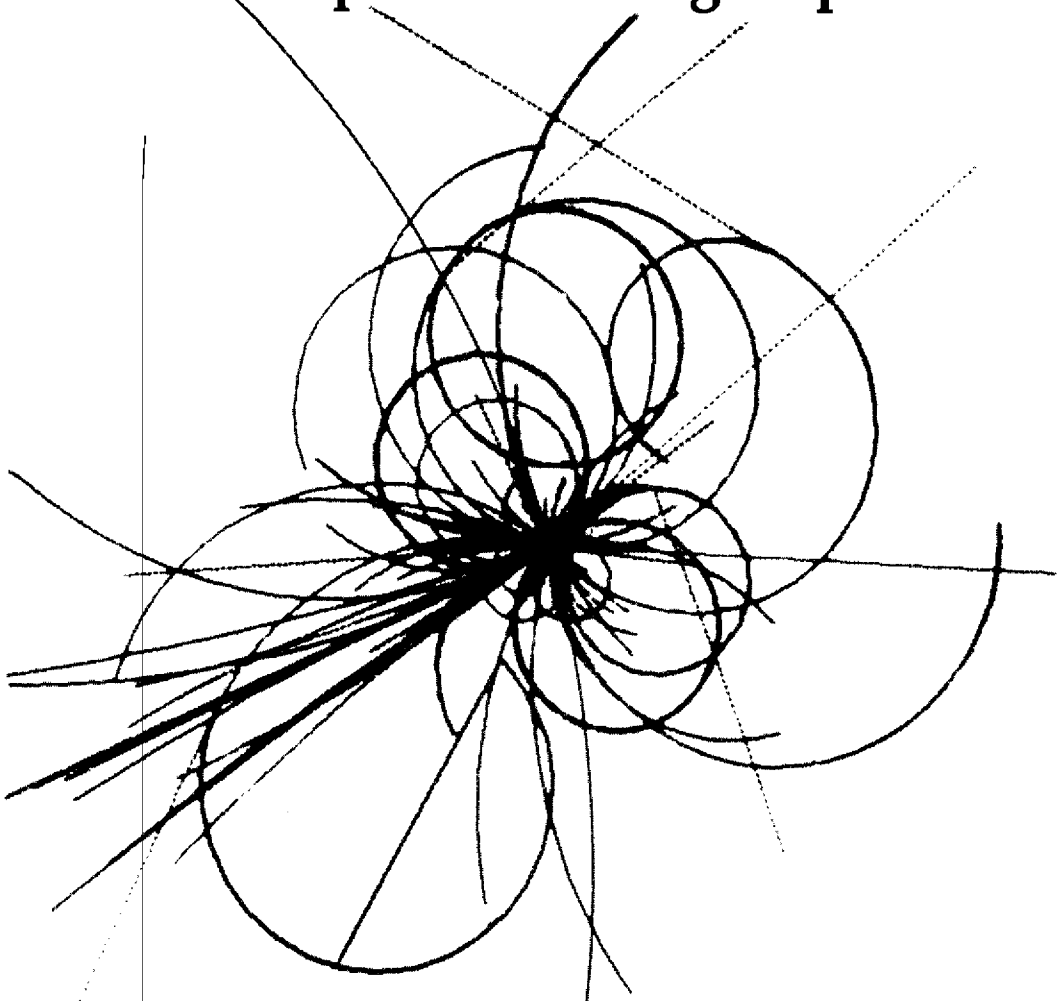
SSC-67

The Superconducting Super Collider



A Theory of the Coherent Beam Beam Effect With Long Range Interactions

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Abstract

We present an extension of the Chao-Ruth treatment of the Vlasov equation which includes both the head on and the long-range collisions. We also simplify the multiple bunches treatment by following the time evolution of a site instead of a bunch. Analytical results are obtained for the dipole mode and some numerical results for the higher modes.

1. INTRODUCTION

Beam-beam collisions are an important consideration in the operation of storage rings. In the contemplated Superconducting Super Collider (SSC), the finite crossing angle α gives rise to long range interactions. The inclusion of the long range forces introduces the possibility of communication between bunches of particles.

In this paper, we exploit the symmetries of this complicated problem. It will be shown that dipole oscillations where the bunches act as macroparticles can be solved analytically for distributed lattices. For higher modes, the Vlasov techniques introduced by Chao and Ruth [1] are generalized to the long range problem. Unfortunately, like them, we did not analyze the possible effect of tune spread. It is reasonable to believe that tune spread will greatly raise the stability of the system.

All the equations of this work are the result of a linearization of the Vlasov equation. This implies that the collisions can be represented by linear operators (i.e., matrices). To keep the language very general, we will delay the introduction of actual expressions for the collision operators. We will start with a group theoritical view of a periodic lattice undergoing beam-beam collisions.

2. The fundamental symmetry

Consider a ring with N interaction sites uniformly distributed. We also assume that the tune between sites is the same all around the ring. Furthermore we assume that the beams contain mN bunches evenly distributed. Figure 1 shows the case N = 3 and m = 4.

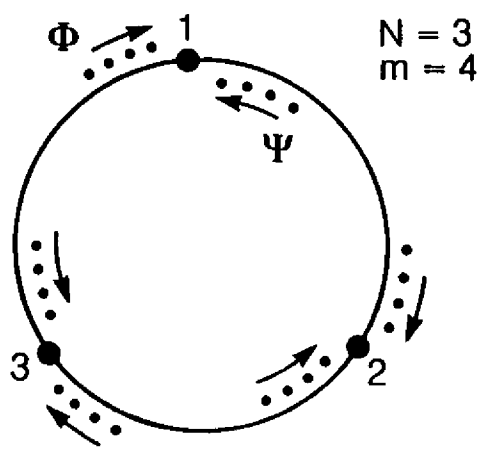

Fig. 1. The schematic representation of a ring with 3 collision site (N=3) and 4 bunches per site and per beam (m=4).

The vectors Φ and Ψ are supposed to contain the physical information of each beam. Formally, we can write an equation for the process of colliding and moving single particles to the next collision site:

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}^{+} = \begin{pmatrix} A & A' \\ B & B' \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}^{-} \tag{2.1}$$

Our operator should be invariant under the exchange of Φ and Ψ. From this we conclude that it has the form:

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}^{+} = IA \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}^{-} + \Delta B \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}^{-} \tag{2.2a}$$

$$\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2b}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{2.2c}$$

The operators A and B act on Φ and Ψ individually. The operator Δ mixes the two beams.

