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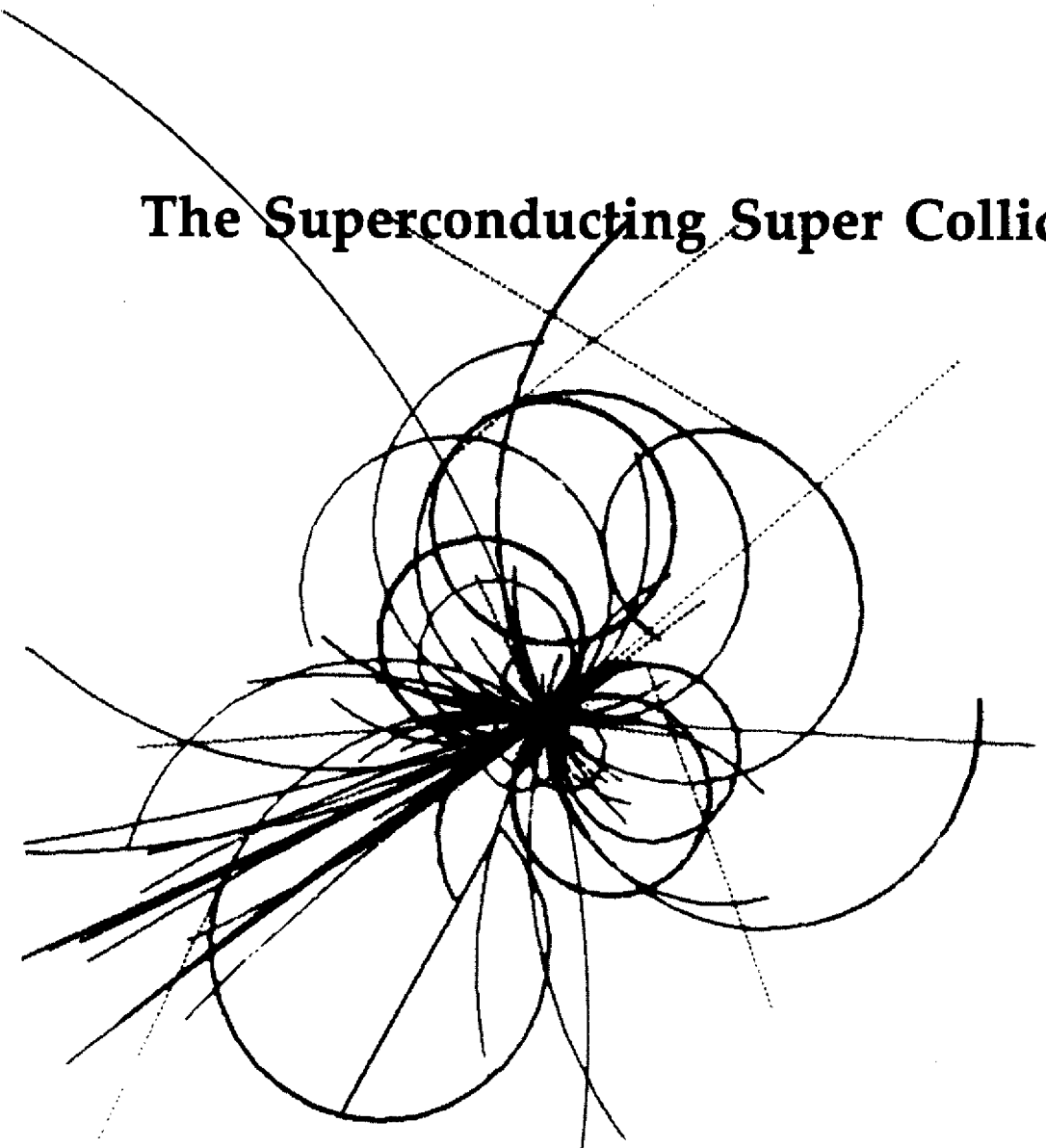

Lie Algebraic Maps and Invariants Produced by Tracking Codes

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*Operated by Universities Research Association, Inc. for the U. S. Department of Energy

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Abstract

We derive the tools necessary for the extraction of a map from standard tracking programs using multipole kicks. Our results applied to programs using approximate Hamiltonians (PATRICIA, RACETRACK) or to the more rigorous symplectic integrators. As an example of a symplectic integrator, we chose the kick code TEAPOT for its simplicity. Our results are nevertheless general because Neri has shown that symplectic integrators are in fact kick codes. Finally, we give the general formulae for geometric invariants which are quartic extensions of the usual Courant-Snyder invariants.

1. Introduction

In the study of circular machines, fast methods of tracking have always been useful. For large, error dominated rings, such as the contemplated SSC, one must take into account the effect of very high order multipoles.

Traditionally, the multipoles have been incorporated into tracking programs as zero length elements or kicks. Unfortunately, these kick codes have tracking as their primary goal and are incapable of giving certain insights into the nature of the non-linear lattice functions. On the other hand, matrix codes such as MARYLIE[1] or TRANSPORT[2,3] have the capability of computing non-linear lattice functions[4] around the design orbit. Unfortunately, they have no way of computing exactly the matrices around an arbitrary orbit. This is due to the perturbative approach of these codes. They expand everything in terms of the design trajectory. Off-momentum effects as well as misalignments are not treated exactly.

In this paper, we intend to show how one computes the non-linear map around any trajectory with the help of a kick code. From the non-linear map, represented by Lie transforms, we will extract the non-linear invariants.[5,6]

The maps computed by our method from kick codes, are accurate to third order around the chosen trajectory, but, unlike those of matrix codes, they are exact in the momentum and the magnitude of the misalignments of the orbit under study.

1.1 The choice of a coordinate system.

The orbit we choose to study will be represented by a vector Z.

$$Z = (X, P_x, Y, P_y, T, P_t) \tag{1.1}$$

The positions X and Y are deviation from the ideal closed orbit of the machine. Often they are chosen in a plane perpendicular to the design momentum, but as we will discover this is not always necessary. The momentum P_x and P_y are also in the plane of (X, Y). They measure the deviation from the ideal momentum P_0 and they are scaled by P_0. The time T is defined as follows:

$$T = c \text{ (Time of flight)}$$

$$c = \text{Speed of light} . \tag{1.2}$$

Finally, P_t is related to the energy deviation:

$$P_t = - \left(\frac{E-E_0}{c\ P_0}\right) . \tag{1.3}$$

1.2 Is (P_x, P_y) better than (X', Y')?

Most kick codes use X' $(=\frac{dx}{ds})$ and Y' instead of the momenta P_x and P_y.

The use of X' and Y' facilitates the expression of the map in a drift space:

$$\Delta X = LX' \tag{1.4a}$$

$$\Delta Y = LY' , \quad L = \text{path length of the design orbit.} \tag{1.4b}$$

Since equation (1.4) is exact, it implies that in terms of the momenta drifts are more complex. In fact, one needs to replace X' and Y' by the expressions:

$$X' = \frac{P_x}{P_s} \tag{1.5a}$$

$$Y' = \frac{P_y}{P_s} \tag{1.5b}$$

$$P_s = (1 - \frac{2P_t}{\beta_0} + P_t^2 - P_x^2 - P_y^2)^{1/2} \ . \tag{1.5c}$$

On the other hand, multipole kicks are much simpler in terms of the momentum:

$$\Delta\vec{P} = \frac{qL}{P_0} [A_s, \vec{P}] \tag{1.6a}$$

$$A_s(\vec{r}) = \text{vector potential} \tag{1.6b}$$

$$\vec{r} = (X, Y) \text{ measured in the plane of the multipole} \tag{1.6c}$$

As seen by the reader, one cannot win. For (X', Y') the drifts are trivial while for a canonical set the kicks are easy.

