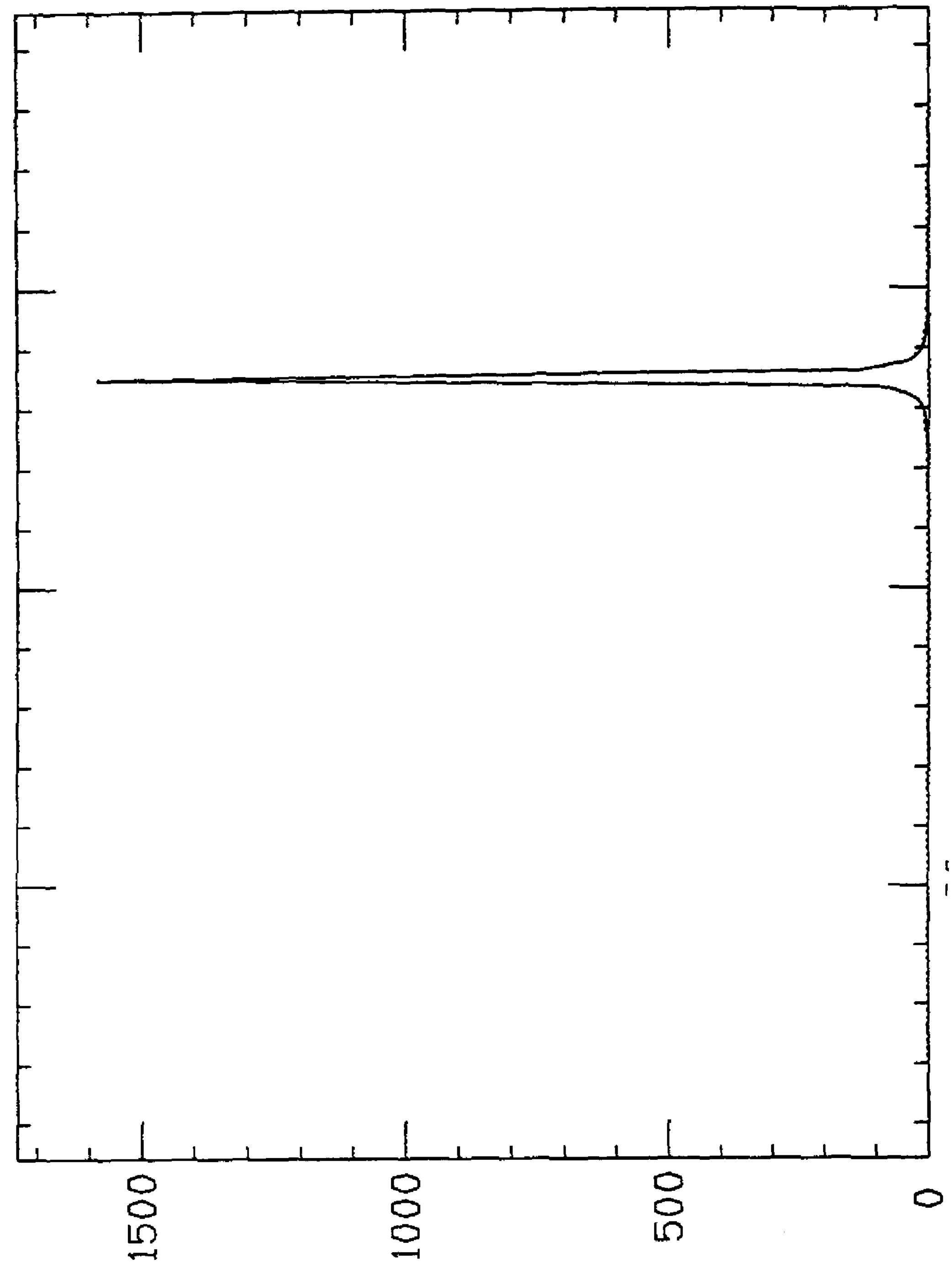


Fig. 1 Loss Power (in W) as function of detuning (in MHz) for mode n = 2 (530 MHz).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1000
500
0