At this moment, it seems that nothing more can be added. However, the N-fold symmetry of this problem can be made apparent if we adopt a point of view used in hydrodynamic calculations. We will make use of the Eulerian view. Instead of labeling the bunches and follow them around the ring, we will label the sites. The vectors Φ and Ψ are decomposed in N smaller vectors representing the state of each site:

$$\Gamma = (\Phi,\Psi) = (\varphi',\psi',\ldots,\varphi^{N},\psi^{N}) \tag{2.3a}$$

$$\gamma^{\kappa} = (\varphi^{\kappa},\psi^{\kappa}). \tag{2.3b}$$

We define the relocation operator Ω. Since the superscripts in (2.3) denote the sites, after each collisions the entries of Γ must be relocated. This can be achieved by Ω:

$$\Omega\Gamma = (\varphi^{N},\ \psi^{2},\ \varphi^{1},\ \psi^{3},\ \ldots,\varphi^{N-1},\ \psi^{1}) \tag{2.4}$$

Finally, we write the collision and the relabeling in terms of the operators: A, B, Δ and Ω.

$$\Gamma^1 = \Omega\, A\Gamma^\circ + \Omega\Delta\, B\Gamma^\circ \ . \tag{2.5}$$

The superscript of Γ is incremented after each collision and relocation. For example, one turn of machine or N collisions is given by:

$$\Gamma^N = (\Omega\, A + \Omega\Delta B)^N\ \Gamma^\circ \ . \tag{2.6}$$

Keeping in mind that A and B commute with Ω and Δ, we see that the study of one turn is related to the study of the products of Ω and Δ. By adopting the Eulerian viewpoint, one does not need to compute the operator for one turn but only for $1/N^{th}$ of a turn. Indeed, the $1/N^{th}$ of a turn operator $\Omega A + \Omega\Delta B$ will determine the stability of the motion.

In addition, it is remarkable that the dimensionality of this problem can be reduced by a factor of N. The products of Ω and Δ form a group of finite order. More details are given in Appendix I. The only important result is given by the following theorem.

Theorem:

There exists a basis for Γ where all the operators in (2.6) decompose simultaneously into blocks of the dimensionality of γ^κ. Hence, in these subspaces, labeled by μ, the operator for the process has the form:

$$C_\mu = \omega_\mu A + \omega_\mu \delta_\mu B \tag{2.7a}$$

$$\omega_\mu = \begin{pmatrix} \cos(\frac{2\pi}{N}\mu) & \sin(\frac{2\pi}{N}\mu) \\ -\sin(\frac{2\pi}{N}\mu) & \cos(\frac{2\pi}{N}\mu) \end{pmatrix} \tag{2.7b}$$

$$\delta_\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{array}{ll} 0 \le \mu \le \frac{N}{2} & \text{N even} \\ 0 \le \mu \le \frac{N-1}{2} & \text{N odd.} \end{array} \tag{2.7c}$$

This result can be derived from group theoritical considerations[2](Appendix I) or by discrete transforms (Appendix II).

We have introduced very little physics at this stage. All the physical content of the collisions is contained in the A and B operators. In the case of long range collisions, we will discover that some simplifying assumptions can further reduce the dimensionality. The problem becomes solvable analytically for the dipole mode and numerically for the higher modes.

III. The study of dipole oscillations.

Let us suppose that each bunch is represented by a position x and a momentum x'. The counter-clockwise bunches will be represented by y and y'. In the simplest model,[3,4] the head-on collision is of the form:

$$\begin{aligned} x'_+ &= x' + \epsilon(x-y). \\ y'_+ &= y' + \epsilon(y-x) \ . \end{aligned} \tag{3.1}$$

The long range collisions, at a distance s from the head-on interaction site have the form:

$$\begin{aligned} x'_+ &= x' + \frac{\epsilon p}{s^2}(x-y) \\ y'_+ &= y' + \frac{\epsilon p}{s^2}(y-x), \end{aligned} \tag{3.2}$$

where p measures the strength of the long range interaction.

As we said before, each site has m bunches in each beam. Furthermore, we assume that J long range collisions are allowed to occur before the separation of the beams takes place. The vector γ^{μ} representing the μth mode of the ring has 2 m components:

$$\gamma^{\mu} = (\varphi_1^{\mu}, \psi_1^{\mu}, \ldots, \varphi_m^{\mu}, \psi_m^{\mu}) \tag{3.3a}$$

$$\varphi_i^{\mu} = (x_i^{\mu}, x_i^{\mu'}) \tag{3.3b}$$

$$\psi_i^{\mu} = (y_i^{\mu}, y_i^{\mu'}) \ . \tag{3.3c}$$

The bunches are shown in Fig. 2.

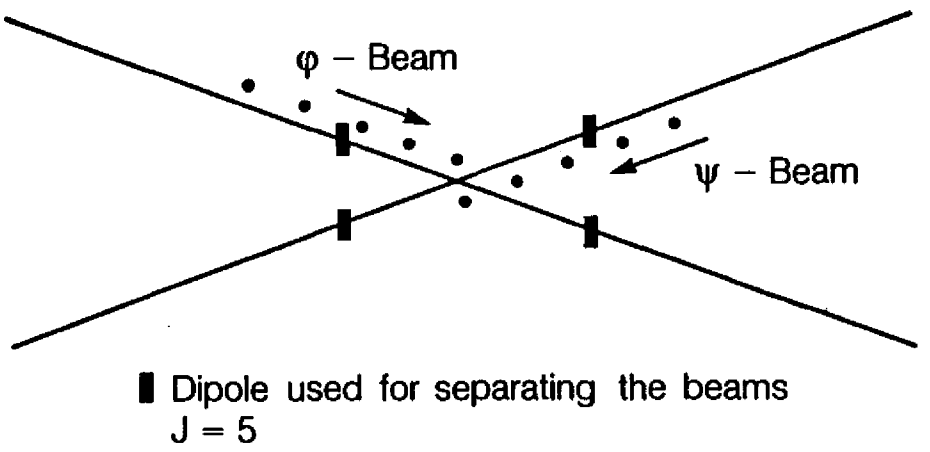

Fig. 2. Configuration of an interaction region.

