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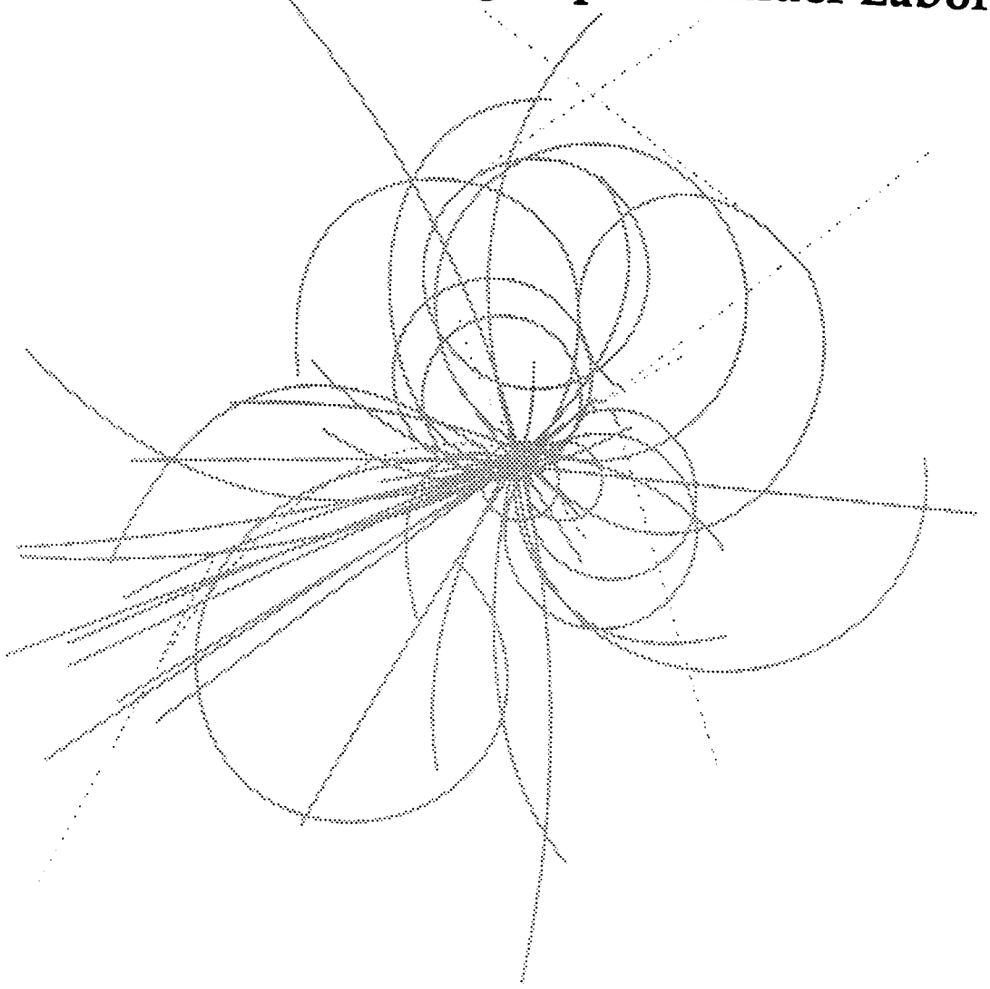

Interpolation of Off-Energy Matrices: Symplectic and Otherwise (A Comparison of Methods)

D. Douglas

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I. Introduction

Several tracking programs simulate the motion of off-momentum particles by using transport matrices adjusted to compensate for the variation of the particle rigidity relative to the rigidity of a design (on energy) particle. The off-momentum matrices are generally determined by one of two methods. First, the matrices may be re-evaluated for exactly the current momentum of the particle. This method is exact and symplectic; no approximations (relative to the linear/kick model of the code) are involved. However, this is considered rather time consuming.

Alternatively, the matrices are computed for a "mesh" of momentum values; matrices at intermediate values are then obtained by interpolation on momentum. In this note we examine the interpolation procedure used in PATRICIA[1), 2)] and show that it does not in general yield a symplectic matrix. We present two alternative interpolation schemes, suggested by Dragt and Leemann, which will yield a symplectic approximate to the exact off-momentum transport matrix. An alternative to symplectic

interpolation, that of resymplectification of a nonsymplectic approximate, is discussed by Furman in another SSC Technical Memorandum[3].

We conclude with a brief computational comparison of a symplectic and the nonsymplectic interpolation schemes to the "benchmark" of calculating the exact off-energy matrix. We find that the interpolation scheme employed by PATRICIA is only marginally adequate for SSC studies but that a more accurate nonsymplectic interpolation may be sufficiently precise to provide a matrix which is symplectic to within the numerical precision of a computer. Also, we find that the symplectic interpolation algorithm considered does produce the advertised exactly symplectic matrix.

II. Nonsymplectic Interpolation

This method of computing off-energy matrices is based on the analyticity of a transfer matrix when considered as a function of the momentum deviation $\delta = \Delta p/p_0$. For a particular momentum deviation δ we write

$$M(\delta) = M(0) + \frac{\delta}{1!} M'(0) + \frac{\delta^2}{2!} M''(0) + \ldots \tag{1}$$

The interpolation method in use in PATRICIA[2] truncates (1) at $O(\delta^3)$ to obtain an approximate $\bar{M}(\delta)$ to $M(\delta)$.

$$\bar{M}(\delta) = M(0) + \delta M'(0) + \frac{\delta^2}{2} M''(0) + \frac{\delta^3}{6} M'''(0) \tag{1'}$$

If the exact $M(\delta)$ is supplied at five momentum values in an arbitrary range $[-\hat{\delta}, \hat{\delta}]$, values for the expansion coefficients $M(0)$, $M'(0)$, $M''(0)$, $M'''(0)$ may be obtained. Specifically, the matrices $M(\delta_i)$ are computed <u>exactly</u> at momentum deviations $\delta_i = -\hat{\delta}, -\hat{\delta}/2, 0, \hat{\delta}/2, \hat{\delta}$ $(i = 1, 2, \ldots 5)$

These exact values are inserted in the system

$$M(\delta_i) = M(0) + \delta_i M'(0) + \frac{\delta_i^2}{2!} M''(0) + \frac{\delta_i^3}{3!} M'''(0) \quad i = 1, 2, \ldots 5$$

Which is solved for M(0), M'(0), etc. It is seen that the $M^{(n)}(0)$ are given by

$$\begin{aligned} M(0) &\quad \text{(specified exactly)} \\ M'(0) &= (1/3)(4B - A) \qquad + O(\hat{\delta}^4) \\ M''(0) &= \frac{1}{\hat{\delta}^2}[M(-\hat{\delta}) - 2M(0) + M(\hat{\delta})] + O(\hat{\delta}^2) \\ M'''(0) &= \frac{8}{\hat{\delta}^2}[A - B] + O(\hat{\delta}^2) \end{aligned} \tag{2}$$

where

$$A = \frac{M(\hat{\delta}) - M(-\hat{\delta})}{2\hat{\delta}}$$

$$B = \frac{M(\delta/2) - M(-\hat{\delta}/2)}{2(\hat{\delta}/2)}$$

The result $\bar{M}(\delta)$ of performing the interpolation (1') using the coefficients (2) will match the exact $M(\delta)$ through $O(\delta^3)$ (provided that the derivatives (2) are computed with sufficient precision that the error terms of $O(\hat{\delta}^n)$ are negligible.) $\bar{M}(\delta)$ will, however, <u>not</u> be symplectic except at $\pm\hat{\delta}$, $\pm 1/2\,\hat{\delta}$ and 0 (where $\bar{M} = M$ by construction).