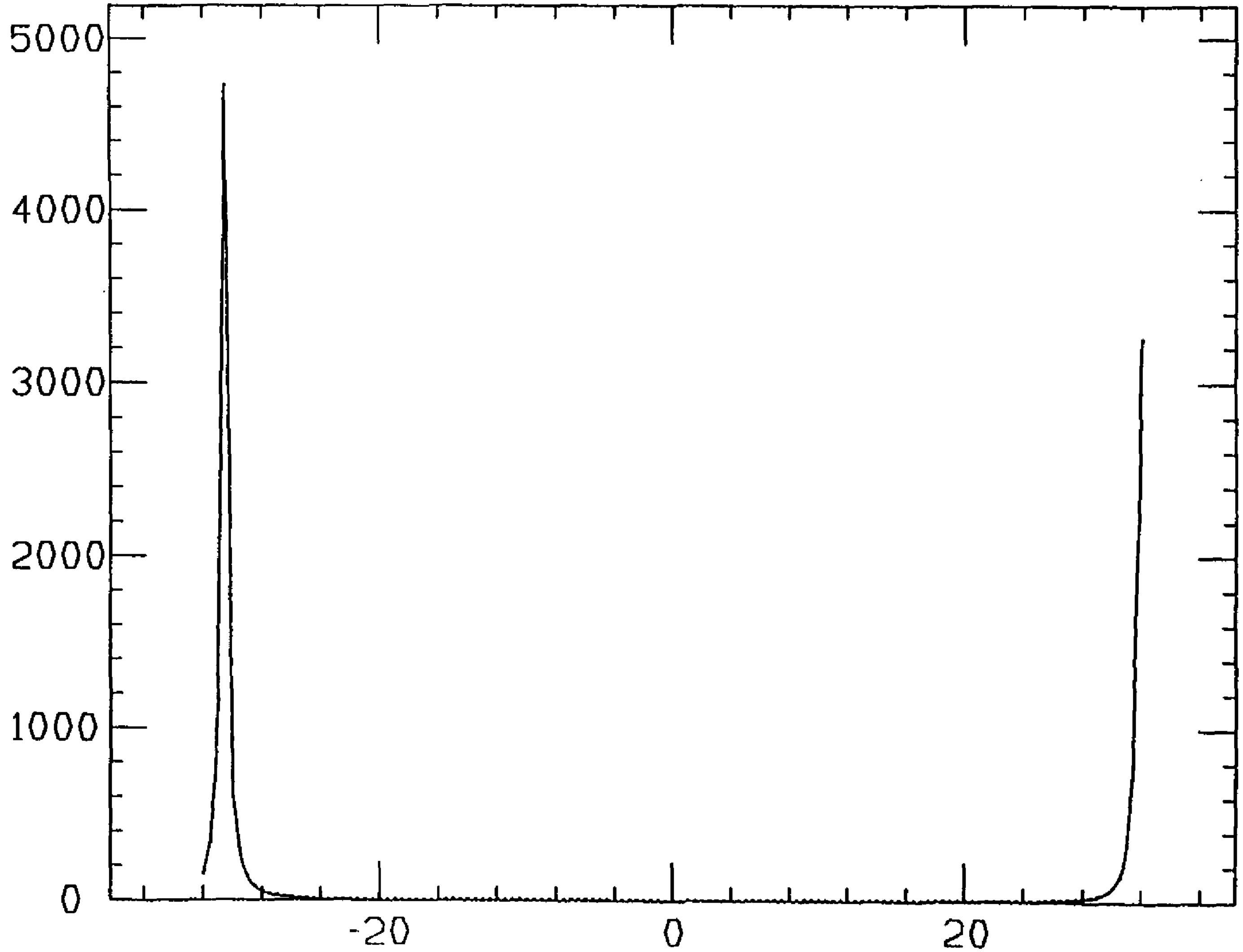