1.3 Making X' and Y' canonical by approximating kicks.

In the case of a large machine, where the influence of a single kick is small, it is conventional to introduce an approximation:

$$X' = \frac{P_x}{P_s} \simeq \frac{P_x}{(1-\frac{2P_t}{\beta_0} + P_t^2)^{1/2}} = \frac{P_x}{(1+\delta)} \tag{1.7a}$$

$$\delta = \frac{P - P_0}{P_0} \ . \tag{1.7b}$$

With this approximation, the equation for a kick becomes:

$$\Delta\vec{r}' = \frac{qL}{P_0(1+\delta)} [A_s, \vec{P}] \ . \tag{1.8}$$

Consistent with this approximation, one writes down a Hamiltonian in terms of $\vec{r}'$. To do this, we start from the correct Hamiltonian H:

$$H = P_s - \frac{q}{P_0} A_s \tag{1.9}$$

But, following (1.7) we approximate (1.9):

$$H = -(1+\delta)\left(1-\frac{P_x^{\,2}+P_y^{\,2}}{(1+\delta)^2}\right)^{1/2} - \frac{q}{P_0} A_s$$

$$\tilde{H} \simeq -(1+\delta) + \frac{1}{2(1+\delta)}(P_x^2+P_y^2) - \frac{q}{P_0} A_s \tag{1.10}$$

If one does not need the time of flight information contained in the δ-dependence of H and $\tilde{H}$, one can replace $\vec{P}$ by $\vec{r}'$. The Hamiltonian K governing the motion of $(\vec{r}, \vec{r}')$ is just:

$$\tilde{K} = \frac{\tilde{H} + (1+\delta)}{(1+\delta)} = \frac{1}{2}(X'^2+Y'^2) - \frac{q}{P_0(1+\delta)} A_s \ . \tag{1.11}$$

The Hamiltonian $\tilde{K}$ is at the root of many traditional codes used in accelerator design. These include PATRICIA[7] and RACETRACK[8]. The non-linear tools which are discussed in the later sections of this paper have been implemented in the program RACETRACK.

The Hamiltonian $\tilde{K}$ introduces errors which are proportional to the integrated strength of A_s. It provides a very accurate chromatic description of very large machines because the use of X' and Y' does not introduce errors solely proportional to the length of the ring. On the other hand, codes using P_x and P_y, introduce errors which are of zeroth order in the length anytime they approximate the drift (1.5) by a Taylor series. (MARYLIE for example).

Of course, for a small ring, the integrated strength of A_s can be large. Dragt noticed in the context of the Los Alamos Proton Storage Ring (PSR) the inadequacy of $\tilde{K}$ in predicting the correct chromatic properties.[9]

1.4 The no-compromise kick code: TEAPOT.

Passing from (X', Y') to (P_x, P_y) involves a square root only. If one is willing to live with it, it is possible to write a kick code which uses the exact formulae for the drift and the multipole kicks. Such a code has no dynamical approximation of the type introduced in section 1.3. Its only approximation is to replace an ideal long element by one or a few thin lens kicks represented by equation (1.6).

It is important to emphasize that it would be possible to regain long elements by splitting them in a large number of kicks and drifts. This is not the case if an incorrect Hamiltonian such as $\tilde{K}$ is used to describe the thin lenses. In fact, if one uses $\tilde{K}$ for a long element, only the linear part of the map is correctly restored.

Inspired by the desire to be exact, Schachinger and Talman of the SSC Central Design Group, have written a code which uses the exact formalism for the lenses. This program, TEAPOT[10] (Thin Element Accelerator Program for Optics and Tracking), has been equipped with an algorithm capable of generating a six dimensional map around any orbit.

As a result of the strict compliance of TEAPOT to the thin lens model, we have been able to use the concatenation routines of the program MARYLIE in the map generating algorithm.

In the next section, we examine the tools necessary to extract a map out of a kick code. We concentrate on the example of TEAPOT, because of its greater complexity and exactness.

2. The elements of a thin lens code.

The present version of TEAPOT contains multipole kicks, drifts and rotations in the plane of the ring. It has also a special linear kick to handle sector bends. The basic ideas of the code are contained in Figure 1.

This figure shows two successive multipole planes $\mathscr{P}_i$ and $\mathscr{P}_{i+1}$. The center of the multipoles are located at P_i and P_{i+1}. The canonical coordinates of the particles at plane $\mathscr{P}_i$ are expressed with respect to P_i in a frame within $\mathscr{P}_i$. The X_i direction points away from the center of the multipole and Y_i is perpendicular to the plane of the ring. The components of P_{i+1} in the frame of $\mathscr{P}_i$ are given by the pair $(\mathscr{S}_{i+}, \mathscr{X}_{i+})$.

The particle drifts towards $\mathscr{P}_{i+1}$. Finally, X_{i+1} and Y_{i+1} are the coordinate of the particle expressed in the plane $\mathscr{P}_{i+1}$ just before entering the next multipole. The reader will notice that we use capital letters for the ray under study, unlike reference 10 where lower case letters are used.

2.1 The tracking in TEAPOT.

First, it is instructive to derive the connection between (X_i, Y_i, T_i) and $(X_{i+1}, Y_{i+1}, T_{i+1})$ in terms of canonical transformations. Consider the phase space vector Z_i defined in Section 1.

$$Z_i = (X_i, P_{xi}, Y_i, P_{yi}, T_i, P_{ti}) \quad . \tag{2.2}$$

The particle drifts in the frame of $\mathscr{P}_i$ a distance Σ_i:

$$\Sigma_i^2 = \mathscr{S}_{i+} + \mathscr{X}_{i+} \tan\phi_{i+} \tag{2.3a}$$

$$X_{i+} = X_i + \frac{P_{xi}}{P_{si}} \Sigma_i \tag{2.3b}$$

$$Y_{i+} = Y_i + \frac{P_{yi}}{P_{si}} \Sigma_i \tag{2.3c}$$

$$T_{i+} = T_i + \left\{ \frac{\frac{1}{\beta_0} - P_{ti}}{P_{si}} \right\} \Sigma_i \tag{2.3d}$$

$$P_{si} = \left(1 - \frac{2}{\beta_0} P_{ti} + P_{ti}^2 - P_{xi}^2 - P_{yi}^2\right)^{1/2} . \tag{2.3e}$$

All the momenta are left invariant by the drift. We then rotate Z_{i+} in the frame of $\mathscr{P}_{i+1}$ and express the component X_{i+1} with respect to the point P_{i+1}:

$$X_{i+1} = \frac{X_{i+}}{\left(1 + \frac{P_{x(i+)}}{P_{s(i+)}} \tan\phi_{i+}\right) \cos\phi_{i+}} - \frac{\mathscr{X}_{i+}}{\cos\phi_{i+}} \tag{2.4a}$$

$$P_{x(i+1)} = \cos\phi_{i+} P_{x(i+)} - \sin\phi_{i+} P_{s(i+)} \tag{2.4b}$$

$$Y_{i+1} = Y_{i+} - \frac{\tan\phi_{i+}\, X_{i+}}{1 + \frac{P_{x(i+)}}{P_{s(i+)}} \tan\phi_{i+}} \frac{P_{y(i+)}}{P_{s(i+)}} \tag{2.4c}$$

$$P_{y(i+1)} = P_{y(i+)}$$

$$T_{i+1} = T_{i+} + \frac{\tan\phi_{i+}}{1 + \frac{P_{x(i+)}}{P_{s(i+)}} \tan\phi_{i+}} \frac{\left(\frac{1}{\beta_0} - P_{ti}\right)}{P_{s(i+)}} X_{i+} \tag{2.4d}$$

Substitution of (2.3) into (2.4) leads to the expression used in the program TEAPOT.