The long-range collisions will depend on the single particle transport map between the long-range sites and the head-on site. It is shown in Appendix III that the collision operator between the 2 m bunches can be expressed as follows if we neglect cross-terms:

$$\Upsilon_+ = \mathcal{M}\Upsilon \tag{3.4a}$$

$$= \exp\,(:H:) \tag{3.4b}$$

$$H = \sum_{j=1}^{m} H_j + \sum_{\substack{\ell=-J \\ \ell\neq 0}}^{J} \sum_{j=1}^{m} H_{jj+\ell} \tag{3.4e}$$

$$H_j = \frac{\varepsilon}{2}\,(x_j - y_j)^2 \tag{3.4e}$$

$$H_{jj+\ell} = \frac{2\varepsilon p}{L^2\ell^2}\,(\rho_\ell\, x_j - \rho_{-\ell}\, y_{j+\ell})^2 \tag{3.4f}$$

Here, :H: is the Lie operator[5] associated to H and ρ_ℓ is the single particle map which transports a particle from the interaction point at s=0 to the long range site at $s=\frac{\ell L}{2}$. It is important to realize that the Lie operators in Equation (3.4) are all expressed in terms of their values at s=0, before the collisions take place. This is done to simplify the analysis. It should be noted that we added some collisions which do not occur. For example, the first bunch φ_1 interacts with ψ_m through the first long-range interaction. This renders the system translationally invariant under the raising of the index j. If m is large this approximation is not very drastic. Indeed it is similar to the Von Karman boundary condition of crystallography. We neglect boundary effects.

The analytical solvability of the problem will depend on ρ_ℓ. There are two models for which the computation is done analytically. The first one is the pseudo-drift. Let us assume that ρ_ℓ is a drift:

$$\rho_\ell x = x + \frac{\ell L}{2}\, x' \tag{3.5a}$$

$$\rho_\ell\, x' = x' \,. \tag{3.5b}$$

Assuming that the beta function β_* at s=0 is smaller than $\frac{\ell L}{2}$, then the second term in Equation (3.5a) is larger than the first one. Using this we replace ρ_ℓ by the pseudo-drift σ_ℓ:

$$\sigma_\ell x = \frac{\ell L}{2} x' \tag{3.6a}$$

$$\sigma x'_\ell = -\frac{2}{\ell L} x \quad . \tag{3.6b}$$

In the second and simplest model, we abandon the exact knowledge of the actual map. We only retain the change in β function resulting from the drift. Again, if $\frac{\ell L}{2}$ is large compared to β_*, we replace ρ_ℓ by the telescopic map θ_ℓ:

$$\theta_\ell x = \sqrt{\beta_\ell}\ \frac{1}{\sqrt{\beta_*}}\ x \tag{3.7a}$$

$$\theta_\ell x' = \sqrt{\beta_*}\ \frac{1}{\sqrt{\beta_\ell}} x' \quad , \text{ with } \beta_\ell = \beta_{-\ell} \ . \tag{3.7b}$$

It turns out, as we will see, that only the details of the stability limit depend on the actual model used for ρ.

For σ_ℓ and θ_ℓ it is possible to evaluate equation (3.4a) in a way which preserves the symplectic nature of $\mathcal{M}$. Because of its simplicity, we will first examine the case of θ_ℓ. For this approximation, equation (3.4a) takes the form:

$$x_{j+} = x_j$$

$$x'_{j+} = x'_j + \epsilon(x_j - y_j) + \frac{\epsilon p}{\beta_*} \sum_{\ell=1}^{J} (2x_j - y_{j+\ell} - y_{j-\ell}) \frac{4\beta_\ell}{L^2 \ell^2} \tag{3.8}$$

The parameter p can be estimated in terms of a gaussian[3,4] model for the bunch.

$$p = \frac{-2\ \sigma_*^2}{\alpha^2}\ ,\ \sigma_* = \text{size of beam.} \tag{3.9}$$

$$\alpha = \text{crossing angle.}$$

Due to the invariance in j of Equation (3.8) it is desirable to canonically transform φ_j and ψ_j:

$$\varphi_\lambda = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} f(i\frac{2\pi}{m}\lambda j)\ \varphi_j \tag{3.10a}$$

$$f(x) = \cos(x) + \sin(x) \tag{3.10b}$$

Under the effect of (3.10), (3.8) becomes:

$$x_{\lambda+} = x_\lambda$$

$$x'_{\lambda+} = x'_\lambda + \epsilon(x_\lambda - y_\lambda) + \frac{2\epsilon p}{\beta_*^2}(q(0)x_\lambda - q(\lambda)y_\lambda) \tag{3.11a}$$

$$q(\lambda) = \sum_{j=1}^{J} \cos(\frac{2\pi}{m}j\lambda)\ \frac{4\beta_j\ \beta_*}{L^2 j^2} \tag{3.11b}$$

Finally, we introduce the phase advance between interaction points. Since all the φ_j and ψ_j are variables at the interaction point, they transformed identically. The same can be said of the decoupled variables φ_λ and ψ_λ. The phase advance matrix $R(\nu_0)$ is just:

$$R(\nu_0) = \begin{pmatrix} \cos(2\pi\nu_0) & \beta_*\sin(2\pi\nu_0) \\ \dfrac{-\sin(2\pi\nu_0)}{\beta_*} & \cos(2\pi\nu_0) \end{pmatrix}. \tag{3.12}$$

We are now in a position to derive the operators A_λ and B_λ of Equation (2.7).

$$A_\lambda = R(v_0) \begin{pmatrix} 1 & 0 \\ \epsilon(1+\frac{2p}{\beta_*^2}q(0)) & 1 \end{pmatrix} \tag{3.13a}$$

$$B_\lambda = R(v_0) \begin{pmatrix} 0 & 0 \\ -\epsilon(1+\frac{2p}{\beta_*^2}q(\lambda)) & 0 \end{pmatrix} \tag{3.13b}$$

These operators fit into Equation (2.7):

$$C^\lambda_\mu = \omega_\mu(A_\lambda + \delta_\mu B_\lambda) \ , \tag{3.14}$$

The operator C^λ_μ is in general a 4x4 matrix acting on $\Gamma^\mu_\lambda = (\varphi^\mu_\lambda, \psi^\mu_\lambda)$. Again we remind the reader that ω_μ and δ_μ mix φ and ψ, while A and B act separately on each two-dimensional vector.