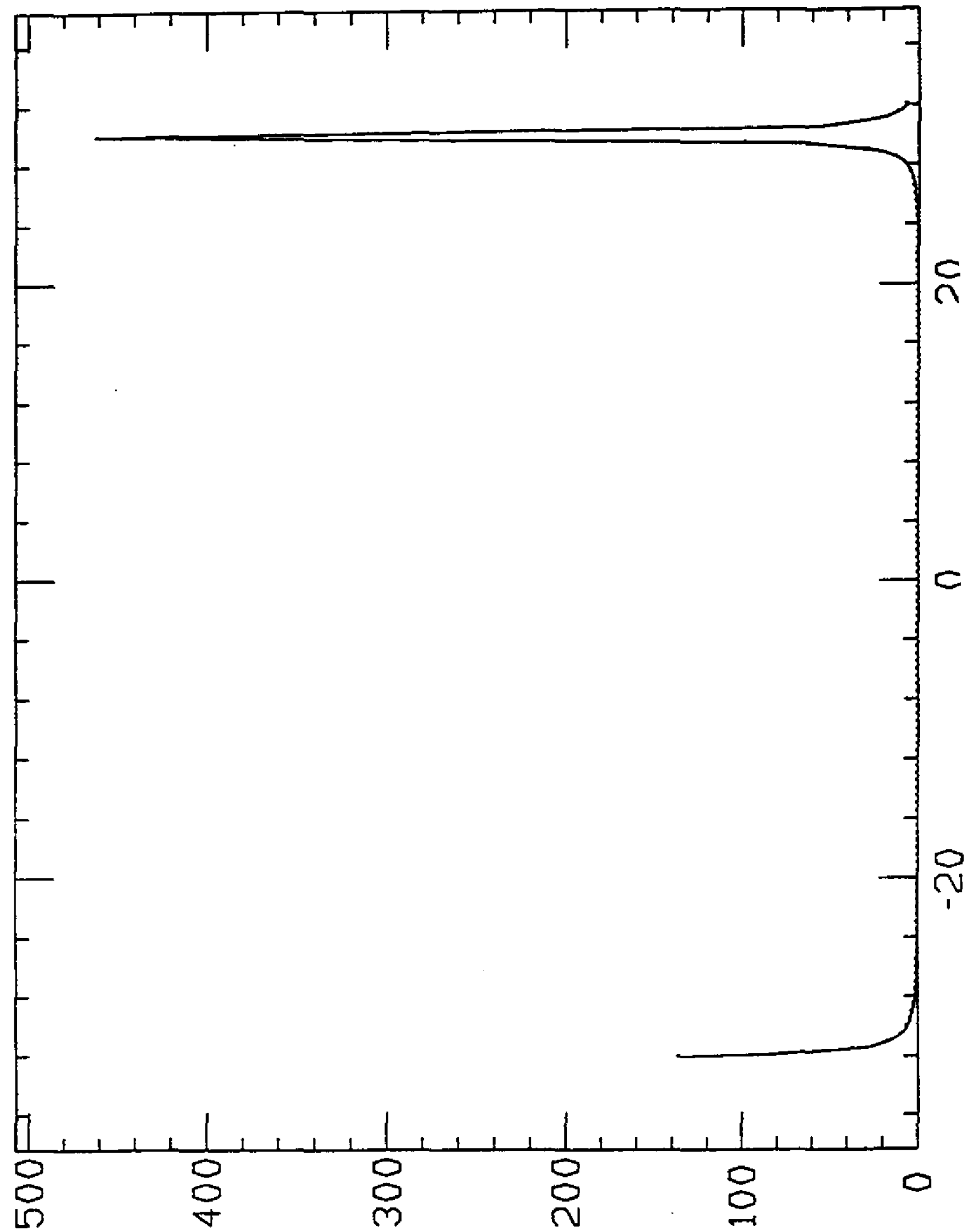