$$X_{i+1} = \frac{1}{\cos\phi_{i+}} \frac{X_i - \mathscr{X}_{i+} + \frac{P_{xi}}{P_{si}} \mathscr{S}_{i+}}{1 + \frac{P_{xi}}{P_{si}} \tan\phi_{i+}} \tag{2.5a}$$

$$P_{x(i+1)} = \cos\phi_{i+} P_{xi} - \sin\phi_{i+} P_{si} \tag{2.5b}$$

$$Y_{i+1} = Y_{i+} + \frac{P_{yi}}{P_{si}} \frac{\mathscr{S}_{i+} - \tan\phi_{i+} (X_i - \mathscr{X}_{i+})}{1 + \frac{P_{xi}}{P_{si}} \tan\phi_{i+}} \tag{2.5c}$$

$$P_{y(i+1)} = P_{yi} \tag{2.5d}$$

$$T_{i+1} = T_i + \left\{ \frac{\frac{1}{\beta_0} - P_{ti}}{P_{si}} \right\} \Sigma_i$$

$$+ \frac{\tan\phi_{i+}}{1+\frac{P_{xi}}{P_{si}}\tan\phi_{i+}} \frac{\left(\frac{1}{\beta_0} - P_{ti}\right)}{P_{si}} \left(X_i + \frac{P_{xi}}{P_{si}} \Sigma_i\right) \tag{2.5e}$$

$$P_{t(i+1)} = P_{ti} \tag{2.5f}$$

Finally, we consider the multipole kick. In terms of the canonical vector Z, kicks are represented by the vector potential $\tilde{A}$:

$$\tilde{A} = \frac{qLB}{P_0} \mathrm{Re} \sum_{n=0}^{\infty} \frac{1}{(n+1)} (ia_n+b_n) (X+iY)^{n+1} \tag{2.6a}$$

$$Z^+ = \exp(:\tilde{A}(Z^-):) Z^- = Z^- + [\tilde{A},Z^-]. \tag{2.6b}$$

The program allows for misalignments in the plane of the multipole. Rotation errors are obtained by modifying a_n and b_n. Translation errors are computed by translating $\tilde{A}$:

$$\tilde{A}_{errors} = \tilde{A}(X^- - \Delta X^-, Y^- - \Delta Y^-). \tag{2.7}$$

It should be added that sector bends are treated by including an extra term to $\tilde{A}$.

$$\tilde{A}_{errors} = A + \alpha(X^- - \Delta X^-) + \beta(Y^- - \Delta Y^-). \tag{2.7a}$$

This completes the description of this kick code in terms of canonical variables. We now look at the expansion around a trajectory.

2.2 The transfer map expanded around a trajectory.

The expansion of a map around a given trajectory involves taking the derivatives of the tracking operations of a given program. One follows a particle along its path and simultaneously differentiates the operations performed by the code. The resulting matrix is concatenated with the matrices of the previous operations. At the end, one gets a first, second or even

third order matrix for the full ring relative to the given trajectory. This brute force method has been incorporated in the programs MATPOT (matrix TEAPOT) and MATRACE (Matrix Racetrack) to third order in (X, P_x, Y, P_y). It should be noted that a Lie algebraic representation can be extracted from the matrix representation of the map.[11]

Here, we will be more elegant and perhaps more confusing. Instead of a direct differentiation of the expressions in section 2.1, we will appeal to elementary geometry in defining the non-linear maps around a given trajectory. The map representing the motion of a particle from one plane to the next will be a product of a rotation, a drift and a second rotation. The parameters defining this map will depend on the particular ray for which the map is computed. Finally, our treatment is six-dimensional and Lie algebraic from the start. With this in mind, the reader is invited to carefully follow the description of these three operations with Figure 2 as a guide.

A. The drift between the multipole planes

Consider an arbitrary trajectory Z_i at the plane $\mathscr{P}_i$, as shown on Fig. 2:

$$Z_i = (X_i, P_{xi}, Y_i, P_{yi}, T_i, P_{ti}) \ . \tag{2.8}$$

Suppose we are interested in a neighboring ray Z:

$$Z = Z_i + \zeta_i ; \ \zeta_i = (x_i, p_{xi}, y_i, p_{yi}, \tau_i, p_{\tau i}) \ . \tag{2.9}$$

For simplicity, we will scale ζ_i by P_{0i}, where P_{0i} is the momentum of Z_i (Z_i is scaled by P_0, the momentum of the design trajectory of the machine).

The variable ζ_i is measured in the plane $\mathscr{P}_i$.

Now, consider a rotation of angle ψ_0 in the plane of the ring. We choose ψ_0 such that the new frame has its x-axis perpendicular to the projection of the ray in the plane of the ring:

$$\tan\psi_0 = \frac{P_{xi}}{P_{si}} \ . \tag{2.10}$$

Such a rotation will depend on ψ_0, P_{yi} and $\frac{V_{0i}}{c}$ the velocity of the ray Z_i. Let us assume that it can be represented by a symplectic operator $\mathscr{R}(\psi_0, P_{yi})$:

$$\zeta_\perp = \mathscr{R}^{-1}(\psi_0, P_{yi})\, \zeta_i \ . \tag{2.11}$$

By writing the inverse of $\mathscr{R}$ in (2.11), we assume that $\mathscr{R}$ rotates from a perpendicular frame to an arbitrary frame. The Lie representation[12] for $\mathscr{R}$ has the form:

$$\mathscr{R} = \mathscr{F}_2\, e^{:f_3:}\, e^{:f_4:}\, \mathscr{G}_2 \tag{2.12}$$

The expressions for $\mathscr{F}_2$, $\mathscr{G}_2$, f_3 and f_4 are given in Appendix 1, Table 2.

As shown in Fig. 2, $\zeta_\perp$ is such that its y-axis makes an angle ψ_0 with the ray Z_i:

$$\psi_0 = \tan^{-1} \frac{P_{yi}}{(P_{si}^{\ 2} + P_{xi}^{\ 2})^{1/2}} \ . \tag{2.13}$$

We can derive a symplectic transformation which correspond to the drift around Z_i. In the frame of $\zeta_\perp$, it is given by the map $\mathscr{D}$.

$$\zeta_\perp^{final} = \mathscr{D}(OB, P_{yi})\ \ \zeta_\perp \ . \tag{2.14}$$

As hinted in (2.14), $\mathscr{D}$ will depend on the distance in the plane OAB as well as the vertical momentum of the ray Z_i. The Lie algebraic representation of $\mathscr{D}$ is given in Appendix 1, Table 1.

Finally, we can use $\mathscr{R}$ to rotate $\zeta_\perp$ into the plane $\mathscr{P}_{i+1}$. This involves a rotation by an angle ψ_1:

$$\psi_1 = \psi_0 - \phi_{i+} \ . \tag{2.15}$$

We can express the full map from ζ_i to ζ_{i+1} as follows:

$$\zeta_{i+1} = \mathscr{R}^{-1}(\psi_0, P_{yi})\, \mathscr{D}(OB, P_{yi})\, \mathscr{R}(\psi_1, P_{yi})\, \zeta_i \ . \tag{2.16}$$

B. The multipole kick.

The Lie algebraic representation of a multipole kick in terms of ζ_i is obtainable from (2.7). In fact, we need only to expand $\tilde{A}_{errors}$ around Z_i in homogeneous polynomials in ζ_i:

$$f_m(\zeta_i) = \tilde{A}^m_{errors}(Z_i;\, \zeta_i) \ ; \ m \geq 2 \ . \tag{2.17a}$$

$$\tilde{A}_{errors}(Z_i + \zeta_i) = \sum_{m=0}^{\infty} \tilde{A}^m(Z_i;\, \zeta_i) \ . \tag{2.17b}$$

The computation of each f_m requires the evaluation of only two derivatives. This is a result of Maxwell's equations. In fact, the computation of these two derivatives is equivalent to computing B_x and B_y with a different set of a_n and b_n coefficients. Hence, it is an extremely fast operation, as fast as the tracking of one ray through the multipole.

This concludes our discussion of the computation of a map around a particular trajectory. In the remaining sections, we illustrate how one obtains invariants from the map around a closed orbit.