The case of the pseudo-drift is slightly more difficult. There we factor the head-on map from the long range kick.[6] To first order, this is permitted.

$$M = M_{LR} M_0 \tag{3.15a}$$

$$M_0 = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \tag{3.15b}$$

$$M_{LR}(\lambda) = \begin{pmatrix} A_{LR} & B_{LR} \\ B_{LR} & A_{LR} \end{pmatrix} \tag{3.15c}$$

$$M = \begin{pmatrix} a_\lambda & b_\lambda \\ b_\lambda & a_\lambda \end{pmatrix}$$

$$a_\lambda = A_{LR}A_0 + B_{LR}B_0$$

$$b_\lambda = A_{LR}B_0 + B_{LR}A_0 \tag{3.15d}$$

The matrices A_0 and B_0 are obtainable from the previous calculations by setting p to zero:

$$A_0 = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} , \qquad B_0 = \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix} . \tag{3.16}$$

The action of the long range kick can be exactly evaluated:

$$x_{j+} = x_j - \varepsilon p(2J\, x'_j + \sum_{\ell=1}^{J} (y'_{j+\ell} + y'_{j-\ell})) \tag{3.17}$$

$$x'_{j+} = x'_j$$

Again we transform to a new set of canonical variables using Equation (3.10):

$$x_{\lambda+} = x_\lambda - 2\, \varepsilon p\, Jx'_\lambda - 2\, \varepsilon p\, q(\lambda)\, y'_\lambda$$

$$x'_{\lambda+} = x'_\lambda \ , \ q(\lambda) = \sum_{j=1}^{J} \cos(\frac{2\pi}{m}\, j\lambda) \ . \tag{3.18}$$

From (3.18), we deduce the expression for A_{LR} and B_{LR}:

$$A_{LR} = \begin{pmatrix} 1 & -2\varepsilon pq(0) \\ 0 & 1 \end{pmatrix} \tag{3.19a}$$

$$B_{LR} = \begin{pmatrix} 0 & -2\varepsilon pq(\lambda) \\ 0 & 0 \end{pmatrix} \ . \tag{3.19b}$$

Finally A_λ and B_λ of equation (3.14) are obtained by advancing the phase using $R(\nu_0)$:

$$A_\lambda = R(\nu_0)a_\lambda$$

$$B_\lambda = R(\nu_0)b_\lambda \tag{3.20}$$

In the next section, we evaluate the eigenvalues of C_μ^λ for the two models and determine the stability diagram.

IV. The stability for the two models.

For each model we must solve the equation:

$$\det\,(C_\mu^\lambda - \Lambda I) = 0 \tag{4.1}$$

The matrix C_μ^λ is a 4 x 4 symplectic matrix. Some standard tricks[7] can be used to reduce (4.1) into a quadratic equation in Λ. We assume that the most unstable modes will reach their threshold when Λ becomes ± 1.[8]

In the case of the telescopic model, one can show that Equation (4.1) implies that Λ reaches ± 1 when the quantity $\epsilon\beta_*$ obeys the equation:

$$8(C\mp K)^2 + 8(1+D)\,S(C\mp\kappa)\,\epsilon\beta_* + 2(D^2 - Q^2 + 2(D-Q))\,S^2(\epsilon\beta_*)^2 = 0 \tag{4.2a}$$

$$C = \cos(2\pi\nu_0),\ S = \sin(2\pi\nu_0),\ \kappa = \cos(\tfrac{2\pi}{N}\mu), \tag{4.2b}$$

$$Q\,(\lambda) = \frac{2D}{\beta_*^2}q(\lambda),\ D = Q(0)\ .$$

In terms of the usual tune shift parameter ξ, (4.2a) can be solved:

$$\xi_\pm^{[1]} = \frac{2}{2+D+Q(\lambda)}\,\xi_\pm \tag{4.3a}$$

$$\xi_\pm^{[2]} = \frac{2}{D-Q(\lambda)}\,\xi_\pm \tag{4.3b}$$

$$\xi_\pm = \frac{-1}{4\pi}\,\frac{C\mp\kappa}{S} \tag{4.3c}$$

$$\xi = \frac{\epsilon\beta_*}{4\pi}\ . \tag{4.3d}$$

The quantities D and Q vanish in the absence of long range collisions. In that case, $\xi_{\pm}$ given by (4.3c) determines the stability of the system. Figure 3 shows the plots of $\xi_{\pm}$ and the resulting stable region for positive ξ and N=3.

In the presence of long range collisions, the most unstable modes determine the boundary of the stable region. To get them we need to find the extrema of $Q(\lambda)$ under the variations of the quasi-continuous parameter λ: $(1 \leq \lambda \leq m;\ m >> 1)$. In Figure 4 and 5, we show the head on and long range dominated regimes. Indeed, if D+Q reaches for some λ the value -2; the system is dominated by long range collisions. Notice that this model predicts that the stability curve is obtained by multiplying $\xi_{\pm}$ by a factor independent of v_0. This is not really true if drifts separate the long range sites. The pseudo-drift model corrects this qualitative discrepancy of the telescopic model.

The pseudo-drift produces a cubic in $\epsilon\beta_*$ for the stability limit:

$$8(C\mp\kappa)^2 + 8(1+D)\ (C\mp\kappa)S\ \ \epsilon\beta_*$$

$$+ \{\pm 8C\kappa(D-Q) + 4S^2(D-Q) + 2S^2(D^2-Q^2) + 8\kappa^2 Q - 8C^2 D\}\ (\epsilon\beta_*)^2$$

$$+ 4CS(Q^2-D^2)\ (\epsilon\beta_*)^3 = 0\ . \tag{4.4a}$$

$$Q = \frac{2p}{\beta_*^2}\, q(\lambda),\ D = Q(0)\ . \tag{4.4b}$$

Neglecting terms quadratic in D or Q, we can approximate the solution for weak long range:

$$\xi_{\pm}^{[1]} = r_a \xi_{\pm} \tag{4.5a}$$

$$\xi_{\pm}^{[2]} = r_b \xi_{\pm} \tag{4.5b}$$

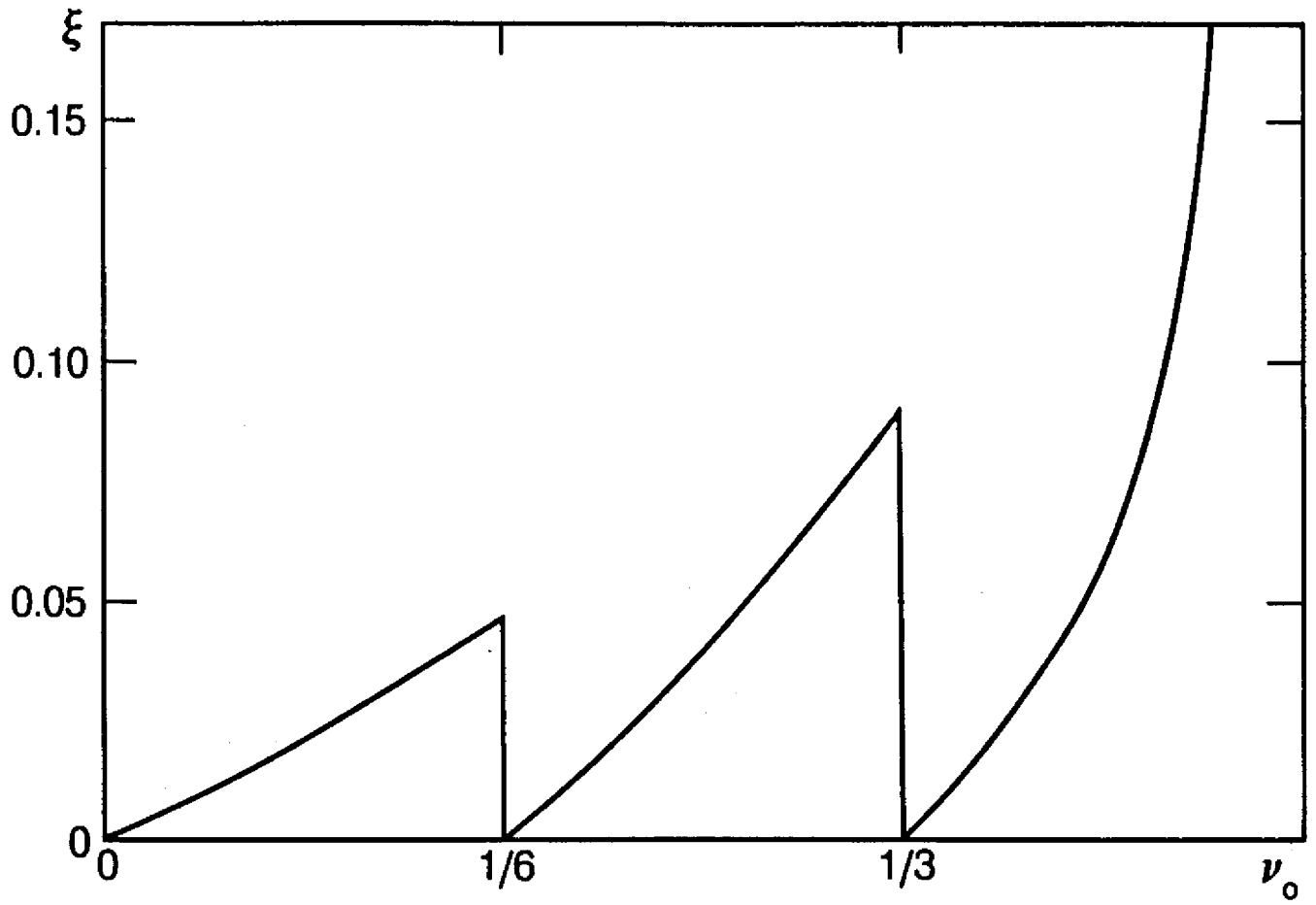

Fig. 3. The graph of equation (4.3c) for N=3.

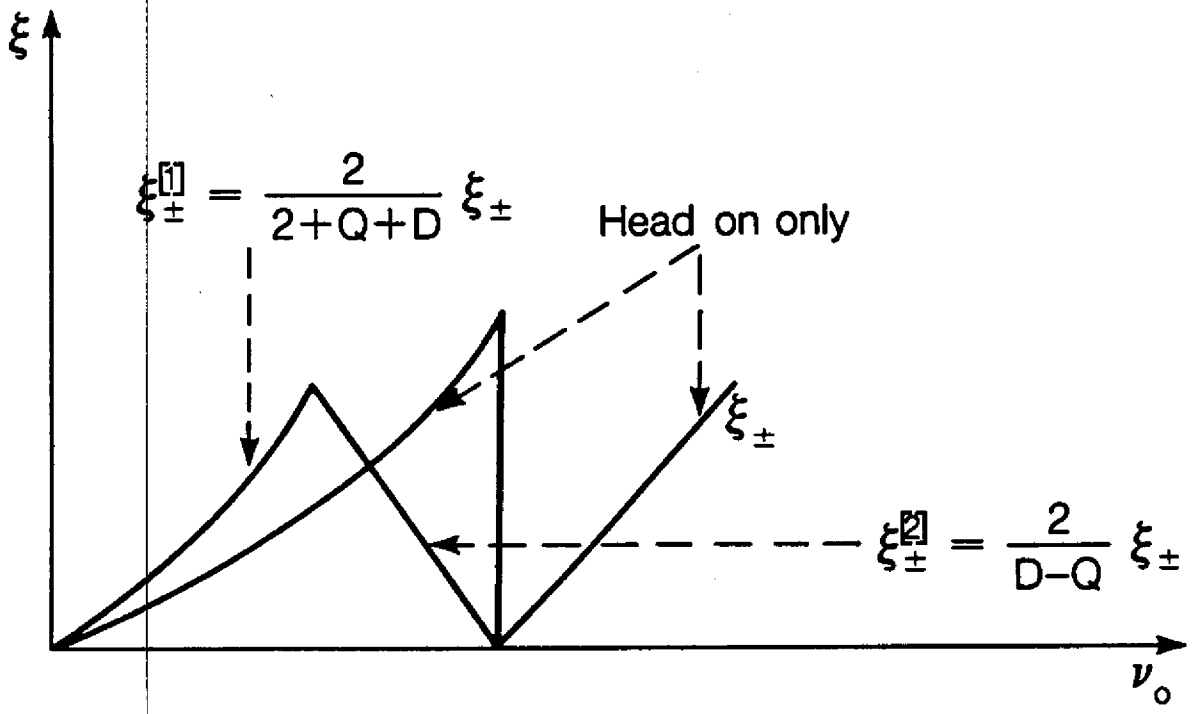

Fig. 4. The graph of (4.3a), (4.3b) and (4.3c) are shown in a weak long range regime for the telescopic model.