Fig. 3 Same for Mode n = 5 (967 MHz).



500
400
300
200
100
0
-20
0
20

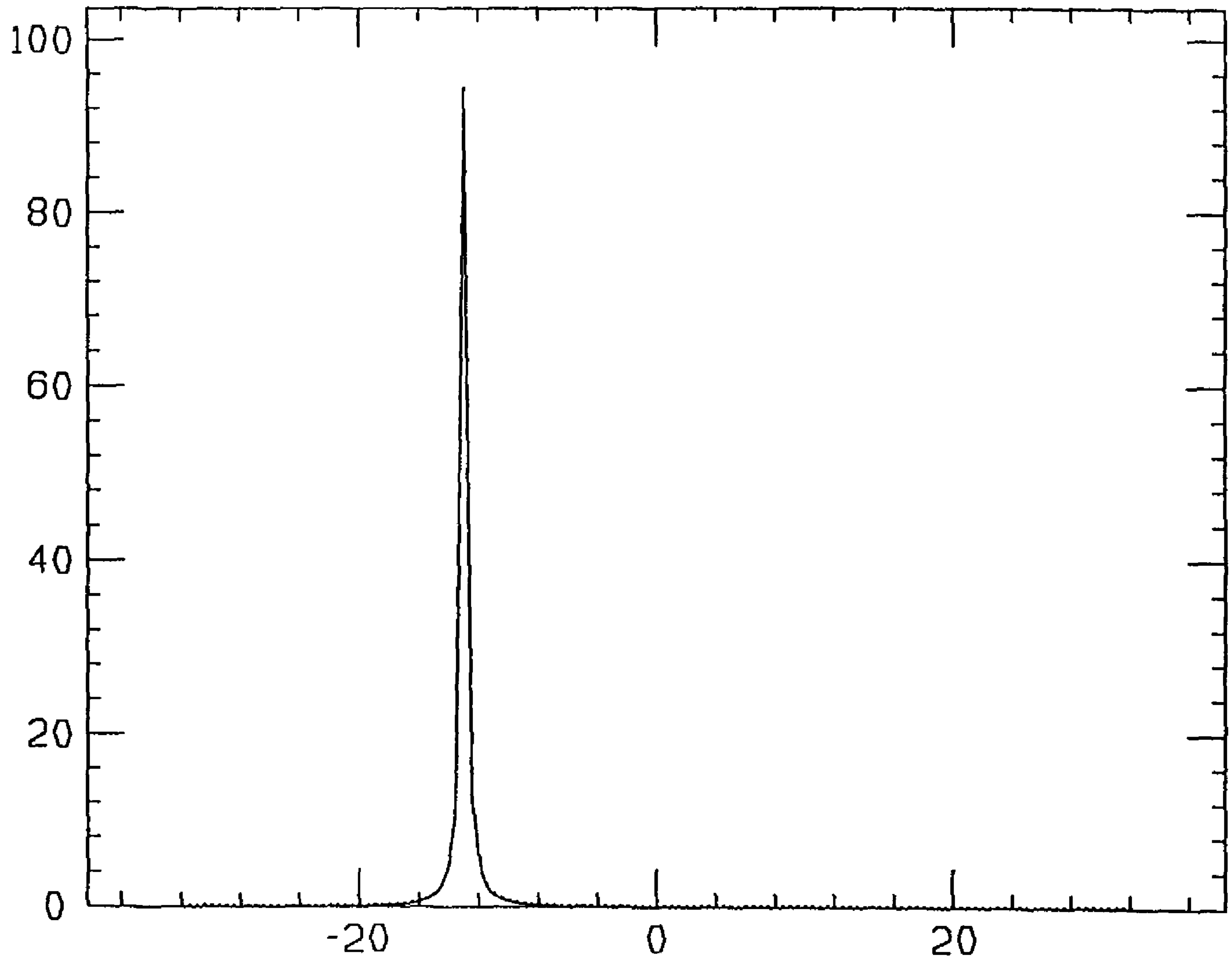
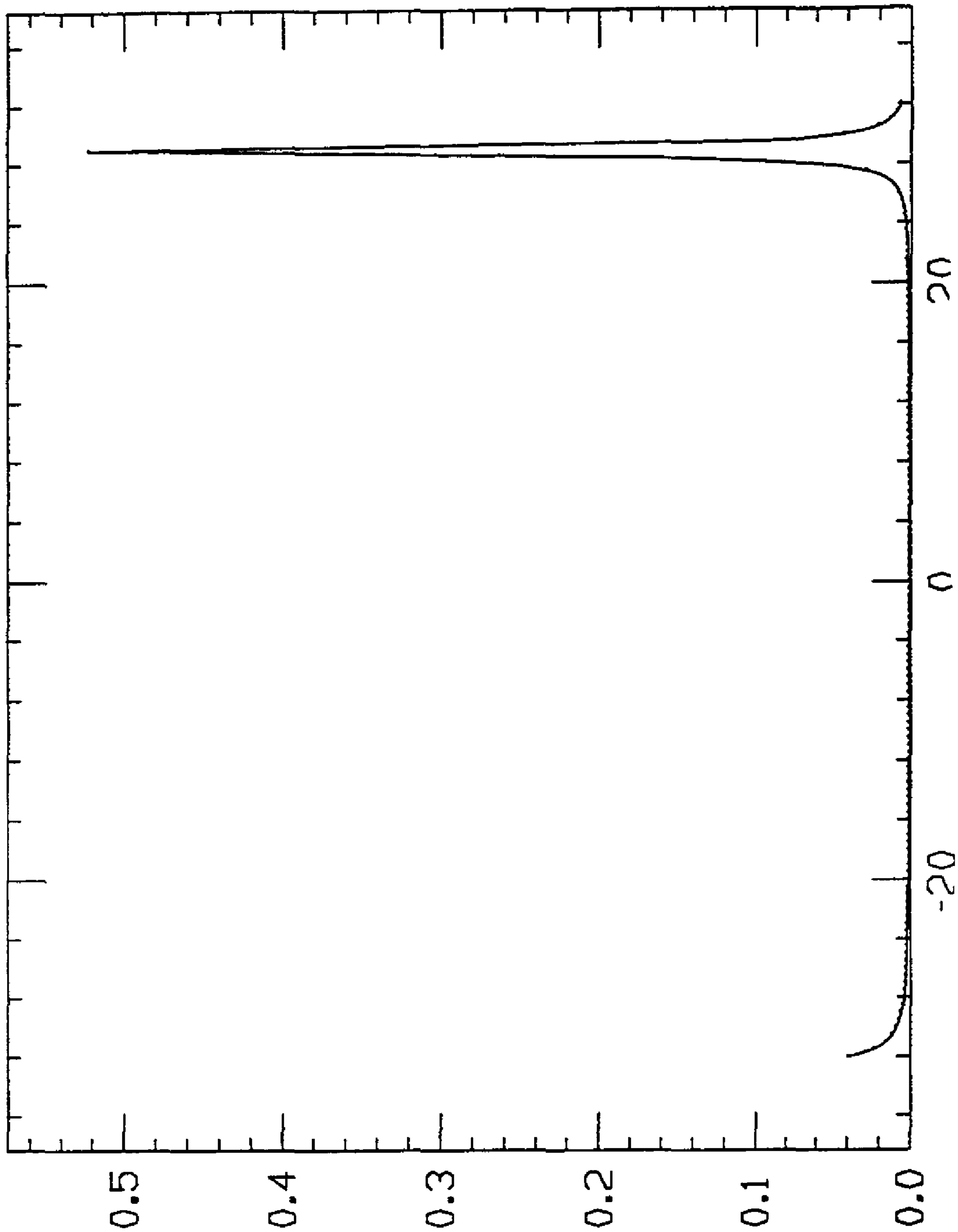