The breakdown in symplecticity is apparent if $\bar{M}$ is inserted into the symplectic condition. Were $\bar{M}$ symplectic, we would have

$$\bar{M}\, J\, \tilde{\bar{M}} = J \tag{3}$$

The expansion (1') gives, upon use in (3)

$$
\begin{aligned}
\overline{M}(\delta) J \tilde{\overline{M}}(0) &= M(0)\ J\ M(0) \\
&+ \delta\left(\frac{1}{2} M(0)\ J\ \tilde{M}'(0) + \tilde{M}'(0)\ J\ \tilde{M}(0)\right) \qquad (4) \\
&+ \delta^2\left(\frac{1}{2} M(0)\ J\ \tilde{M}''(0) + M'(0)\ J\ \tilde{M}'(0) + \frac{1}{2} M''(0)\ J\ \tilde{M}(0)\right) \\
&+ \delta^3\left(\frac{1}{6} M(0)\ \tilde{M}'''(0) + \frac{1}{2} M'(0)\ J\ \tilde{M}''(0) + \frac{1}{2} M''(0)\ J\ \tilde{M}'(0) + \frac{1}{6} M'''(0)\ J\ \tilde{M}(0)\right) \\
&+ \delta^4\left(\frac{1}{6} M'(0)\ J\ \tilde{M}'''(0) + \frac{1}{4} M''(0)\ J\ \tilde{M}''(0) + \frac{1}{6} M'''(0) J\ \tilde{M}'(0)\right) \\
&+ \delta^5\ [\ldots] \\
&+ \ldots
\end{aligned}
$$

As $M(0)\ J\ \tilde{M}(0) = J$ ($M(0)$ <u>is</u> symplectic)

we must have all coefficients of δ equal to zero. Examination of the symplectic condition (3) using the full expansion (1) shows that in fact the coefficients of δ, δ^3 and δ^3 do vanish. However, if the full expansion (1) is used in (3), we find the coefficient of δ^4 is as follows:

$$
\begin{aligned}
&\frac{1}{24} M(0)\ J\ \tilde{M}''''(0) + \frac{1}{6} M'(0)\ J\ \tilde{M}'''(0) + \frac{1}{4} M''(0)\ J\ \tilde{M}''(0) \\
&\qquad + \frac{1}{6} M'''(0)\ J\ \tilde{M}'(0) + \frac{1}{24} M''''(0)\ J\ \tilde{M}(0)
\end{aligned}
$$

Comparison to (4) shows the truncated expansion misses the terms

$$\frac{1}{24} M(0)\ J\ \tilde{M}''''(0) \quad \text{and} \quad \frac{1}{24} M''''(0)\ J\ \tilde{M}(0)$$

For the exact matrix (1) these are not in general zero. Consequently, $\bar{M}$ violates the symplectic condition in fourth order.

(The same expansion (1') is clearly also in violation of (3) in orders 5, 6, ... etc. because terms involving $M''''(0)$, $M'''''(0)$, ... are absent.)

The truncated interpolation scheme (1') is therefore not symplectic. Numerical estimates of this violation are made below.

III. Symplectic Interpolation Schemes

A. Method of Dragt[4] It is possible to represent a symplectic matrix M by use of a Cayley transformation. If we write

$$M = \frac{(I + JS)}{(I - JS)} \quad (I - JS)^{-1} (I + JS) = (I + JS) (I-JS)^{-1} \tag{5}$$

where S is a real symmetric matrix, it follows that $MJ\tilde{M} = J$, that is, M is symplectic. Similarly, if we compute

$$JS = \frac{(M - I)}{(M + I)} \quad (M + I)^{-1} (M - I) = (M + I) (M + I)^{-1}$$

where M is symplectic, the result S will be well defined (provided M is not "near" -I), real, and symmetric.

Consider the case of symplectic matrix with a "weak" parametric dependence:

$$M = M(\delta) = M(0) + \delta M'(0) + \ldots$$

if we compute the "reduced" matrix

$$M_R (\delta) = M^{-1}(0) M(\delta)$$

we see that

$$M_R(\delta) \cong I$$

for small δ

We may therefore represent $M_R(\delta)$ using the Cayley transformation (5), as $M_R(\delta)$ will be "far" from -I. The parametric dependence of M_R will lie in S, as in the following equation.

$$M_R(\delta) = \frac{I + JS(0)}{I - JS(\delta)} \tag{6}$$

At any specific δ, S is obtained from

$$S(\delta) = \frac{M_R(\delta) - I}{M_R(\delta) + I} . \tag{6'}$$

Equations (6) fully specify a procedure to relate the symplectic matrix $M_R(\delta)$ with its Cayley generator $S(\delta)$.

Equations (6) also suggest the basis for a method of symplectic interpolation to obtain a symplectic approximate to an arbitrary off-momentum symplectic matrix $M(\delta)$.

This matrix will be described in terms of a symmetric, momentum dependent matrix $S(\delta)$ by the following transformation.

$$M(\delta) = M(0)\left(\frac{I + JS(\delta)}{I - JS(\delta)}\right) \tag{7}$$

The matrix value at zero momentum, M(0), is factored out to insure the Cayley transformation represents a matrix far from -I.

We proceed by computing M(0) exactly at parameter values $\pm\hat{\delta}$, $\pm\hat{\delta}/2$, 0 (just as with the nonsymplectic interpolation scheme). The reduced matrix

$M_R(\delta) = M^{-1}(0)\, M(\delta)$ is constructed at each interpolation point. Equation (6') is then used to obtain $S(\pm\,\hat{\delta})$, $S(\pm\,\hat{\delta}/2)$ and $S(0)$. Given these, $S(\delta)$ may be approximated by the interpolation scheme described in Section I.

For, given the five matrixes $S(\pm\,\hat{\delta})$, $S(\pm\,\hat{\delta}/2)$ and $S(0)$, we may compute $S(0)$, $S'(0)$, $S''(0)$ and $S'''(0)$. We may then construct the symmetric matrix

$$\bar{S}(\delta) = S(0) + \delta\, S'(0) + \frac{\delta^2}{2} S''(0) + \frac{\delta^3}{6} S'''(0) \tag{8}$$

(which is an approximation to the exact $S(\delta)$ accurate to $O(\delta^3)$). This symmetric approximate $\bar{S}(\delta)$ is then used in the Cayley transformation (6) to obtain an approximation $\bar{M}_R(\delta)$ from which a matrix $\bar{M}(\delta) = M(0)\, \bar{M}_R(\delta)$ is constructed. The key point is that because the interpolation is done on the Cayley generator S in such a way that $\bar{S}$ is symmetric, the result $\bar{M}_R$ must, by construction, be symplectic! The approximation $\bar{M}(0)$ to the off-momentum matrix is, consequently, also symplectic (as it is the product of the two symplectic matrices $M(0)$ and $M_R(\delta)$).

Summarizing, we write the symplectic approximate as follows.

$$M(\delta) = M(0) \left(\frac{I + J\,\bar{S}\,(\delta)}{I - J\,\bar{S}\,(\delta)} \right) \tag{9}$$

It should be observed that this method is computationally less efficient than simple interpolation because evaluation of (9) for $\bar{M}$ requires not only simple interpolation to obtain $\bar{S}(\delta)$, but also a matrix inversion and several matrix products to construct $\bar{M}$ from $\bar{S}$. The overhead in initially computing S', S'', S''', as used in (8) is also higher than that for obtaining M', M'', M''' used in (1') because before the derivatives of S can be obtained, S itself must be computed from (6') at each of the interpolation points.