3. A non-linear Floquet representation.

As mentioned earlier, it is quite important to obtain the exact behavior resulting from misalignments or chromatic effects. In the study of the contemplated SSC, the chromatic effects of high multipole errors must be

computed to all orders in δ or the misalignments. However, most of the properties around the resulting closed orbit can be explained by a relatively low order study of the optics around it. In Appendix 2, we apply this technique to a linear problem. The linear problem being solvable exactly, is an ideal example of the perturbative approach introduced in this section. This example is presented for pedagogical purposes only. In the following two sections, we will derive some of the non-linear canonical transformations implemented in the program MARYLIE; they are used in obtaining invariants and tune shifts. The reader will notice references to equations in Appendix 2 ((B1), (B2), etc...); this allows the reader to establish connections between the non-linear case and the equivalent linear computation in Appendix 2. An exact link between the two computations is obtained if g_3 in equation (3.3) is set to zero. The reader is advised to set g_3 to zero in the first reading of section 3 and 4. (No sextupolar terms in the map).

Suppose we have a fixed point $\vec{f}$ of the map $\mathscr{M}$. $\mathscr{M}$ may represent the off-momentum motion with magnet errors for one turn of the machine ($\theta = 0 \rightarrow \theta = 2\pi$). This map has a Lie algebraic representation:

$$\mathscr{M} = \exp(:f_2:)\ \exp(:f_3:)\ \exp(:f_4:) \tag{3.1a}$$

$$f_m(\xi_0) = \sum_{i+j+k+\ell=m} A_{ijk\ell}\ x_0^i p_{x0}^j y_0^k p_{y0}^\ell \tag{3.1b}$$

Here the Lie operators are expressed in terms of the initial restricted phase space vector ξ_0. For simplicity, we will not look at the full phase space ζ_0.

Using some standard techniques, it is possible to find a canonical transformation which transforms f_2 into a very simple operator[13]:

$$\exists\ \mathscr{A}_2 \text{ such that } \mathscr{A}_2 f_2 = -\pi\nu_x(x_0^2 + p_{x0}^2) - \pi\nu_y(y_0^2 + p_{y0}^2) \tag{3.2}$$

This transformation, when applied to $\mathscr{M}$, leads to the result:

$$\mathscr{M}_F = \mathscr{A}_2 \mathscr{M} \mathscr{A}_2^{-1} = \exp(:\mathscr{A}_2 f_2:) \exp(:\mathscr{A}_2 f_3:) \exp(:\mathscr{A}_2 f_4:)$$

$$= \exp(:g_2:) \exp(:g_3:) \exp(:g_4:) \ . \tag{3.3}$$

We will say that $\mathscr{M}_F$ is the map $\mathscr{M}$ in linear Floquet variables. At this stage, it is possible to introduce the usual action-angle variables:

$$x = \sqrt{2\ I}\ \cos\phi \tag{3.4a}$$

$$p_x = -\sqrt{2\ I}\ \sin\phi \tag{3.4b}$$

$$y = \sqrt{2\ J}\ \cos\psi \tag{3.4c}$$

$$p_y = -\sqrt{2\ J}\ \sin\psi \ . \tag{3.4d}$$

All the polynomials g_m are rewritten as follows (B6):

$$g_m = \sum_{\substack{a+b=m \\ |c|+|d|=m,m-2,\ldots}} r^m_{abcd}\ I^{\frac{a}{2}}\ J^{\frac{b}{2}} \exp\{i(c\phi+d\psi)\} \tag{3.5a}$$

or, conveniently,

$$g_m = \sum r^m_{abcd}\ |abcd\rangle \ . \tag{3.5b}$$

In particular, g_2 has a simple form (B1a), (B1b):

$$g_2 = -2\pi\nu_x I - 2\pi\nu_y J = -2\pi\nu_x |2000\rangle - 2\pi\nu_y |0200\rangle \ . \tag{3.6}$$

Moreover, the vector $|abcd\rangle$ is an eigenvector of $:g_2:$. The application of $:g_2:$ on this vector gives the relation:

$$:g_2: \ |abcd\rangle = \exp(i(c\mu_x+d\mu_y)\ )\ |abcd\rangle \tag{3.7a}$$

$$\mu_i = 2\pi\nu_i \ . \tag{3.7b}$$

The map $\mathscr{A}_2$ originally simplified the linear part of $\mathscr{M}$ allowing us to compute $\mathscr{M}_F$ in this new basis. Can we continue our simplification and reduce the non-linear terms to their simplest form? To explore this, we apply a canon- ical transformation $\mathscr{A}_3$ on $\mathscr{M}_F$.

$$\mathscr{A}_3 = \exp(:F_3:) \tag{3.8a}$$

$$\mathscr{M}_{F_2} = \mathscr{A}_3 \, \mathscr{M}_F \, \mathscr{A}_3^{-1}$$

$$= \exp(:g_2:) \exp(:-g_2:) \, \mathscr{A}_3 \exp(:g_2:) \exp(:g_3:) \exp(:g_4:) \, \mathscr{A}_3^{-1}. \tag{3.8b}$$

With the help of standard Lie algebraic tools, we rewrite (3.8b) accurately to the order of g_4:

$$\mathscr{M}_{F_2} = \exp(:g_2:) \exp(:\exp(:-g_2:) F_3 - F_3 + g_3:)$$

$$\times \exp(:g_4 + \tfrac{1}{2}[\exp(:-g_2:) F_3, g_3 - F_3] + \tfrac{1}{2}[F_3, g_3]:) \tag{3.9}$$

The map $\mathscr{M}_{F_2}$ can be made simpler by removing its second order content. This is done by requiring that its third order polynomial vanishes:

$$(1 - \exp(:-g_2:)) \, F_3 = g_3 \, . \tag{3.10}$$

It is easy to see that (3.10) can be inverted away from integer and third integer resonances. This inversion is easily carried out using the eigenbasis representation:

$$F_3 = (1 - \exp(:-g_2:))^{-1} \, g_3$$

$$F_3 = (1 - \exp(:-g_2:))^{-1} \sum_{\substack{a+b=3 \\ |c|+|d|=3,1}} \Gamma^3_{abcd} \, |abcd\rangle \tag{3.11}$$

$$= \sum_{\substack{a+b=3 \\ |c|+|d|=3,1}} \frac{\Gamma^3_{abcd} \, |abcd\rangle}{1 - \exp(-i2\pi(c\nu_x + d\nu_y))}$$

The map $\mathscr{M}_{F_2}$ is purged from all quadratic contributions (B1a):

$$\mathscr{M}_{F_2} = \exp(:g_2:)\ \exp(:h_4:) \tag{3.12a}$$

$$h_4 = g_4 + \frac{1}{2}[F_3,\ g_3]$$

$$= g_4 + \frac{1}{2}\sum \frac{\Gamma^3_{abcd}\ \Gamma^3_{\alpha\beta\kappa\delta}}{1 - \exp(-i2\pi(c\nu_x+d\nu_y))}\ [|abcd\rangle,\ |\alpha\beta\kappa\delta\rangle] \tag{3.12b}$$

$$[|abcd\rangle,\ |\alpha\beta\kappa\delta\rangle] = \frac{i}{2}\ (c\alpha-\kappa a)\ |a+\alpha-2\ b+\beta\ c+\kappa\ d+\delta\ \rangle$$

$$+ \frac{i}{2}\ (d\beta-\delta b)\ |a+\alpha\ b+\beta-2\ \ c+\kappa\ d+\delta\ \rangle. \tag{3.12c}$$

All of h_4 can be lumped into one sum (B6):

$$h_4 = \sum_{\substack{a+b=4 \\ |c|+|d|= 4,2,0}} H^4_{abcd}\ |abcd\rangle\ . \tag{3.13}$$

As we did before we will attempt to simplify $\mathscr{M}_{F_2}$ with a canonical transformation $\mathscr{A}_4$ (B5):

$$\mathscr{A}_4 = \exp\ (:F_4:)$$

$$\mathscr{M}_{F_3} = \mathscr{A}_4 \mathscr{M}_{F_2}\ \mathscr{A}_4^{-1} = \exp(:g_2:)\ \exp(:h_4+\exp(:-h_2:)\ F_4 - F_4:)\ . \tag{3.14}$$