Fig. 5   Same for Mode n = 9 (1600 MHz).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0.4
0.3
0.2
0.1
0.0
-20
0
20

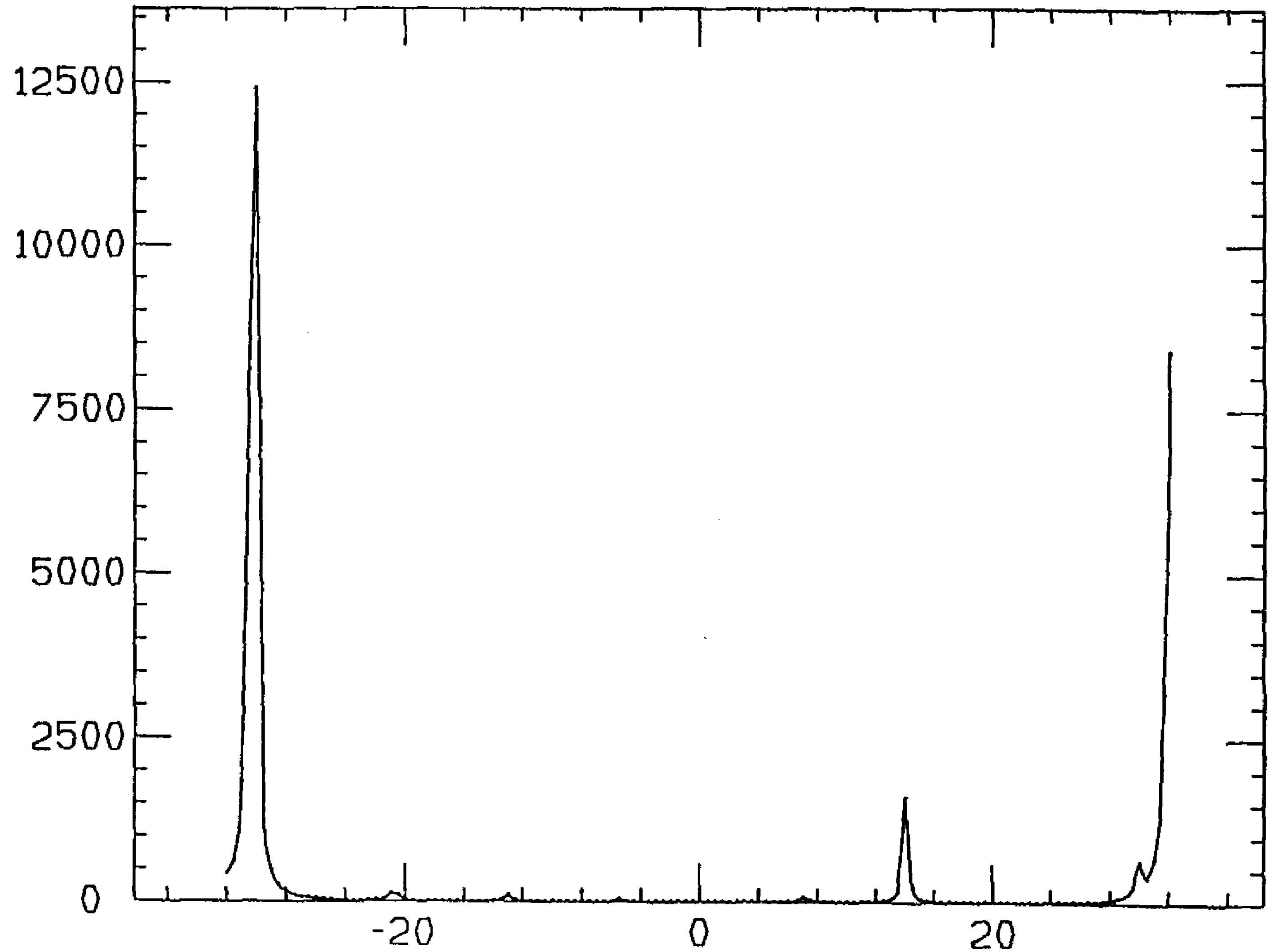

Fig. 7 Same for the sum of all 18 (trapped) modes of the SSC cavities.

Appendix: Energy Loss During Injection

Injection into the SSC is presently planned in "box-car" fashion $N = 15$ batches of $n = 1130$ bunches each from the HEB booster. Ten s of 10 empty places, and a bigger one of 190 are left between those b total of 16950 bunches is thus injected into the $M = 17280$ possible (corresponding to each sixth rf bucket). For simplicity, we shall n this small difference and assume that each batch contains $n = M/N =$ bunches.

We also assume that the batches are injected from the booster at time-intervals ΔT. Then the total energy loss during injection is o by summing over all states containing from one to N adjacent batches

$$\Delta E = \frac{\Delta T}{N} \sum_{k=1}^{N} \sum_{p=-\infty}^{\infty} \mathrm{Re}\, Z(p\omega_o)\, |\tilde{I}_k(p\omega_o)|^2 ,$$

where
$$\tilde{I}_k(\omega) = \tilde{I}_b(\omega) \sum_{\ell=1}^{nk} e^{j\omega \ell T/M} \quad .$$

i) Summing this geometric series for "non-harmonics" $p(\mathrm{mod}M) \neq 0$ yi

$$|\tilde{I}_k(p\omega_o)|^2 = \tilde{I}_b^2(p\omega_o) \frac{\sin^2(\pi nkp/M)}{\sin^2(\pi p/M)} \quad .$$

Inverting the order of summation, we obtain the contribution of the lines between bunch-frequency harmonics

$$\Delta E_I = \frac{\Delta T}{N} \sum_{p=-\infty}^{\infty}{}' \mathrm{Re}\, Z(p\omega_o) \frac{\tilde{I}_b^2(p\omega_o)}{\sin^2(\pi p/M)} \sum_{k=1}^{N} \sin^2(\pi nkp/M) \ ,$$

where $\sum'$ means that multiples of M are excluded from the sum.