B. Method Of Leemann[5] As an alternative to the method of interpolating on the symmetric matrix generating a Cayley transformation, Leemann has suggested interpolating on the coefficients of a second-degree polynomial generating function. The method is well illustrated by considering a simple 2 x 2 symplectic matrix which is generally represented by

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ \frac{(1+\alpha^2)}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{10}$$

The transformation

$$\begin{pmatrix} \bar{x} \\ \bar{p}_x \end{pmatrix} = M \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{11}$$

may be alternately represented by

$$\bar{x} = \frac{\partial F(x, \bar{p}_x)}{\partial p_x} \tag{11'}$$

$$p_x = \frac{\partial F(x, \bar{p}_x)}{\partial x}$$

provided we take as the generating function F the polynomial

$$F(x, \bar{p}_x) = F_{02}x^2 + F_{11}x\bar{p}_x + F_{02}\bar{p}_x^2 \tag{12}$$

In F, the coefficients required to reproduce (11) with (10) as M are

$$F_{20} = -\frac{M_{21}}{2M_{22}} = \frac{(1+\alpha^2)^2\sin\mu}{2\beta\,(\cos\mu - \alpha\sin\mu)}$$

$$F_{11} = \frac{1}{M_{22}} = \frac{1}{(\cos\mu - \alpha\sin\mu)} \tag{12'}$$

$$F_{02} = \frac{M_{12}}{2\,M_{22}} = \frac{\beta\sin\mu}{2\,(\cos\mu - \alpha\sin\mu)}$$

In terms of the F_{ij}, M is

$$M = \begin{pmatrix} F_{11} + \dfrac{4F_{02}F_{20}}{F_{11}} & -\dfrac{2F_{02}}{F_{11}} \\ -\dfrac{2F_{20}}{F_{11}} & \dfrac{1}{F_{11}} \end{pmatrix} \tag{13}$$

In the case where M is parameter (e.g. momentum) dependent, $M(\delta)$ is to be computed by interpolation on the F_{ij}. Specifically, we compute $M(\delta)$ at the by now familiar values $\delta = \pm\hat{\delta}, \pm\hat{\delta}/2$, and 0, and using (12') compute each $F_{ij}(\delta)$ at the "interpolation points" $\delta = \pm\hat{\delta}, \pm\hat{\delta}/2, 0$. To obtain an approximation to $M(\delta)$ at intermediate δ values, an approximate $\overline{F}_{ij}(\delta)$ to each coefficient $F_{ij}(\delta)$ is computed by interpolation amongst $F_{ij}(\pm\hat{\delta})$, $F_{ij}(\pm\hat{\delta}/2)$, and $F_{ij}(0)$. An approximate $\overline{M}(\delta)$ for $M(\delta)$ is then assembled using (13):

$$\overline{M}(\delta) = \begin{bmatrix} \overline{F}_{11}(\delta) + \dfrac{4\overline{F}_{02}(\delta)\overline{F}_{20}(\delta)}{\overline{F}_{11}(\delta)} & -\dfrac{2\overline{F}_{02}(\delta)}{\overline{F}_{11}(\delta)} \\ -\dfrac{2\overline{F}_{20}(\delta)}{\overline{F}_{11}(\delta)} & \dfrac{1}{\overline{F}_{11}(\delta)} \end{bmatrix} \tag{14}$$

The result is, by construction, a symplectic approximate to the exact $M(\delta)$.

Observe that construction of $\overline{M}(\delta)$ requires interpolation on only 3 functions F_{20}, F_{11} and F_{02}, and computation of multiples and ratios of the F_{ij}. The computational efficiency of this method will therefore almost certainly exceed that of Cayley - transformation based interpolations, and may approach or equal that of simple nonsymplectic interpolation. However, use of (11') breaks down in the event that $M_{22} = 0$ (see (12')). In such cases, an alternate representation must be used.

For example, if M describes an insertion with 90° phase advance matched to values $\beta \neq 0$, $\alpha = 0$ we have

$$M = \begin{pmatrix} 0 & \beta \\ -\frac{1}{\beta} & 0 \end{pmatrix}$$

For this case, it is appropriate to represent (11) using a generating function

$$F = F(x, \bar{x}) = \frac{1}{\beta} x \bar{x} .$$

Then, the equivalent to (11') becomes

$$p_x = \frac{\partial F(x, \bar{x})}{\partial x} \qquad (11'')$$

$$\bar{p}_x = -\frac{\partial F(x, \bar{x})}{\partial \bar{x}}$$

The interpolation procedure would in this case be carried out by interpolation on β.

More generally, if M represents an element or beamline through which the cosine-like trajectory is zero, then M must be represented by equations of the form (11") (with generating function $F = F(x, \bar{x})$). Consequently, the method of Leemann will require certain numerical checks on the matrix under consideration to determine what form of generating function F is appropriate[6]. This logic will produce a certain overhead (especially in the case of a general 6 x 6, fully coupled, symplectic matrix) if used in a tracking program.

IV. Numerical Considerations

A short program (Appendix A) has been written to compare the accuracy and symplecticity of the standard interpolation scheme to the Cayley transformation based scheme. The program constructs the matrices $M(0)$, $M'(0)$, $M''(0)$, $M'''(0)$ and $S(0)$, $S'(0)$, $S''(0)$, $S'''(0)$ required for the two interpolation schemes. It then provides a result $M(\delta)$ (for any requested δ) by 1) exact recomputation, 2) standard (nonsymplectic) interpolation, and 3) symplectic interpolation. The quantity $MJ\tilde{M}$ is also computed and displayed to provide a test of symplecticity. We have found that computations of the numerical derivatives $M'(0)$, $M''(0)$, ... requires substantial precision and considerable caution on the choice of interpolation points $\pm\hat{\delta}$. This program is therefore in double precision and has been implemented on the CDC-6600 (to provide 96 bits in the mantissa, 120 bits overall). We hope to implement it in quad precision (REAL *16) on the VAX in the future. (For remarks on precision see Appendix A).

Results from this program, which studies the example of an SSC-like 2 x 2 quadrupole matrix (ℓ= 5m, B' = 150 T/m, $B\rho$ = 66712.8 T-m) are presented in Appendix B (program output) and Tables 1 and 2. Table 1 shows that both interpolation schemes are accurate through $O(\delta^3)$, as stated. (Deviations scale as δ^4.) Table 2 shows that the recomputed matrix and the Cayley-interpolated matrix are exactly symplectic, while the standard interpolation scheme fails to provide a symplectic result past $O(\delta^3)$ (Deviations scale as δ^4.) This could be of concern in the SSC parameter range ($\delta \lesssim .005$) especially during longer tracking runs. However, since the error goes as δ^4, it is possible that if the interpolation were carried out to $O(\delta^4)$ [so

that the error became $O(\delta^5)$] the error would lie beyond the precision of the computer[7], so that the resultant interpolated matrix would be numerically (if not analytically) symplectic. However, an $O(\delta^4)$ interpolation based on the Taylor's series expansion (1') may be impossible to do (at least in 64-bit arithmetic) because of roundoff error encountered while computing the derivatives. (See comments in Appendix A).

V. Conclusions

We have found that the standard interpolation scheme employed in PATRICIA is not symplectic and may be inadequate for the SSC tracking problem. We have described two symplectic approximation schemes and numerically demonstrated that one provides the advertised symplectic third order approximate.