Here, we discover that it is not possible to remove h_4 completely (B6), (B7). In (3.13), the terms such that $|c|+|d|$ equals zero are not removable by canonical transformation. We separate them from the rest of h_4:

$$h_4 = \sum_{\substack{a+b=4 \\ |c|+|d|=4,2}} H^4_{abcd}\ |abcd\rangle + H^4_{4000}\ |4000\rangle$$

$$+ H^4_{0400}\ |0400\rangle + H^4_{2200}\ |2200\rangle \tag{3.13}$$

As before, we can compute F_4:

$$F_4 = \sum_{\substack{a+b=4 \\ |c|+|d|=4,2}} \frac{H^4_{abcd}}{1\ -\exp(-i2\pi(c\nu_x+d\nu_y))}\ |abcd\rangle\ . \tag{3.14}$$

The final map is given by

$$\mathscr{M}_{F_3} = \exp(:g_2:)\ \exp(:T_4:) \tag{3.15a}$$

$$T_4 = H^4_{4000}\ |4000\rangle + H^4_{0400}\ |0400\rangle + H^4_{2200}\ |2200\rangle$$

$$= H^4_{4000} I^2 + H^4_{0400}\ J^2 + H^4_{2200} IJ \quad . \tag{3.15b}$$

One notices that g_2 and T_4 are all expressed in terms of the action variables I and J. Therefore, $:g_2:$ and $:T_4:$ commute and the exponents in $\mathscr{M}_{F_3}$ can be combined (B8):

$$\mathscr{M}_{F_3} = \exp(:g_2 + T_4:) \ . \tag{3.16}$$

It is a simple exercise to show that T_4 produces tune shifts with amplitude. In fact, the tune shift are just (B9):

$$\begin{pmatrix} \Delta\nu_x \\ \Delta\nu_y \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} I \\ J \end{pmatrix} \tag{3.17a}$$

$$A = -\frac{H^4_{4000}}{\pi},\ B = -\frac{H^4_{2200}}{2\pi},\ C = -\frac{H^4_{0400}}{\pi}. \tag{3.17b}$$

Finally, it is clear that near certain resonances the canonical transformations represented by F_3 and F_4 will be ill defined. This is closely related to the absence of true non-linear invariants of the map $\mathscr{M}$. In the next section, we derive a cubic and quartic correction to the linear invariants. The map $\mathscr{M}$ is also rewritten in terms of a pseudo-Hamiltonian H.[14]

4. The non-linear invariants and the Pseudo-Hamiltonian.

In the previous section, we computed the canonical transformation leading to $\mathscr{M}_{F_3}$. One can express the original map $\mathscr{M}$ in terms of $\mathscr{M}_{F_3}$:

$$\mathscr{M} = \mathscr{A}^{-1}\ \mathscr{M}_{F_3}\ \mathscr{A} \tag{4.1a}$$

$$\mathscr{A} = \exp(:F_4:)\ \exp(:F_3:)\ \mathscr{A}_2 \ . \tag{4.1b}$$

Let us write two functions with the help of $\mathscr{A}$:

$$I_4 = \mathscr{A}^{-1} I = \mathscr{A}^{-1}\left(\frac{x_0^2 + p_{x0}^2}{2}\right) \tag{4.2a}$$

$$J_4 = \mathscr{A}^{-1} J = \mathscr{A}^{-1}\left(\frac{y_0^2 + p_{y0}^2}{2}\right) \tag{4.2b}$$

It is easy to check that I_4 and J_4 are quartic invariants of $\mathscr{M}$:

$$\begin{aligned}
\mathscr{M} I_4 &= \mathscr{A}^{-1} \mathscr{M}_{F_3} \mathscr{A} \mathscr{A}^{-1} I \\
&= \mathscr{A}^{-1} \mathscr{M}_{F_3} I \\
&= \mathscr{A}^{-1} \exp(:g_2 + T_4:)\ I \\
&= \mathscr{A}^{-1} I = I_4;\ [g_2,\ I] = [T_4,\ I] = 0\ .
\end{aligned} \tag{4.3}$$

Because g_2 and T_4 are polynomials in I and J, their Poisson bracket with I or J is zero. As an example, we can compute the cubic invariant I_3:

$$\begin{aligned}
I_3 &= \mathscr{A}_2^{-1}\exp(:F_3:)I \\
&= \mathscr{A}_2^{-1}\ (I+[F_3,\ I]) + \ldots \\
&= \mathscr{A}_2^{-1}(I+[F_3,\ |2000>]) \\
&= \mathscr{A}_2^{-1}\ I + \mathscr{A}_2^{-1} \sum_{\substack{a+b=3 \\ |c|+|d|=3,1}} \frac{-\Gamma^3_{abcd}\ ic|abcd>}{1 - \exp(-i2\pi(c\nu_x+d\nu_y))} \\
&= \mathscr{A}_2^{-1} I + \mathscr{A}_2^{-1}\ \Delta I_3
\end{aligned} \tag{4.4}$$

The quantity $\mathscr{A}_2^{-1}I$ is the quadratic invariant. The term $\mathscr{A}_2^{-1}\Delta I_3$ is the cubic correction to $\mathscr{A}_2^{-1}I$.

Finally, we can compute the pseudo-Hamiltonian H (B10):

$$\mathscr{M} = \mathscr{A}^{-1}\exp(:g_2+T_4:)\,\mathscr{A}$$

$$= \exp(\mathscr{A}^{-1}:g_2+T_4:\mathscr{A})$$

$$= \exp(:\mathscr{A}^{-1}(g_2+T_4):) \tag{4.5}$$

From (4.5), we can read off H (B10):

$$H = -\mathscr{A}^{-1}g_2 - \mathscr{A}^{-1}T_4$$

$$= 2\pi\nu_x I_4 + 2\pi\nu_y J_4 - H^4_{4000}I_4^2 - H^4_{0400}J_4^2 - H^4_{2200}I_4 J_4 \tag{4.6}$$

All the manipulation described in section 3 and 4 are performed by the program MARYLIE. MARYLIE can compute invariants, pseudo-Hamiltonian and generating function representations of any map provided by a kick code.[14] It handles the full six dimensional representation of the map $\mathscr{M}$.

Conclusion

We have shown how it is possible to extract the Lie Algebraic polynomials corresponding to the operations of a kick code. In particular, we have worked out the tools necessary to compute the six-dimensional lattice functions of an exact kick code. These techniques apply to simple codes like TEAPOT or to more complicated symplectic integrators. Indeed, Neri[15] has shown that the various symplectic integrators amount to a succession of kicks, rotations and drifts.

Our techniques are also applicable to codes using the approximate formula $\vec{p} = (1+\delta)\vec{r}'$.

It is the author's belief that the most important contribution of the Lie algebraic methods to accelerator design resides in the analytical powers of the Lie algebraic tools which permit the extraction of tune shifts and invariants from complicated maps; in other words the Lie algebraic manipulations capability of MARYLIE are more important than its ability to generate the map of a particular magnet. We hope that the reader will see that it is possible to extract from ordinary tracking codes the Lie algebraic polynomials of a transfer map.

Acknowledgment

The author is grateful to Richard Talman, Lindsay Schachinger, Albin Wrulich, Liam Healy and Filippo Neri for discussing certain points of this paper. The author also acknowledges the work done by Alex Dragt's group at the University of Maryland. Many of the tools advertised in this paper are being implemented by Dragt's group. In particular, they have extended the resonance basis subroutine to cover the full six-dimensional phase space.

Appendix I The Drift $\mathscr{D}$ and the Rotation $\mathscr{R}$.