Using the identity $\sin^2\alpha k = (1-\cos 2\alpha k)/2$, we find that the const contributes N/2 to the sum over k, while the p-dependent term vanish exactly. Thus

$$\Delta E_1 = \frac{\Delta T}{2} \sum_{p=-\infty}^{\infty} \text{Re } Z(p\omega_o) \frac{\tilde{I}_b^2(p\omega_o)}{\sin^2(\pi p/M)} \tag{A5}$$

ii) For harmonics of the bunch frequency $\omega_b = M\omega_o$, all the terms in Eq. (A2) are in phase and simply add up, so

$$\tilde{I_k}(p\omega_b) = nk\tilde{I_b}(p\omega_b) \ , \tag{A6}$$

and

$$\sum_{k=1}^{N} |\, \tilde{I}_k \ (p\omega_b)|^2 = I_b(p\omega_b)n^2 \sum_{k=1}^{N} k^2 \tag{A7}$$

With the identity

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)\ (2N+1)}{6} \tag{A8}$$

one finds the "harmonic" contribution (with M = Nn) to the energy loss

$$\Delta E_2 = \frac{\Delta T}{6} (M+n)(2M+n) \sum_{p=-\infty}^{\infty} \text{Re } Z(p\omega_b) \ \tilde{I}_b^2 \ (p\omega_b) \ . \tag{A9}$$

Adding the two contributions (A5) and (A9), one obtains the total energy loss during injection

$$\Delta E = \frac{\Delta T}{2} \sum_{p=-\infty}^{\infty} \left[\text{Re } Z(p\omega_b) \ \tilde{I}_b^2 \ (p\omega_b) \ \frac{(M+n)(2M+n)}{3} + \sum_{q=1}^{M-1} \frac{\text{Re}Z(p\omega_b + q\omega_o) \ \tilde{I}_b^2 \ (p\omega_b + q\omega_o)}{\sin^2(\pi q/M)} \right] , \tag{A10}$$

where we have regrouped the "non-harmonic" terms. The denominator in that latter part is small only either for q << M or for M-q << M, where the sine can be replaced by its argument. The non-harmonic terms thus are large only near the bunch-frequency harmonics.

If the impedance does not vary rapidly (see below), we can replace last term by

$$\Delta E_2 = \Delta T \frac{M^2}{\pi^2} \sum_{p=-\infty}^{\infty} Z(p\omega_b)\, \tilde{I}_b^2\, (p\omega) \sum_{q=1}^{[\frac{M-1}{2}]} \frac{1}{q^2} \ .$$

Since M is much larger than unity, we can replace the sum over q by th function $\zeta(2) = \pi^2/6$.

The total energy loss becomes then approximately

$$\Delta E = \frac{\Delta T M^2}{3} F \sum_{p=-\infty}^{\infty} \mathrm{Re}Z\, (p\omega_o)\, \tilde{I}_b^2\, (p\omega_b) \ ,$$

where

$$F = (1 + \frac{n}{M})(1 + \frac{n}{2M}) + 1/2 \ .$$

The first term is the contribution of the bunch-frequency harmonics an close to unity for M >> n. The contribution of the "non-harmonic" lin about one half and is thus a non-negligible part of the losses during injection even for the case of a "slowly" varying impedance (the bunch always shorter than a bucket, so the single-bunch spectrum $\tilde{I}_b(\omega)$ will significantly over a bunch-frequency interval.

For resonant impedances which are much narrower than the width of "non-harmonic" tails of the spectrum, i.e., a few times a revolution frequency, this approximation will no longer hold. The "non-harmonic" contribution could then become much larger if a spectral line just fal a resonant peak.

For the rf cavity of the SSC, higher modes have frequencies betwee MHz and about 2GHz, and Q values of the order of 5×10^4. Hence their width is between 10 and 40 kHz, corresponding to 3 to 12 times the rev frequency. The assumption of "slowly varying" impedance is thus only marginally fulfilled. However, the Q-values have been reduced by a factor of 20 to approximate the frequency spread[1] of the rf cavity cells. Under this assumption, the resonances are wide enough for Eq. (A-12) to hold, and the energy loss during injection is only about half of the energy loss of the machine with all bunches filled.