It is possible that the nonsymplectic interpolation scheme described herein will provide a numerically symplectic result in 64-bit arithmetic, if extended to the next higher order. However, this may not be possible, as numerical error may dominate computation of the fourth derivative term (especially in 64-bit arithmetic). A possible scheme to circumvent this problem would be to establish the derivatives $M'(0)$. ... $M''''(0)$ in 128-bit arithmetic, and then to truncate to 64-bits for tracking calculations.

Work on this topic continues. We will perform timing tests on each scheme described here as well as on the scheme of Furman[3] to establish which of the following is both computationally efficient and symplectic: 1) recomputation of matrix, 2) nonsymplectic interpolation to $O(\delta^3)$ and $O(\delta^4)$, 3) symplectic interpolation (based on Cayley transformations and based on generating functions), and 4) resymplectification.

We hope to examine interpolations schemes based on methods other than the truncated Taylor's series (2). It may be more efficient, for example, to do a several-point Lagrange interpolation on matrix elements than it is to compute derivatives numerically and use a Taylor's series.

VI. References and Notes

1) H. Wiedemann, PEP-220 (Sept. 1976)

2) S. Khiefets, BNL-31761 (August 1982)

3) M. Furman, SSC-TM-4001 (January 1985)

4) A. Dragt (private communication)

5) Ch. Leemann (private communication)

6) For matrices with $M_{22} = 0$ (cosine-like trajectory zero) $F = F(x,\bar{x})$ is appropriate. Otherwise, $F = F(x,\bar{p}_x)$ is appropriate.

7) At least in 64-bit arithmetic

Appendix A: Program Listing, CDC-6600 Version

Comments

1) This code studies a 2 x 2 focussing quadrupole matrix. Other cases could be done by altering the subroutine MATGEN.

2) Establishing M', M", ... requires some delicacy, for equations (2) involve numerical differentiation of $M(\delta)$. Caution must be exercised to choose δ sufficiently small that the $O(\hat{\delta}^4)$ errors in M' and $O(\hat{\delta}^2)$ errors in M" and M'" are negligible , but $\hat{\delta}$ must not be so small that the derivatives (especially M"') are dominated by roundoff errors.

In this regard, we found it necessary to work in double precision on the CDC-6600 (96-bit mantissa in 120-bit arithmetic). With this large word, we determined that if $\hat{\delta} \lesssim 10^{-5}$, the errors in the derivatives (2) were adequately small (so as not to influence the results for M(0)). However, we found that to avoid roundoff error domination of the calculation of M"'(0), we needed $\hat{\delta} > 10^{-8}$. Our test runs (Appendix B) were therefore done with $\hat{\delta} = 10^{-6}$.

3) Computation of S'(0), S"(0), ... from $S(\hat{\delta})$, $S(\hat{\delta}/2)$, ... (for the symplectic interpolation) had an additional sensitive point. By construction these must be symmetric matrices, and, in fact, we saw this to be the case to computer precision. However, roundoff errors produced "assymmetries" which were inflated (even with the 120-bit word on the

CDC-6600) by the numerical differentiation (2). The resulting derivatives of $S(\delta)$ were therefore slightly assymmetric. When $S(\delta)$ was constructed via (1'), it therefore failed to be exactly symmetric, leading to a nonsymplectic result for $\bar{M}(\delta)$.

This problem was avoided by symmetrising $S(\hat{\delta})$, $S(\hat{\delta}/2)$, ... at the outset. The interpolation point generators were reset to $1/2[S(\delta_i) + \tilde{S}(\delta_i)]$ to remove the assymmetries which arose due to roundoff errors (see subroutine SOLGEN). The derivatives obtained from the symmetrised generators were, in turn, exactly symmetric, along with the interpolated $S(\delta)$. $\bar{M}(\delta)$ was then exactly symplectic.

4) Because $M(\delta)$ for the quadrupole is far from $-I$, we have dispensed with the "reduction by $M(0)$" described in Section III.A. That is, the direct Cayley transform (5) was used in this code, rather than the "scaled" transform (7), for there was no danger of $M(\delta)$ approaching $-I$ and thereby rendering S indeterminate.

5) The aforementioned problems with roundoff errors suggest two subjects worthy of further consideration.

   a. Precision <u>greater</u> than 64-bits appears to be necessary, if the derivatives used in interpolation are to be established numerically as described above.

b. Because of roundoff problems, it may be difficult to extend this interpolation method to $O(\delta^4)$. It might not be possible to evaluate $M''''(0)$ with sufficient accuracy, even with REAL* 16 (128-bit) variables.