First, we must derive the map of equation (2.14). According to a standard result of Hamiltonian theory, the Hamiltonian[16] for a drift is given by $-P_z$:

$$H = -P_z = -(1-\frac{2}{\beta} P_{ti} + P_{ti}^2 - P_{xi}^2 - P_{yi}^2)^{1/2} \quad . \tag{A1}$$

Calling ζ_i the phase space variables measuring deviations around Z_i, we can produce a Hamiltonian K for the motion of ζ_i:

$$K = -\left(1-\frac{2}{\beta} p_{\tau i} + p_{\tau i}^2 - p_{xi}^2 - (p_{yi} + P_{yi})^2\right)^{1/2} - \left(\frac{p_{\tau i}}{\beta} + p_{yi}P_{yi}\right)\frac{1}{\sqrt{1-P_{yi}^2}} \quad . \tag{A2}$$

The Lie algebraic polynomials for $\mathscr{D}$ are readily available from K. In fact, we need only to expand K in power[17] of ζ_i.

$$k = \sum_{m=2}^{\infty} K_m \tag{A3a}$$

$$f_m = -\ OB\ K_m \tag{A3b}$$

$$\mathscr{D}(OB,\ P_{yi}) = \exp(:f_2:)\ \exp(:f_3:)\ \ldots\ . \tag{A3c}$$

The coefficients of K_2, K_3 and K_4 are listed in Table 1.

Because we have the Hamiltonian K, we are in a position to derive the rotation $\mathscr{R}$. Consider the rotation from $\zeta_\perp$ to ζ_i in Fig. 2. Calling $\vec{q}$ the vector $(x_\perp, y_\perp, \tau_\perp)$, we drift $\vec{q}$ from the plane of $\zeta_\perp$ to $\mathscr{P}_i$.

$$\vec{q}_{\perp i} = \exp(-:dK:)\ \vec{q}_\perp \tag{A4a}$$

$$d = \tan\psi_0\ x_{\perp i} \tag{A4b}$$

Equation (A4b) parametrizes the plane $\mathscr{P}_i$. It permits us to solve for $x_{\perp i}$. Notice that d is the distance along the OB direction of figure 2 and that $p_{s\perp}$

in the following equations is the total momentum along the same direction:

$$x_{\perp i} = \frac{x_\perp}{1 - \frac{p_{x\perp}}{p_{s\perp}} \tan \psi_0} , \tag{A5a}$$

$$p_{s\perp} = \left((1 - \frac{2}{\beta} p_{\tau\perp} + p_{\tau\perp}^2 - p_{x\perp}^2 - (p_{y\perp} + P_{yi})^2\right)^{1/2}. \tag{A5b}$$

Finally, we can use (A4) and (A5) to compute the vector $\vec{q}_i$:

$$x_i = \frac{x_{\perp i}}{\cos \psi_0} \tag{A6a}$$

$$Y_i = \left[\frac{p_{y\perp} + P_{yi}}{p_{s\perp}}\right] d + y_\perp \tag{A6b}$$

$$t_i = \left[\frac{(\frac{1}{\beta} - P_{\tau i})}{p_{s\perp}}\right] d + \tau_\perp . \tag{A6c}$$

The momentum $p_{x\perp}$ must be redefined. This involves a simple rotation:

$$p_{xi} = \cos \psi_0 \, p_{x\perp} + \sin \psi_0 \, \{p_{s\perp} - (1 - P_{yi}^2)^{1/2}\} \tag{A7a}$$

$$p_{yi} = p_{y\perp} \tag{A7b}$$

$$p_{\tau i} = p_{\tau\perp} . \tag{A7c}$$

From equation (2.12), it is possible to find the Lie algebraic polynomial[18] for $\mathscr{R}(\psi, P_{yi})$. They are listed in Table 2.

Appendix 2. A linear example of the non-linear perturbation theory.

Consider a map $\mathscr{M}$ defined as follows:

$$\mathscr{M} = \exp(:-\mu J:)\ \exp(:ax^2:), \tag{B1a}$$

$$J = \frac{x^2 + p^2}{2} \quad . \tag{B1b}$$

The parameter "a" is assumed to be small. Nevertheless it is possible to write $\mathscr{M}$ with a single Lie operator. This is most easily seen by looking at the matrix representation of $\mathscr{M}$.

$$\begin{pmatrix} \mathscr{M}x \\ \mathscr{M}p \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu + 2\ a\ \cos\mu & \cos\mu + 2\ a\ \sin\mu \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} . \tag{B2}$$

It is well known from the Courant-Snyder theory that the quadratic invariant of M is just:

$$I_M = \frac{1}{2}\ (\gamma x^2 + 2\alpha xp + \beta p^2) \tag{B3a}$$

$$\alpha = -\frac{a\ \sin\ \mu}{[1-(\cos\mu + a\ \sin\mu)^2]^{1/2}},\ \beta = -\frac{\alpha}{a}\ ,\ \gamma = \frac{\alpha(-\sin\mu + 2a\cos\mu)}{a\ \sin\mu} \tag{B3b}$$

In terms of I_m, the map $\mathscr{M}$ has the form:[10]

$$\mathscr{M} = \exp(:-\tilde{\mu}\ I_m:) \tag{B4a}$$

$$\cos\tilde{\mu} = \cos\mu + a\ \sin\mu\ . \tag{B4b}$$

We can regain the results of (B3) and (B4) by the perturbative techniques of section 3 and 4. Consider a transformation ordered in power of "a":

$$\mathscr{A} = \ldots \mathscr{A}_2\ \mathscr{A}_1 = \ldots \exp(:a^2 F_2:)\ \exp(:aF_1:)\ . \tag{B5}$$

First, we expand ax^2 in terms of the Floquet representation:

$$ax^2 = a2J\cos^2\phi = \frac{a}{2} J\{e^{i2\phi} + e^{-i2\phi}+2\}$$

$$= \frac{a}{2} \{|22\rangle + |2\text{-}2\rangle + 2|20\rangle\} \tag{B6}$$

The vector $|20\rangle$ in (B6) is a tune shift term. The canonical transformation cannot remove $|20\rangle$:

$$F_1 = \frac{1}{2} \frac{1}{1- \exp(:-\mu J:)} \{|22\rangle + |2\text{-}2\rangle\}$$

$$= \frac{1}{2} \left\{\frac{|22\rangle}{1 - \exp(-i2\mu)} + \frac{|2\text{-}2\rangle}{1 - \exp(i2\mu)}\right\} \tag{B7}$$

To first order in "a", we have the equality:

$$\mathscr{A}_1 \mathscr{M} \mathscr{A}_1^{-1} = \exp(:-\mu|20\rangle:) \exp(:a|20\rangle:)$$

$$= \exp(:(-\mu+a)J:) \ . \tag{B8}$$

From (B8), we deduce the first order shift of μ. The result agrees with (B4b):

$$\tilde{\mu} = \mu - a + O(a^2) \ \ldots \ . \tag{B9}$$

Finally, we compute the pseudo-Hamiltonian of $\mathscr{M}$ using F_1

$$H = \mathscr{A}_1^{-1}(\mu-a)J = (\mu-a) \{J-a[F_1, J]\}$$

$$= (\mu-a) \left\{J - \frac{a}{2}\left(\frac{-2i|22\rangle}{1 - \exp(-i2\mu)} + \frac{2i|2\text{-}2\rangle}{1 - \exp(i2\mu)}\right)\right\}$$

$$= (\mu-a) \left\{J + aJ \frac{(\sin(2\phi) - \sin(2\phi+2\mu))}{1 - \cos(2\mu)}\right\}$$

$$= \frac{(\mu-a)}{2} \left\{\left(1 - \frac{a \sin(2\mu)}{1-\cos(2\mu)}\right)x^2+\left(1 + \frac{a \sin(2\mu)}{1 - \cos(2\mu)}\right) p^2 - 2axp\right\} \tag{B10}$$

From (B10), we deduce the value of the Twiss parameters:

$$\alpha = -a + O(a^2) \ldots \tag{B11}$$

$$\beta = 1 + \frac{a \cos\mu}{\sin\mu} + O(a^2) \ldots$$

$$\gamma = 1 - \frac{a \cos\mu}{\sin\mu} + O(a^2) \ldots \quad .$$

These values agree to first order with (B3). This concludes the treatment of this example.