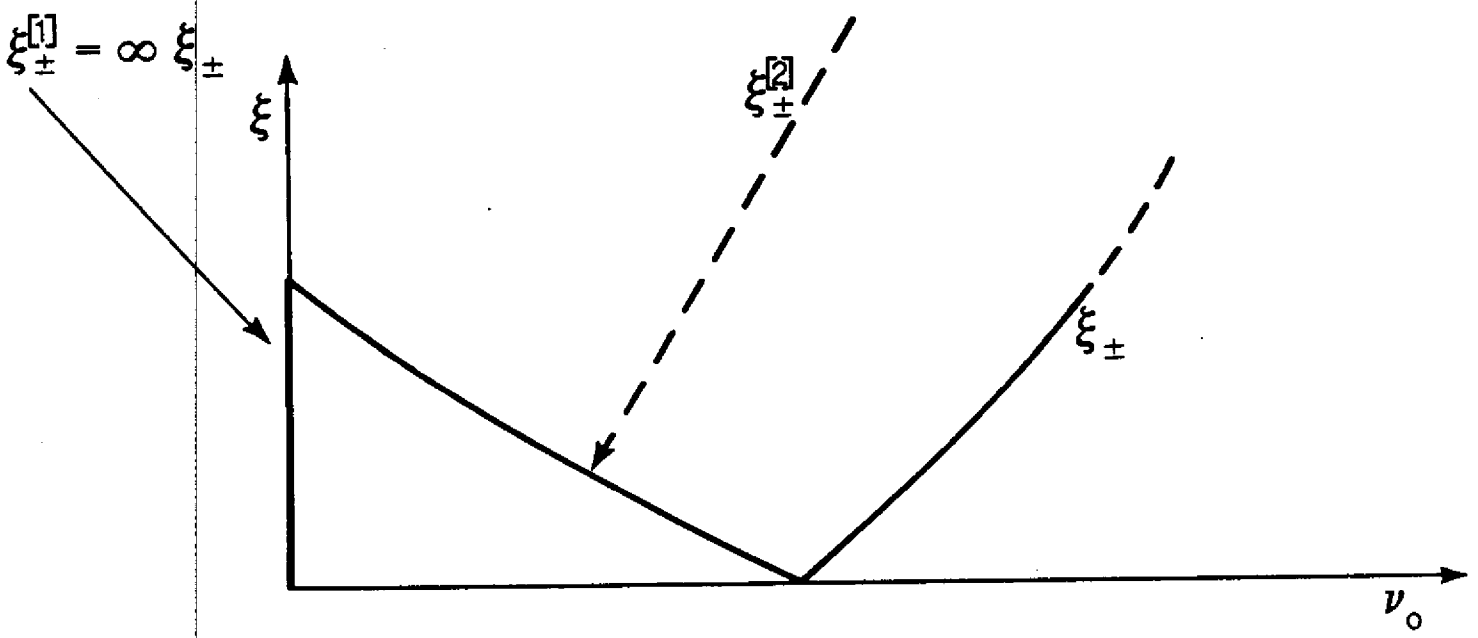

Fig. 5. The graph of (4.3a) and (4.3b) are shown in a long range dominated system.

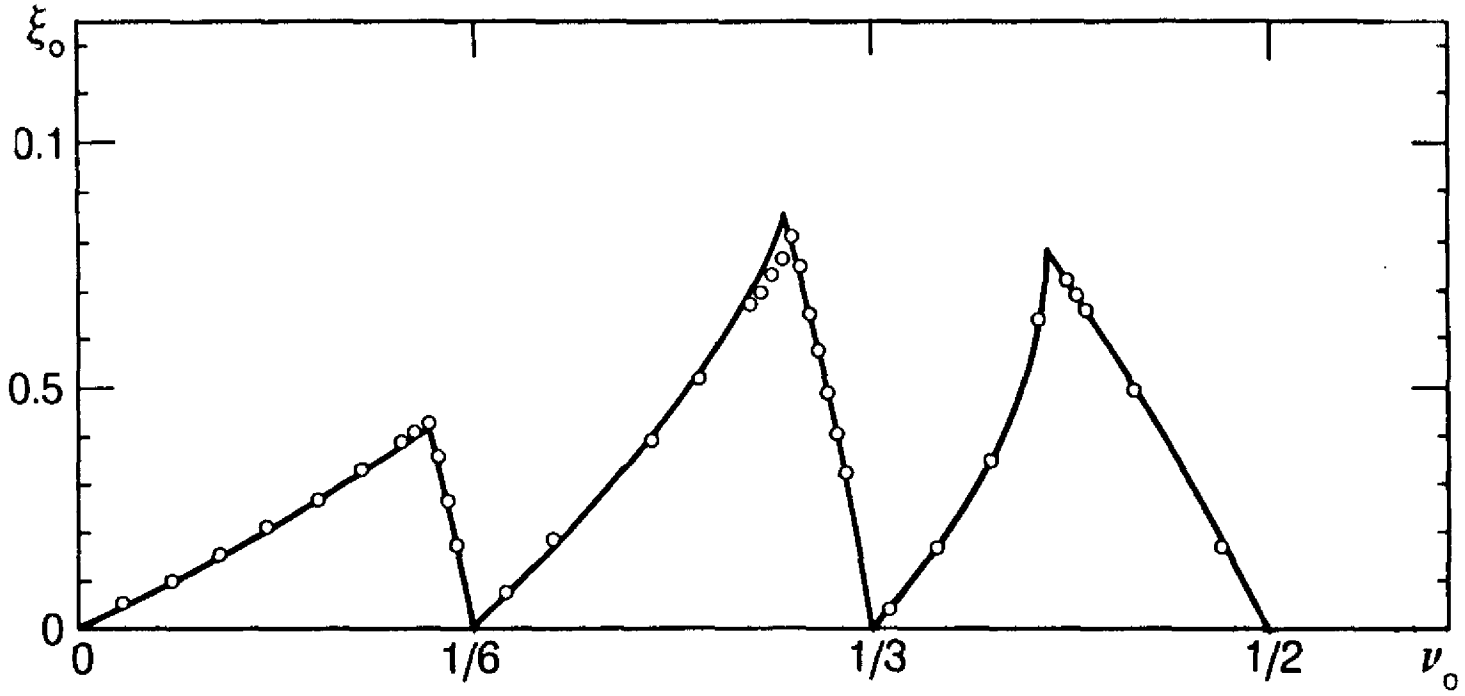

Fig. 6. The solid curves are the prediction of the pseudo-drift model for N=3, α=50μrad and J=4. The circles are the results of M.A. Furman's simulation where N=3, m=5, J=4 and α=50μrad.