It is therefore suggested that we examine alternate nonsymplectic interpolations schemes which can be used to obtain highly accurate (e.g., $O(\delta^4)$, $O(\delta^5)$) approximations. Possible schemes of this type include multi-point Lagrange interpolation schemes.

```
      PROGRAM SYMMAT(INPUT,OUTPUT,
     &               TAPE5=INPUT,
     &               TAPE6=OUTPUT)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION DELTA(100)
      DIMENSION REX(2,2),RNS(2,2),RI(2,2),RJ(2,2),SGEN(2,2),RSY(2,2)
      DIMENSION AOFF(2,2,5),DINS(2,2,4),DISY(2,2,4),SM(2,2,5)
      DATA (RI(1,K),K=1,2)/1.D0,0.D0/
      DATA (RI(2,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(1,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(2,K),K=1,2)/-1.D0,0.D0/
      CALL CONNECT(5)
      CALL CONNECT(6)
      WRITE(6,9000)
 9000 FORMAT(1H ,* INPUT BPRIME, LENGTH, BRHO  *)
      READ(5,*) BPRIME,AL,BRHO
      WRITE(6,9001)
 9001 FORMAT(1H ,* INPUT DHAT, THE DIFFERENTIATION HALF-RANGE *)
      READ(5,*) DHAT
      WRITE(6,9002)
 9002 FORMAT(1H ,* INPUT NUMBER OF CASES NKASE *)
      READ(5,*) NKASE
      WRITE(6,9003)NKASE,NKASE
 9003 FORMAT(1H ,* INPUT *,I4,* DELTA VALUES FOR THE *,I4,* CASES *)
      READ(5,*) (DELTA(K),K=1,NKASE)
      WRITE(6,9004) BPRIME,AL,BRHO,DHAT,NKASE
 9004 FORMAT(1H //1H ,*  PARAMETERS ARE */
     &1H /1H ,* BPRIME = *,D32.26/
     &1H /1H ,* AL     = *,D32.26/
     &1H /1H ,* BRHO   = *,D32.26/
     &1H /1H ,* DHAT   = *,D32.26/
     &1H /1H ,* NKASE  = *,I4//)
      PRINT(6,*) # DELTA VALUES #
      DO 1 K=1,NKASE
      PRINT(6,*) #   DELTA(#,K,#)  = #,DELTA(K)
    1 CONTINUE
C
C  DO THE EXACT CALCULATION
C
      WRITE(6,9005)
 9005 FORMAT(1H //1H ,*   EXACT OFF MOMENTUM MATRIX CALCULATIONS *//)
      DO 100 J=1,NKASE
      DEL=DELTA(J)
      CALL MATGEN(REX,DEL,BPRIME,AL,BRHO)
      WRITE(6,9006) J,DEL,(REX(1,K),K=1,2),(REX(2,M),M=1,2)
 9006 FORMAT(1H //1H ,* FOR CASE *,I4,* DELTA = *,D32.26//
     &1H ,2(D32.26,2X)/1H ,2(D32.26,2X)/)
      CALL SYMCON(REX)
  100 CONTINUE
C
C  SET UP OFF MOMENTUM MATRICES AT DIFFERENTIATION VALUES
C
      DINC=DHAT/2.D0
      DEL=-1.D0*DHAT
```

```
      DO 200 LYN=1,5
      CALL MATGEN(REX,DEL,BPRIME,AL,BRHO)
      DO 201 J=1,2
      DO 201 K=1,2
  201 AOFF(K,J,LYN)=REX(K,J)
      DEL=DEL+DINC
  200 CONTINUE
      WRITE(6,9007)
 9007 FORMAT(1H ///1H ,*  DIFFERENTIATION POINT MATRICES  *///)
      DO 210 LYN=1,5
      WRITE(6,9008) (AOFF(1,J,LYN),J=1,2),(AOFF(2,K,LYN),K=1,2)
 9008 FORMAT(1H //1H ,2(D32.26,2X)/1H ,2(D32.26,2X)///)
  210 CONTINUE
C
C  NEXT, DO THE STANDARD NONSYMPLECTIC INTERPOLATION FOR EACH CASE
C
      WRITE(6,9009)
 9009 FORMAT(1H ///1H ,*  NONSYMPLECTIC INTERPOLATION  *///)
C
C       SET UP THE DERIVATIVE MATRICES
C
      CALL INTSET(AOFF,DINS,DHAT)
      WRITE(6,9010)
 9010 FORMAT(1H ///1H ,* MATRICES USED IN STANDARD INTERPOLATION *///)
      DO 220 LYN=1,4
      WRITE(6,9011) (DINS(1,J,LYN),J=1,2),(DINS(2,K,LYN),K=1,2)
 9011 FORMAT(1H //1H ,2(D32.26,2X)/1H ,2(D32.26,2X)///)
  220 CONTINUE
C
C       DO THE CASES
C
      DO 300 J=1,NKASE
      DEL=DELTA(J)
      CALL INTERP(RNS,DINS,DEL)
      WRITE(6,9016) J,DEL,(RNS(1,K),K=1,2),(RNS(2,M),M=1,2)
 9016 FORMAT(1H //1H ,* FOR CASE *,I4,* DELTA = *,D32.26//
     &1H ,2(D32.26,2X)/1H  ,2(D32.26,2X)/)
      CALL SYMCON(RNS)
  300 CONTINUE
C
C  NEXT, DO THE CAYLEY TRANSFORMATION BASED SYMPLECTIC INTERPOLATION
C  FOR EACH CASE
C
      WRITE(6,9020)
 9020 FORMAT(1H ///1H *   SYMPLECTIC INTERPOLATION    *////)
C
C  SET UP THE INTERPOLATION MATRICES
C
      CALL SOLGEN(AOFF,SM)
      WRITE(6,9019)
 9019 FORMAT(1H //1H ,*  DIFFENTIATION POINT GENERATOR MATRICES  *///)
      DO 400 LYN=1,5
      WRITE(6,9021) (SM(1,J,LYN),J=1,2),(SM(2,K,LYN),K=1,2)
 9021 FORMAT(1H /1H ,2(D32.26,2X)/1H ,2(D32.26,2X)///)
  400 CONTINUE
```

```
C
C       GET THE DERIVATIVES OF THE GENERATOR
C
      CALL INTSET(SM,DISY,DHAT)
      WRITE(6,9022)
 9022 FORMAT(1H ///1H ,*  DERIVATIVES OF GENERATOR MATRICES *///)
      DO 401 LYN=1,4
      WRITE(6,9023) (DISY(1,J,LYN),J=1,2),(DISY(2,K,LYN),K=1,2)
 9023 FORMAT(1H //1H ,2(D32.26,2X)/1H ,2(D32.26,2X)///)
  401 CONTINUE
C
C     DO THE CASES
C
      DO 499 J=1,NKASE
      DEL=DELTA(J)
      CALL INTERP(SGEN,DISY,DEL)
      CALL SYMPM(SGEN,RSY)
      WRITE(6,9030) J,DEL,(RSY(1,M),M=1,2),(RSY(2,K),K=1,2)
 9030 FORMAT(1H //1H ,* FOR CASE *,I4,* DELTA = *,D32.26//
     &1H ,2(D32.26,2X)/1H ,2(D32.26,2X)///)
      CALL SYMCON(RSY)
  499 CONTINUE
      END
C
C
C
C
      SUBROUTINE MATGEN(RM,DEL,BP,AL,BRHO)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION RM(2,2)
      RK=DSQRT( BP/(BRHO*(1.DO+DEL)) )
      RM(1,1)=DCOS(RK*AL)
      RM(2,2)=RM(1,1)
      RM(1,2)=DSIN(RK*AL)/RK
      RM(2,1)=-1.DO*RK*RK*RM(1,2)
      RETURN
      END
C
C
C
C
      SUBROUTINE INTSET(AM,BM,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION AM(2,2,5),BM(2,2,4)
      DO 10 J=1,2
      DO 10 K=1,2
   10 BM(K,J,1)=AM(K,J,3)
      DO 20 J=1,2
      DO 20 K=1,2
   20 BM(K,J,2)=( 8.DO*( AM(K,J,4)-AM(K,J,2) )
     &              - ( AM(K,J,5)-AM(K,J,1) )  )/(6.DO*D)
      DO 30 J=1,2
      DO 30 K=1,2
   30 BM(K,J,3)=( AM(K,J,1)+AM(K,J,5)-2.DO*AM(K,J,3) )/(D*D)
      DO 40 J=1,2
```

```
      DO 40 K=1,2
   40 BM(K,J,4)=4.D0*( AM(K,J,5)-AM(K,J,1)
     &                 -2.D0*(AM(K,J,4)-AM(K,J,2))   )/(D*D*D)
      RETURN
      END
C
C
C
C
      SUBROUTINE INTERP(R,B,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION R(2,2),B(2,2,4)
      DO 10 J=1,2
      DO 10 K=1,2
   10 R(J,K)=B(J,K,1)+D*(B(J,K,2)+D*(.5D0*B(J,K,3)+D*B(J,K,4)/6.D0))
      RETURN
      END
C
C
C
C
      SUBROUTINE SOLGEN(A,B)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION RI(2,2),RJ(2,2),A(2,2,5),B(2,2,5)
      DIMENSION T1(2,2),T2(2,2)
      DATA (RI(1,K),K=1,2)/1.D0,0.D0/
      DATA (RI(2,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(1,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(2,K),K=1,2)/-1.D0,0.D0/
      DO 1000 LYN=1,5
      D=1.D0/((A(1,1,LYN)+1.D0)*(A(2,2,LYN)+1.D0)-A(1,2,LYN)*A(2,1,LYN))
      T1(1,1)=(A(2,2,LYN)+1.D0)*D
      T1(2,2)=(A(1,1,LYN)+1.D0)*D
      T1(1,2)=-1.D0*D*A(1,2,LYN)
      T1(2,1)=-1.D0*D*A(2,1,LYN)
      DO 10 J=1,2
      DO 10 K=1,2
      T2(J,K)=0.D0
   10 B(J,K,LYN)=0.D0
      DO 20 J=1,2
      DO 20 K=1,2
      DO 20 M=1,2
   20 T2(J,K)=T2(J,K)-RJ(J,M)*(A(M,K,LYN)-RI(M,K))
      DO 30 J=1,2
      DO 30 K=1,2
      DO 30 M=1,2
C
C  SET UP THE GENERATOR B, INCLUDING SYMMETRIZATION
C
   30 B(J,K,LYN)=B(J,K,LYN)+.5D0*(T2(J,M)*T1(M,K)+T2(K,M)*T1(M,J))
 1000 CONTINUE
      RETURN
      END
C
C
```

```
C
C
      SUBROUTINE SYMCON(A)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION A(2,2),RJ(2,2),T1(2,2),T2(2,2)
      DATA (RJ(1,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(2,K),K=1,2)/-1.D0,0.D0/
      DO 10 J=1,2
      DO 10 K=1,2
      T1(K,J)=0.D0
   10 T2(K,J)=0.D0
      DO 20 J=1,2
      DO 20 K=1,2
      DO 20 M=1,2
   20 T1(K,J)=T1(K,J)+RJ(K,M)*A(J,M)
      DO 30 J=1,2
      DO 30 K=1,2
      DO 30 M=1,2
   30 T2(K,J)=T2(K,J)+A(K,M)*T1(M,J)
      WRITE(6,9000) (T2(1,K),K=1,2),(T2(2,M),M=1,2)
 9000 FORMAT(1H //1H ,*              SYMPLECTIC CONDITION FOR THIS MATRIX
     & GIVES *//1H ,8X,2(D32.26,2X)/1H ,8X,2(D32.26,2X)////)
      RETURN
      END
C
C
C
C
      SUBROUTINE SYMPM(S,A)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION S(2,2),A(2,2),RI(2,2),RJ(2,2),T1(2,2),T2(2,2),T3(2,2)
      DATA (RI(1,K),K=1,2)/1.D0,0.D0/
      DATA (RI(2,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(1,K),K=1,2)/0.D0,1.D0/
      DATA (RJ(2,K),K=1,2)/-1.D0,0.D0/
      DO 10 J=1,2
      DO 10 K=1,2
      T1(K,J)=RI(K,J)
      T2(K,J)=RI(K,J)
   10 A(K,J)=0.D0
      DO 20 J=1,2
      DO 20 K=1,2
      DO 20 M=1,2
   20 T1(K,J)=T1(K,J)+RJ(K,M)*S(M,J)
      DO 30 J=1,2
      DO 30 K=1,2
      DO 30 M=1,2
   30 T2(K,J)=T2(K,J)-RJ(K,M)*S(M,J)
      DET=1.D0/(T2(1,1)*T2(2,2)-T2(1,2)*T2(2,1))
      T3(1,1)=T2(2,2)*DET
      T3(2,2)=T2(1,1)*DET
      T3(1,2)=-1.D0*T2(1,2)*DET
      T3(2,1)=-1.D0*T2(2,1)*DET
      DO 40 J=1,2
      DO 40 K=1,2
```

```
      DO 40 M=1,2
   40 A(K,J)=A(K,J)+T1(K,M)*T3(M,J)
      RETURN
      END
C
C
C
C
```