Table 1

The Lie Algebraic Coefficients and the Linear Matrix

of $\mathscr{L}(OB, P_{yi}) = \exp(:-OB\ k:)\ b = P_{yi},\ a = (1 - P_{yi}^2)^{1/2}$

Marylie Index[20]	$x^i p_x^j y^k p_y^\ell \tau^m p_\tau^n$	Coefficients of $-K_3$ and $-K_4$
51	0 2 0 1 0 0	$-(b/2a^3)$
53	0 2 0 0 0 1	$-1/(2\beta a^3)$
74	0 0 0 3 0 0	$-(b/a^3 + b^3/a^5)/2$
76	0 0 0 2 0 1	$-(1/a^3 + 3b^2/a^5)/(2\beta)$
79	0 0 0 1 0 2	$(b/a^3 - 3b/(\beta^2 a^5))/2$
83	0 0 0 0 0 3	$-(\gamma^{-2} + \beta^2 b^2)/2\beta^3 a^5)$
140	0 4 0 0 0 0	$-1/(8a^3)$
149	0 2 0 2 0 0	$-(1/a^3 + 3b^2/a^5)/4$
151	0 2 0 1 0 1	$-3b/(2\beta a^5)$
154	0 2 0 0 0 2	$(1/a^3 - 3/(\beta^2 a^5))/4$
195	0 0 0 4 0 0	$-(1/(8a^3) + 3b^2/(4a^5) + 5b^4/(8a^7))$
197	0 0 0 3 0 1	$-(3b/a^5 + 5b^3/a^7)/(2\beta)$
200	0 0 0 2 0 2	$(1 + 3b^2/a^2)/(4a^3) - 15b^2/(4\beta^2 a^7) - 3/(4\beta^2 a^5)$
204	0 0 0 1 0 3	$-5b/(2\beta^3 a^7) + 3b/(2\beta a^5)$
209	0 0 0 0 0 4	$(-\beta^4(b^4 + 4b^2) - (5 - \gamma^{-2})\beta^2)/(8\beta^4 a^7)$

$M_{ii} = 1,\ M_{12} = OB/a,\ M_{34} = OB(1/a + b^2/a^3),$

$M_{36} = OB\ b/(\beta a^3),\ M_{56} = OB\ (\gamma^{-2} + \beta^2 b^2)/(\beta^2 a^3),\ M_{54} = OB\ b/(\beta a^3).$

Table 2

The Lie Algebraic Coefficients and the Linear Matrix of

$\mathscr{R}(\psi, P_{yi})$ $b = P_{yi}$, $a = (1 - P_{yi}^2)^{1/2}$, $\mathscr{R} = \mathscr{F}_2\, e^{:f_3:} e^{:f_4:}\, \mathscr{G}_2$.

$A = -b(\tan^5\psi/2 + 3\tan^3\psi/2 + \tan\psi)/(2\beta a^5)$, $B = -(\tan\psi + \tan^3\psi/2)/(2\beta a^3)$.

Marylie Index	$x^i p_x^j y^k p_y^\ell \tau^m p_\tau^n$	Coefficients of f_3 and f_4
34	1 2 0 0 0 0	$-\tan\psi/(2a)$
35	1 1 1 0 0 0	$-b\tan^2\psi/a^2$
38	1 1 0 0 0 1	$-\tan^2\psi/(\beta a^2)$
43	1 0 0 2 0 0	$-\tan\psi/(2a) - b^2(\tan^3\psi + \tan\psi)/(2a^3)$
45	1 0 0 1 0 1	$-b(\tan^3\psi + \tan\psi)/(\beta a^3)$
48	1 0 0 0 0 2	$-(\tan\psi(\gamma^{-2} + \beta^2 b^2) + \tan^3\psi)/(2\beta^2 a^3)$
105	1 3 0 0 0 0	$-\tan^2\psi/(4a^2)$
107	1 2 0 1 0 0	$-(3\tan^3\psi + 2\tan\psi)\, b/(4a^3)$
109	1 2 0 0 0 1	$-(\tan\psi + 3\tan^3\psi/2)/(2\beta a^3)$
114	1 1 0 2 0 0	$-\tan^2\psi/(4a^2) - b^2(3\tan^4\psi + 5\tan^2\psi)/(4a^4)$
116	1 1 0 1 0 1	$-b(3\tan^4\psi + 5\tan^2\psi)/(2\beta a^4)$
119	1 1 0 0 0 2	$\{\tan^2\psi/a^2 - (3\tan^4\psi + 5\tan^2\psi)/(\beta^2 a^4)\}/4$
130	1 0 0 3 0 0	$b^2\beta A + b\beta B$
132	1 0 0 2 0 1	$3bA + B$
135	1 0 0 1 0 2	$3A/\beta - b\beta B$
139	1 0 0 0 0 3	$-(\gamma^{-2} + \beta^2 b^2)\tan\psi/(2\beta^3 a^5)$
		$+ \tan^3\psi(1-3/(\beta^2 a^2))/(4\beta a^3) - \tan^5\psi/(4\beta^3 a^5)$

$F_{ii} = 1$, $F_{24} = -b\tan\psi/a$, $F_{26} = -\tan\psi/(\beta a)$, $F_{31} = b\tan\psi/a$, $F_{51} = \tan\psi/(\beta a)$.

$G_{11} = 1/\cos\psi$, $G_{22} = \cos\psi$, $G_{ii} = 1$ $i=3,6$.

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11. One can tranform a non-linear matix of the TRANSPORT type into a Lie algebraic representation. See for example: E. Forest, D. Douglas and B. Leemann, Study of the Aberrations of a Periodic Arc Using the Lie Algebraic Techniques, SSC-72, January 1986, also to appear in the Proceedings of the Second International Conference on Charged Particle Optics.

12. David Douglas first derived this result for $P_{yi}=0$. Liam Healy recently noticed the intimate connection between $\mathscr{R}$ and the Hamiltonian written in polar (r, θ, z) coordinates. This connection allows for a general derivation of $\mathscr{R}$ in an arbitrary medium. Here, is appendix I, we restrict ourselves to the case of a drift medium which is solvable by means of high school geometry.

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14. This work has been implemented in a version of MARYLIE used at the SSC Central Design Group. Efforts are under way to include these tools in the standard version of the Code MARYLIE. It is also planned to extend these techniques to a 5th order MARYLIE.

15. Neri has recently shown that the symplectic integrators of Ruth are just kick codes with a fancy splitting of the long elements into thin lenses. He rederived the results of Ruth with the help of the Campbell-Baker-Hausdorff theorem and proceeded to compute even higher order integrators. (Private communication). See also: R.D. Ruth, A Canonical Integration Technique IEEE Transactions on Nuclear Science, NS-30, No. 4, August 83, p. 2669.

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18. It is possible to write a Hamiltonian for the process described by (A6) and (A7). (Liam Healy, Private Communication).