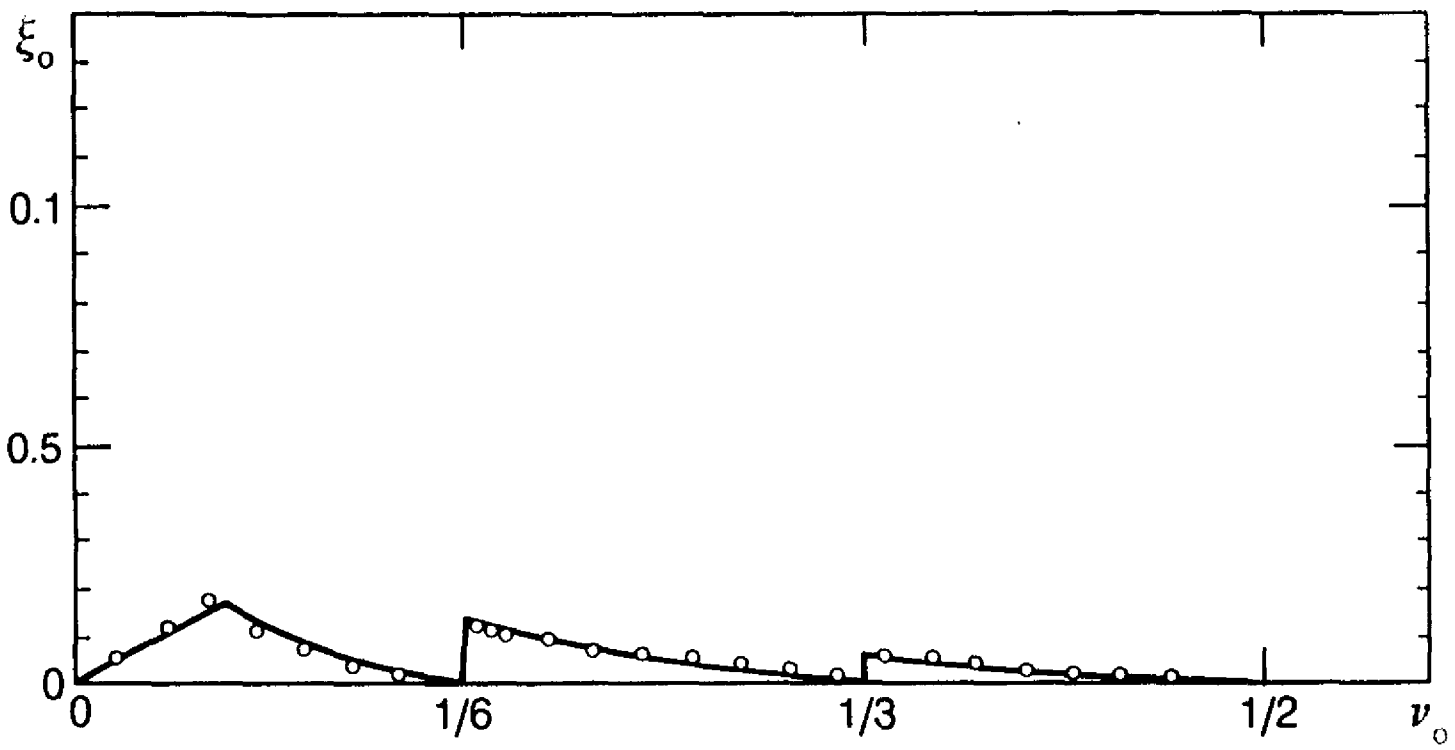

Fig. 7. Same configuration as Fig. 6, but a smaller crossing angle is used, α=10μrad. Again we have a good agreement between the theory and Furman's simulation.

$$r_a = 1-D+\{(D-Q)(2S^2\pm 4C\kappa) + 4\kappa^2 Q - 4C^2 D\}\frac{1}{4S^2} \ldots \tag{4.5c}$$

$$r_b = (r_a+D-1)^{-1} \tag{4.5d}$$

The coefficient r_a modifies the existing head on limit while r_b becomes infinite in the absence of long range collisions. Again the stability limit is determined by the values of λ which extremize the function $q(\lambda)$. By analyzing r_a around the point obeying $C\mp\kappa = 0$, we get the result:

$$0 \le \nu_0 \le \frac{1}{N} \qquad r_a = r_a(q=d) = 1 \tag{4.6a}$$

$$\frac{1}{N} \le \nu_0 \le 0.5 \qquad r_a = r_a(q=q_{min}) \tag{4.6b}$$

$$0 \le \nu_0 \le 0.5 - \frac{1}{N} \qquad r_b = r_b(q=q_{min}) \tag{4.6c}$$

$$0.5 - \frac{1}{N} \le \nu \le 0.5 \qquad r_b = r_b(q=d) = \frac{1}{D}. \tag{4.6d}$$

Remarkably, the results of (4.6) are exact even if the full cubic equation is considered.

In the limit of large long range, equation (4.4) turns into a quadratic. For this equation, we find that r_a in (4.6b) becomes infinite and r_b turns into 1/D. The results are shown in Figures 6 and 7. The circles are the result of an actual simulation with real drifts in between long range sites.[4] The agreement is quite remarkable. In the next sections, we will look at the inclusion of higher modes using the Vlasov equation.

5. The Long Range Vlasov Equation.

In the rest of this paper, we will present a straight forward extension of a model proposed by Chao and Ruth. In their model, the head on kick between

two bunches represented by the distribution functions $\varphi(x,x';s)$ and $\psi(y,y';s)$ is:

$$\Delta x' = \frac{2\pi Nr}{L_x\gamma} \int \psi H_1(x-y)\, dydy' \ . \tag{5.1}$$

Here r is the classical radius of the particles, L_x is the horizontal width of the beams and γ is the Lorentz factor. Finally $H_1(x-y)$ is a step function:

$$H_1(x) = \begin{matrix} -1 \\ 1 \end{matrix} \quad \begin{cases} x < 0 \\ x > 0 \end{cases} \tag{5.2}$$

Assuming that the interaction takes place at s=0, the Vlasov equation[9] becomes:

$$\partial_s\varphi + x'\partial_x\varphi - \partial_{x'}\varphi\ \{K(s)x - A\delta(s) \int H_1(x-y)\ \psi\ dydy'\} = 0$$

$$A = \frac{2\pi Nr}{L_x\gamma}\ ,\ \partial_Z = \frac{\partial}{\partial Z}\ ,\ K(s) = \text{lattice gradient.} \tag{5.3}$$

A similar equation can be written for ψ.

The kick (5.1) assumes that two ultra-relativistic ribbons are colliding. For the long range, we assume that two localized ultra-relativistic beams pass by each other. The long range kick is given by:[10]

$$\Delta x' = \int \psi(y,y';s)\ \chi_s(x-y)\ dydy' \tag{5.4a}$$

$$\chi_s = \frac{AL_x}{\pi}\left(\frac{1}{\alpha s+x-y} - \frac{1}{\alpha s}\right) \tag{5.4b}$$

α = crossing angle; s = path length from the head on site.