Appendix B Output of Test Run of Program SYMMAT

```
runit
  INPUT BPRIME, LENGTH, BRHO
150 5 66712.8
  INPUT DHAT, THE DIFFERENTIATION HALF-RANGE
.000001
  INPUT NUMBER OF CASES NKASE
12
  INPUT   12 DELTA VALUES FOR THE   12 CASES
.064 .032 .016 .008 .004 .002 .001 .0005 .00025 .000125 .0000625 .00003125


   PARAMETERS ARE

  BPRIME =  .150000000000000000000000000D+03

  AL     =  .500000000000000000000000000D+01

  BRHO   =  .667128000000000000000000000D+05

  DHAT   =  .100000000000000000000000000D-05

  NKASE  =   12


  DELTA VALUES
    DELTA(1)   = .064
    DELTA(2)   = .032
    DELTA(3)   = .016
    DELTA(4)   = .008
    DELTA(5)   = .004
    DELTA(6)   = .002
    DELTA(7)   = .001
    DELTA(8)   = .0005
    DELTA(9)   = .00025
    DELTA(10)   = .000125
    DELTA(11)   = .0000625
    DELTA(12)   = .00003125
```

EXACT OFF MOMENTUM MATRIX CALCULATIONS

FOR CASE 1 DELTA = .64000000000000000000000000D-01

.97370109586311188302719754D+00	.49560911599537117626374713D+01
-.10473208470091940236317287D-01	.97370109586311188302719754D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0.	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	0.

FOR CASE 2 DELTA = .32000000000000000000000000D-01

.97288933045432477652009637D+00	.49547333538112517578168733D+01
-.10795000834301845734271230D-01	.97288933045432477652009637D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0.	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	0.

FOR CASE 3 DELTA = .16000000000000000000000000D-01

.97246436091069797104417149D+00	.49540224654385243822853791D+01
-.10963427626228987985335902D-01	.97246436091069797104417149D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0.	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	0.

FOR CASE 4 DELTA = .8000000000000000000000000D-02

.9722468405737988908065195 7D+00 .4953658581928787261347301 2D+01
-.1104962727830017355968977 8D-01 .9722468405737988908065195 7D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .1000000000000000000000000D+01
-.1000000000000000000000000D+01 0.

FOR CASE 5 DELTA = .4000000000000000000000000D-02

.9721367865704181015144682 4D+00 .4953474471668382116187443 1D+01
-.1109323738536064843655030D-01 .9721367865704181015144682 4D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .1000000000000000000000000D+01
-.1000000000000000000000000D+01 0.

FOR CASE 6 DELTA = .2000000000000000000000000D-02

.9720814316118773157988598 9D+00 .4953381866858486535610868 6D+01
-.1111517177461659077340082 9D-01 .9720814316118773157988598 9D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .1000000000000000000000000D+01
-.1000000000000000000000000D+01 0.

FOR CASE 7 DELTA = .10000000000000000000000000D-02

.97205367157274388163937563D+00 .49533354260753434848854942D+01
-.11126171527134913789500185D-01 .97205367157274388163937563D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

FOR CASE 8 DELTA = .50000000000000000000000000D-03

.97203977084134790850648456D+00 .49533121709685437462378178D+01
-.11131679571069377141477957D-01 .97203977084134790850648456D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

FOR CASE 9 DELTA = .25000000000000000000000000D-03

.97203281528867615257445894D+00 .49533005347211955041268486D+01
-.11134435638493863205707298D-01 .97203281528867615257445894D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

-.20194839173657902218540251D-27 .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

FOR CASE 10 DELTA = .125000000000000000000000D-03

.972029336214467672370635 16D+00 .495329471442213808665520 27D+01
-.111358141840135447117730 70D-01 .972029336214467672370635 16D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .100000000000000000000000D+01
-.100000000000000000000000D+01 0.

FOR CASE 11 DELTA = .625000000000000000000000D-04

.972027596352754005032698 30D+00 .495329180372852630684280 71D+01
-.111365035847806871201994 08D-01 .972027596352754005032698 30D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .100000000000000000000000D+01
-.100000000000000000000000D+01 0.

FOR CASE 12 DELTA = .312500000000000000000000D-04

.972026726340727146731621 75D+00 .495329034824566997943439 06D+01
-.111368483171730172101636 31D-01 .972026726340727146731621 75D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .100000000000000000000000D+01
-.100000000000000000000000D+01 0.