19. See reference 16, p. 2623.

20. This index refers to the indexing system used in the program MARYLIE.

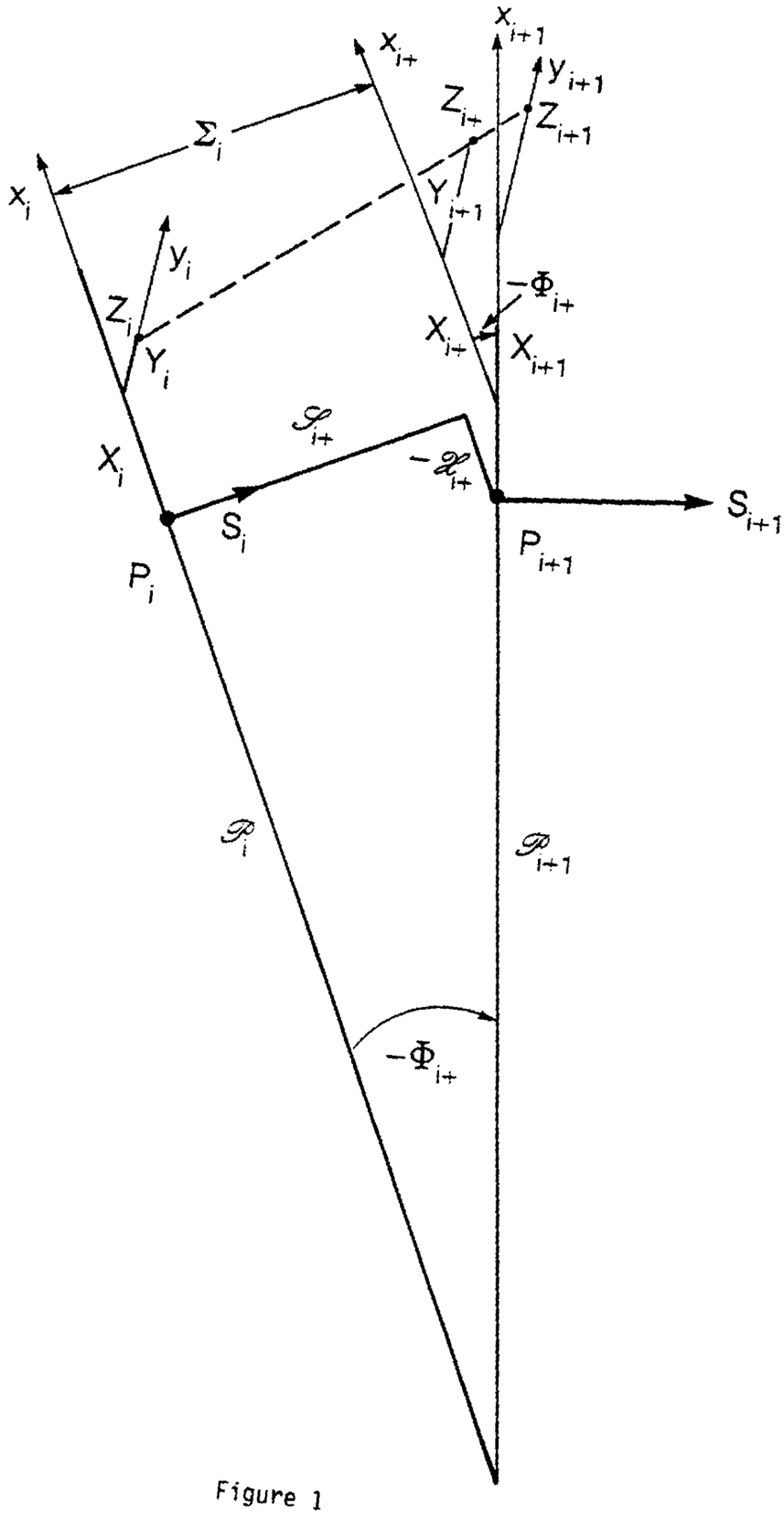
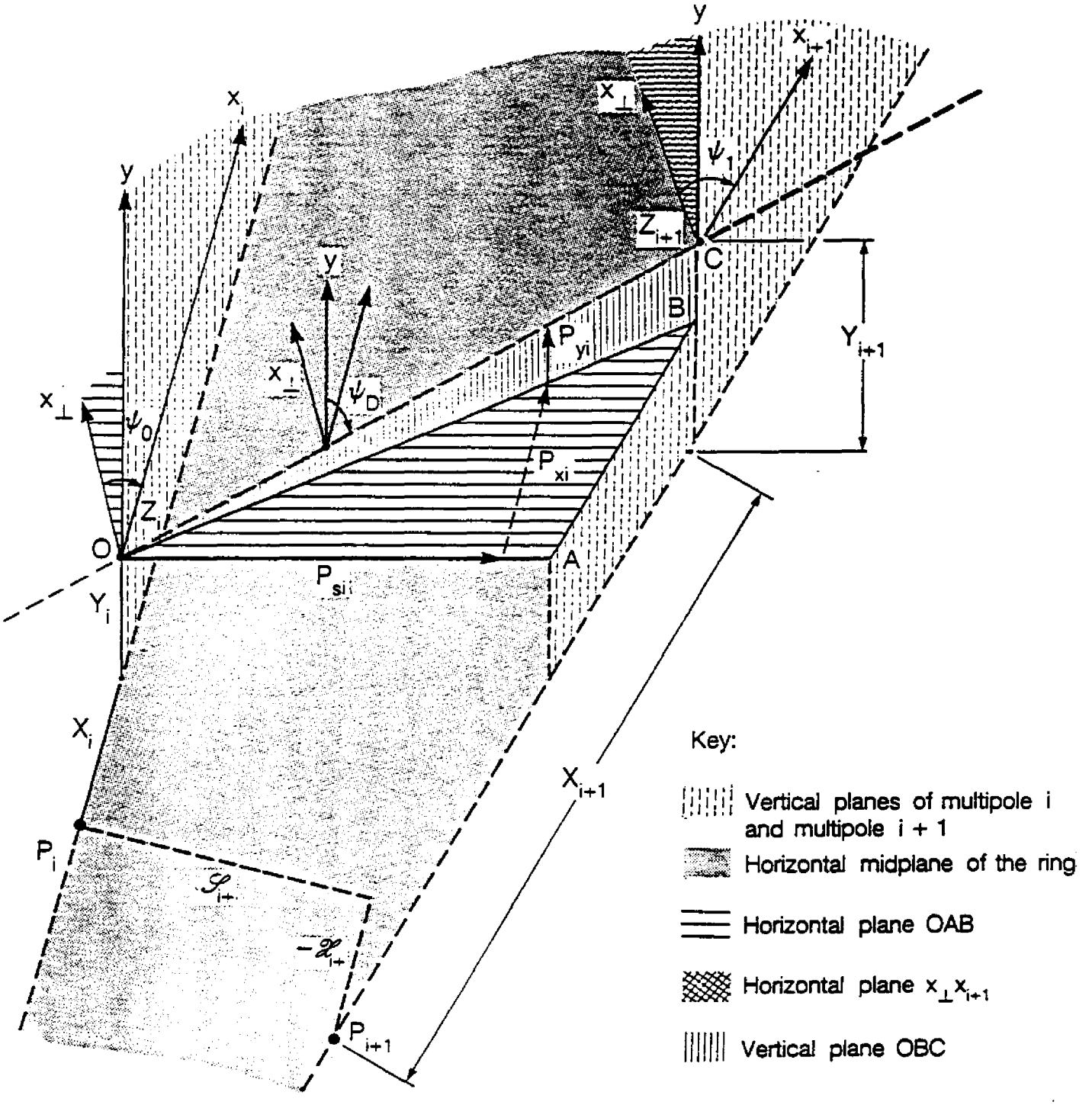

x_{i+1}
x_{i+}
y_{i+1}
Z_{i+}
Z_{i+1}
Σ_i
x_i
Y_{i+1}
y_i
$-\Phi_{i+}$
Z_i
X_{i+}
X_{i+1}
Y_i
$\mathscr{S}_{i+}$
X_i
$-\mathscr{X}_{i+}$
S_{i+1}
S_i
P_{i+1}
P_i
$\mathscr{P}_i$
$\mathscr{P}_{i+1}$
$-\Phi_{i+}$

Figure 1



y
x_{i+1}
$x_\perp$
ψ_1
Z_{i+1}
C
B
Y_{i+1}
P_{yi}
P_{xi}
x_i
y
$x_\perp$
ψ_0
Z_i
O
A
P_{si}
Y_i
X_i
X_{i+1}
P_i
$\mathscr{S}_{i+}$
$-\mathscr{X}_{i+}$
P_{i+1}
Key:
Vertical planes of multipole i and multipole i + 1
Horizontal midplane of the ring
Horizontal plane OAB
Horizontal plane $x_\perp x_{i+1}$
Vertical plane OBC

Figure 2

Figure Captions

1. The geometry for straight line propagation through a drift space in the program TEAPOT.

2. The geometry involved in the computation of the non-linear map around an arbitrary trajectory in the program TEAPOT.

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