DIFFERENTIATION POINT MATRICES

.97202582843156981703878324D+00	.49532888460922499564580722D+01
-.11137204103379072702331669D-01	.97202582843156981703878324D+00

.97202584235308193974754467D+00	.49532888693821876178086752D+01
-.11137198587140385274802566D-01	.97202584235308193974754467D+00

.97202585627458020640183647D+00	.49532888926721020547846603D+01
-.11137193070907161867253346D-01	.97202585627458020640183647D+00

.97202587019606461702234469D+00	.49532889159619932674207660D+01
-.11137187554679402471566091D-01	.97202587019606461702234469D+00

.97202588411753517162975537D+00	.49532889392518612557517304D+01
-.11137182038457107079622897D-01	.97202588411753517162975537D+00

NONSYMPLECTIC INTERPOLATION

MATRICES USED IN STANDARD INTERPOLATION

.9720258562745802064018364 7D+00	.4953288892672102054784660 3D+01
-.1113719307090716186725334 6D-01	.9720258562745802064018364 7D+00

.2784298267726790466811917 3D-01	.4657980564960051138797729 7D-01
.1103246098280053050553286 4D-01	.2784298267726790466811917 3D-01

-.5542413513432415431335795 4D-01	-.9289735951806218782728348 2D-01
-.2185604744787335198822792 1D-01	-.5542413513432415431335795 4D-01

.1654883889758192707275925 6D+00	.2779063759813418572417830 1D+00
.6494328539518541847653229 2D-01	.1654883889758192707275925 6D+00

FOR CASE 1 DELTA = .64000000000000000000000000D-01

.97370152883521020262704661D+00 .49560918883562210382994580D+01
-.10473039337746733292047615D-01 .97370152883521020262704661D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000125642678797101258D+01
-.10000000125642678797101258D+01 0.

FOR CASE 2 DELTA = .32000000000000000000000000D-01

.97288935835031899865851838D+00 .49547334007441707160772377D+01
-.10794989938821551141328828D-01 .97288935835031899865851838D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000008018718269954705D+01
-.10000000008018718269954705D+01 0.

FOR CASE 3 DELTA = .16000000000000000000000000D-01

.97246436268152617354406308D+00 .49540224684178966876028673D+01
-.10963426934639518054973634D-01 .97246436268152617354406308D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000506489952739280D+01
-.10000000000506489952739280D+01 0.

FOR CASE 4 DELTA = .80000000000000000000000000D-02

.97224684068534991053615450D+00 .49536585821164716979602588D+01
-.11049627234736069181316997D-01 .97224684068534991053615450D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000031823965466566D+01
-.10000000000031823965466566D+01 0.

FOR CASE 5 DELTA = .40000000000000000000000000D-02

.97213678657741768583601333D+00 .49534744716801589894670227D+01
-.11093237382627165183402657D-01 .97213678657741768583601333D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000001994296849412D+01
-.10000000000001994296849412D+01 0.

FOR CASE 6 DELTA = .20000000000000000000000000D-02

.97208143161231565886395644D+00 .49533818668592240543248498D+01
-.11115171774445409359365699D-01 .97208143161231565886395644D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000124810434805D+01
-.10000000000000124810434805D+01 0.

```
FOR CASE     7 DELTA =  .10000000000000000000000000D-02

 .97205367157277130530706398D+00   .49533354260753896255951780D+01
-.11126171527124204374226238D-01   .97205367157277130530706398D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                   .10000000000000007805971102D+01
        -.10000000000000007805971102D+01  0.


FOR CASE     8 DELTA =  .50000000000000000000000000D-03

 .97203977084134962333398081D+00   .49533121709685466314488588D+01
-.11131679571068707473769882D-01   .97203977084134962333398081D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                   .10000000000000000488050888D+01
        -.10000000000000000488050888D+01  0.


FOR CASE     9 DELTA =  .25000000000000000000000000D-03

 .97203281528867625977666487D+00   .49533005347211956844938487D+01
-.11134435638493821341619251D-01   .97203281528867625977666487D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                   .10000000000000000030509991D+01
        -.10000000000000000030509991D+01  0.
```

```
FOR CASE    10 DELTA =  .12500000000000000000000000D-03

 .97202933621446767907129893D+00    .49532947144221380979288211D+01
-.11135814184013542095069472D-01    .97202933621446767907129893D+00

            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000001907203D+01
        -.10000000000000000001907203D+01   0.
```

```
FOR CASE    11 DELTA =  .62500000000000000000000000D-04

 .97202759635275400545143716D+00    .49532918037285263075472941D+01
-.11136503584780686956676736D-01    .97202759635275400545143716D+00

            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000119215D+01
        -.10000000000000000000119215D+01   0.
```

```
FOR CASE    12 DELTA =  .31250000000000000000000000D-04

 .97202672634072714675777386D+00    .49532903482456699794783858D+01
-.11136848317173017199951002D-01    .97202672634072714675777386D+00

            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000007449D+01
        -.10000000000000000000007449D+01   0.
```

SYMPLECTIC INTERPOLATION

DIFFENTIATION POINT GENERATOR MATRICES

.564759545377604504966781 69D-02	.197215226305252951352932 14D-30
.197215226305252951352932 14D-30	.251177686147843033012627 30D+01

.564759261666223252200543 62D-02	0.
0.	.251177685555670551574634 25D+01

.564758977955127049810408 59D-02	.197215226305252951352932 14D-30
.197215226305252951352932 14D-30	.251177684963498665657278 84D+01

.564758694244315897366781 40D-02	.197215226305252951352932 14D-30
.197215226305252951352932 14D-30	.251177684371327375259662 76D+01

.564758410533789794440067 72D-02	.394430452610505902705864 28D-30
.394430452610505902705864 28D-30	.251177683779156680380887 66D+01

DERIVATIVES OF GENERATOR MATRICES

.56475897795512704981040859D-02	.19721522630525295135293214D-30
.19721522630525295135293214D-30	.25117768496349866565727884D+01

-.56742190735469056396221807D-02	.23008443068946177657842083D-24
.23008443068946177657842083D-24	-.11843431763146720460236687D-01

.11401997860322270073845236D-01	.19721522630525295135293214D-18
.19721522630525295135293214D-18	.23820789572688275028344913D-01

-.34367581261474027226275498D-01	-.78886090522101180541172857D-12
-.78886090522101180541172857D-12	-.71866446012487955307933228D-01

```
FOR CASE     1 DELTA =  .64000000000000000000000000D-01

 .97370154152789080363942854D+00    .49560919043017230976677213D+01
-.10473031781585189242724738D-01    .97370154152789093968847165D+00


              SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.


FOR CASE     2 DELTA =  .32000000000000000000000000D-01

 .97288935916026455994942554D+00    .49547334017617646233201059D+01
-.10794989456690344207641162D-01    .97288935916026457694835863D+00


              SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.


FOR CASE     3 DELTA =  .16000000000000000000000000D-01

 .97246436273268100365552841D+00    .49540224684821688225510590D+01
-.10963426904190294281283586D-01    .97246436273268100577988750D+00


              SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.
```

FOR CASE 4 DELTA = .80000000000000000000000000D-02

.97224684068856394817932834D+00	.49536585821205099690267584D+01
-.11049627232822999492218316D-01	.97224684068856394844483149D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0.	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	0.

FOR CASE 5 DELTA = .40000000000000000000000000D-02

.97213678657761909282569889D+00	.49534744716804120488048410D+01
-.11093237382507284228045927D-01	.97213678657761909285888182D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

-.10097419586828951109270126D-27	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	-.19721522630525295135293214D-30

FOR CASE 6 DELTA = .20000000000000000000000000D-02

.97208143161232826335126474D+00	.49533818668592398913202763D+01
-.11115171774437906958870454D-01	.97208143161232826335541171D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0.	.10000000000000000000000000D+01
-.10000000000000000000000000D+01	-.39443045261050590270586428D-30

FOR CASE 7 DELTA = .10000000000000000000000000D-02

.97205367157277209359903975D+00 .49533354260753906160411212D+01
-.11126171527123735166812629D-01 .97205367157277209359955792D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

FOR CASE 8 DELTA = .50000000000000000000000000D-03

.97203977084134967261735673D+00 .49533121709685466933693460D+01
-.11131679571068678138805992D-01 .97203977084134967261742146D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

FOR CASE 9 DELTA = .25000000000000000000000000D-03

.97203281528867626285722416D+00 .49533005347211956883641295D+01
-.11134435638493819507910649D-01 .97203281528867626285723224D+00

SYMPLECTIC CONDITION FOR THIS MATRIX GIVES

0. .10000000000000000000000000D+01
-.10000000000000000000000000D+01 0.

```
FOR CASE   10 DELTA =  .12500000000000000000000000D-03

 .97202933621446767926382520D+00    .49532947144221380981706777D+01
-.11135814184013541980459501D-01    .97202933621446767926382621D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.


FOR CASE   11 DELTA =  .62500000000000000000000000D-04

 .97202759635275400546346634D+00    .49532918037285263075624020D+01
-.11136503584780686949514784D-01    .97202759635275400546346647D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.


FOR CASE   12 DELTA =  .31250000000000000000000000D-04

 .97202672634072714675852488D+00    .49532903482456699794793285D+01
-.11136848317173017199503726D-01    .97202672634072714675852490D+00


            SYMPLECTIC CONDITION FOR THIS MATRIX  GIVES

        0.                                      .10000000000000000000000000D+01
        -.10000000000000000000000000D+01  0.


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TABLE 1 : Sample of Matrix Elements @ Various Momenta, and Scaling of Errors With Momentum

δ	(A) $M_{11}(\delta)$ exact	(B) $M_{11}(\delta)$ nonsymplectic	error : (B) − (A)	(C) $M_{11}(\delta)$ symplectic	error (C) − (A)
.064	.9737 0109 5863 1118 8302 720	.9737 0152 8835 2102 0262 705	4.53×10^{-7}	.9737 0154 1527 8908 0363 943	4.45×10^{-7}
.032	.9728 8933 0454 3247 7652 010	.9728 8935 8350 3189 9865 852	$2.79\times10^{-8} = 4.46\times10^{-7}/2^{4}$	.9728 8935 9160 2645 5994 943	$2.87\times10^{-8} = 4.59\times10^{-7}/2^{4}$
.016	.9724 6436 0910 6979 7104 417	.9724 6436 2681 5261 7354 406	$1.77\times10^{-9} = 4.53\times10^{-7}/2^{8}$	.9724 6436 2732 6810 0665 553	$1.82\times10^{-9} = 4.66\times10^{-7}/2^{8}$
.008	.9722 4684 0573 7988 9060 652	.9722 4684 0685 3499 1053 615	$1.12\times10^{-10} = 4.59\times10^{-7}/2^{12}$	.9722 4684 0688 5639 4817 933	$1.15\times10^{-10} = 4.71\times10^{-7}/2^{12}$
.004	.9721 3678 6570 4181 0151 447	.9721 3678 6577 4176 8583 601	$7.00\times10^{-12} = 4.59\times10^{-7}/2^{16}$	.9721 3678 6577 6190 9282 570	$7.20\times10^{-12} = 4.72\times10^{-7}/2^{16}$
.002	.9720 8143 1611 8773 1579 886	.9720 8143 1612 3156 5886 396	$4.38\times10^{-13} = 4.59\times10^{-7}/2^{20}$	.9720 8143 1612 3282 6335 126	$4.51\times10^{-13} = 4.73\times10^{-7}/2^{20}$
.001	.9720 5367 1572 7438 8163 938	.9720 5367 1572 7713 0530 706	$2.75\times10^{-14} = 4.61\times10^{-7}/2^{24}$	.9720 5367 1572 7720 9359 904	$2.82\times10^{-14} = 4.73\times10^{-7}/2^{24}$
.0005	.9720 3977 0841 3479 0850 648	.9720 3977 0841 3496 2333 398	$1.72\times10^{-15} = 4.61\times10^{-7}/2^{28}$	.9720 3977 0841 3496 7261 736	$1.76\times10^{-15} = 4.72\times10^{-7}/2^{28}$
.00025	.9720 3281 5288 6761 5257 446	.9720 3281 5288 6762 5977 666	$1.07\times10^{-16} = 4.60\times10^{-7}/2^{32}$	.9720 3281 5288 6762 6285 722	$1.10\times10^{-16} = 4.72\times10^{-7}/2^{32}$
.000125	.9720 2933 6214 4676 7237 064	.9720 2933 6214 4676 7407 130	$6.70\times10^{-18} = 4.60\times10^{-7}/2^{36}$	.9720 2933 6214 4676 7926 383	$6.89\times10^{-18} = 4.73\times10^{-7}/2^{36}$
.0000625	.9720 2759 6352 7540 0503 270	.9720 2759 6352 7540 0545 144	$4.19\times10^{-19} = 4.61\times10^{-7}/2^{40}$	.9720 2759 6352 7540 0546 347	$4.31\times10^{-19} = 4.73\times10^{-7}/2^{40}$
.00003125	.9720 2672 6340 7271 4673 162	.9720 2672 6340 7271 4675 777	$2.61\times10^{-20} = 4.59\times10^{-7}/2^{44}$	.9720 2672 6340 7271 4675 852	$2.69\times10^{-20} = 4.73\times10^{-7}/2^{44}$

(Data from Appendix B)

TABLE 2: Sample of Elements of $MJ\tilde{M}$ @ Various Momenta,

Scaling of Errors (in Nonsymplectic Case) with δ

δ	Ⓐ $(MJM)_{12}$, exact	Ⓑ $(MJM)_{12}$, symplectic	Ⓑ - Ⓐ error	Ⓒ $(MJM)_{12}$, nonsymplectic	Ⓒ - Ⓐ error
.064	1.	1.	0.	1.0000 0001 2564 2678 7971 0215	1.26×10^{-8}
.032	1.	1.	0.	1.0000 0000 0801 8718 2699 5471	$8.02\times10^{-10} = 1.28\times10^{-8}/2^{4}$
.016	1.	1.	0.	1.0000 0000 0050 6489 9527 3928	$5.06\times10^{-11} = 1.30\times10^{-8}/2^{8}$
.008	1.	1.	0.	1.0000 0000 0003 1823 9654 6657	$3.18\times10^{-12} = 1.30\times10^{-8}/2^{12}$
.004	1.	1.	0.	1.0000 0000 0000 1994 2968 4941	$1.99\times10^{-13} = 1.30\times10^{-8}/2^{16}$
.002	1.	1.	0.	1.0000 0000 0000 0124 8104 3481	$1.25\times10^{-14} = 1.31\times10^{-8}/2^{20}$
.001	1.	1.	0.	1.0000 0000 0000 0007 8059 7110	$7.81\times10^{-16} = 1.31\times10^{-8}/2^{24}$
.0005	1.	1.	0.	1.0000 0000 0000 0000 4880 5089	$4.88\times10^{-17} = 1.31\times10^{-8}/2^{28}$
.00025	1.	1.	0.	1.0000 0000 0000 0000 0305 0999	$3.05\times10^{-18} = 1.31\times10^{-8}/2^{32}$
.000125	1.	1.	0.	1.0000 0000 0000 0000 0019 0720	$1.91\times10^{-19} = 1.31\times10^{-8}/2^{36}$
.0000625	1.	1.	0.	1.0000 0000 0000 0000 0001 1922	$1.19\times10^{-20} = 1.31\times10^{-8}/2^{40}$
.00003125	1.	1.	0.	1.0000 0000 0000 0000 0000 0745	$7.45\times10^{-22} = 1.31\times10^{-8}/2^{44}$