As we did previously, we will concentrate on an individual site (see Figure 2). For this system, with m bunches in each beam and J long range sites on each side of the head on, the Vlasov equation becomes:

$$\partial_s\varphi_i + x'\ \partial_x\varphi_i - \partial_{x'}\varphi_i\{Kx - \sum_{\ell=-J}^{J} \delta(s-\tfrac{\ell L}{2}) \int \chi_s \Psi_{i+\ell}(y,y';\ -s)\ dydy'\} = 0$$

$$\text{where } \chi_0 = AH_1(x-y)\ . \tag{5.5}$$

Notice that all the approximations used in the dipole mode analysis are used once more. At we did before, it is possible to rewrite (5.5) in terms of a local equation at s=0 provided that cross-terms are neglected. It is shown in Appendix V that the resulting equation is:

$$\partial_s\varphi_i + x'\partial_x\varphi_i - \partial_{x'}\varphi_i\, Kx + \delta(s)\, \partial_Z\varphi_i \cdot \sum_{\ell=-J}^{J} \partial_{x'}(\rho_\ell^{-1}\, Z)$$

$$\int \chi_s(\rho_\ell x - \rho_{-\ell} y)\psi_{i+\ell}(y,y';s)dydy' = 0 \qquad (5.6)$$

$$Z = (x,x').$$

Again this very complex equation is solvable under Chao and Ruth's approximations for the telescopic and the pseudo-drift models. In this paper. we will only look at the pseudo-drift. It is a more complex case than the telescopic map.

For the pseudo-drift, we replace ρ_ℓ by σ_ℓ and equation (5.6) becomes:

$$\partial_s\varphi_i + x'\partial_x\varphi_i - \partial_{x'}\, \varphi_i\, Kx + \delta(s)\partial_x\varphi_i \sum_{\ell=1}^{J} G(x'+y')[\psi_{i+\ell}+\psi_{i-\ell}]dydy' = 0$$

$$G(x'+y') = \frac{AL_x}{\pi\alpha} \sum_{n=1}^{\infty} (-\frac{1}{\alpha})^n\, (x'+y')^n \quad . \qquad (5.7)$$

The next step is to linearize (5.7) around an equilibrium distribution f. We must have:

$$\varphi_i = f + \Delta\varphi_i,\ \psi_i = f + \Delta\psi_i. \qquad (5.8)$$

Substitution of (5.8) into (5.7) produces two equations:

$$\partial_s f + [f,\ H_o + V + W] = 0 \qquad (5.9a)$$

$$\partial_s\Delta\varphi_i + [\Delta\varphi_i,\ H_o + V + W] - \delta(s)\partial_x f \sum_{\ell=1}^{J} \int G(x'+y')\ (\Delta\psi_{i+\ell}+\Delta\psi_{i-\ell})\ dydy'$$

$$+ \partial_{x'}f\ \delta(s) \int \chi_0(x-y)\ \Delta\psi_i dydy' = 0\ . \qquad (5.9b)$$

The function H_0, V and W are defined as follows:

$$H_0 = \frac{x'^2}{2} + K\,\frac{x^2}{2} \tag{5.10a}$$

$$V(x';s) = -2J\,\delta(s) \int_0^{x'} d\mu \int G(\mu+y')\, f\, dydy' \tag{5.10b}$$

$$W(x;s) = -\,\delta(s) \int_0^{x} d\mu \int \chi_0(\mu-y)\, f\, dydy' \;. \tag{5.10c}$$

Equation (5.9a) and (5.9b) are the basic equations found in Chao and Ruth's treatment. What will now follow is a marriage of their techniques with what we described previously for the dipole mode; all of that leading to the "good old" operator C^{λ}_{μ}.

VI. The transfer map technique.

Again the dimensionality of the problem is gigantic. We reduce it by applying the transformation (3.10) on $\Delta\varphi_i$ and $\Delta\psi_i$. The resulting equation is

$$\partial_s \Delta\varphi_\lambda + [\Delta\varphi_\lambda,\, H_0 + V + W] - \delta(s)\, \partial_x f\, q(\lambda) \int G(x'+y')\, \Delta\psi_\lambda dydy'$$
$$+ \partial_{x'} f\, \delta(s) \int \chi_0(x-y)\, \Delta\psi_\lambda dydy' = 0 \;. \tag{6.1}$$

The function $q(\lambda)$ is given again by (3.18). At this stage we must assume a certain distribution for f. The simplest one is the water bag distribution. Assuming that the perturbed Hamiltonian $H_0 + V + W$ is integrable, a stationary f obeying (5.9a) depends only on the action variable J_x. The water bag distribution is given by:

$$f(J_x) = H(\frac{\varepsilon_x}{2} - J_x)/\pi\varepsilon_x,\quad H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \;, \tag{6.2a}$$

$$\partial_{J_x} f = -\delta(J_x - \frac{\epsilon_x}{2})/\pi\epsilon_x \ . \tag{6.2b}$$

As Chao and Ruth did, we assume that the Hamiltonian $H_o + V + W$ is quadratic. In that case, it is possible to relate (ϕ_x, J_x) to (x,x'):

$$x = \sqrt{2\beta J_x} \cos \phi_x \tag{6.3a}$$

$$x' = - \sqrt{\frac{2J_x}{\beta}} \sin \phi_x \tag{6.3b}$$

The function β is slightly different from β_*. On the other hand, the Twiss parameter α is still zero due to the symmetry around s=0 of V + W. Taking advantage of (6.2) and (6.3) we can decompose $\Delta\phi_\lambda$ and $\Delta\psi_\lambda$ in multipole modes.

$$\Delta\varphi_\lambda = \delta(J_x - \frac{\epsilon_x}{2}) \sum_{\ell=-\infty}^{\infty} f_\ell(s) \ e^{i\ell\phi_x} \tag{6.4a}$$

$$\Delta\psi_\lambda = \delta(J_x - \frac{\epsilon_x}{2}) \sum_{\ell=-\infty}^{\infty} g_\ell(s) \ e^{i\ell\phi_x} \ . \tag{6.4b}$$

Integrating (6.1) around s=0 leads to the following equations:

$$f_\ell^+ = f_\ell^- - \left[\frac{AL_x q(\lambda)}{2\pi^3\alpha\sqrt{\beta\epsilon_x}}\right]\sum N_{\ell k} g_k^- - \left[\frac{A\sqrt{\beta/\epsilon_x}}{2\pi^2}\right]\sum M_{\ell k} g_k^- \tag{6.5a}$$

$$N_{\ell k} = \int_0^{2\pi}\int_0^{2\pi} e^{-i\ell\phi} \cos\phi \ \tilde{G}(-\sqrt{\epsilon_x/\beta}\ (\sin\phi + \sin\phi_y)) e^{ik\phi_y} \ d\phi d\phi_y \tag{6.5b}$$

$$\tilde{G} = \frac{\pi\alpha}{AL_x} G$$

$$M_{\ell k} = \int_0^{2\pi}\int_0^{2\pi} e^{-i\ell\phi}\sin\phi\, H_1(\cos\phi-\cos\phi_y)e^{ik\phi_y}\, d\phi d\phi_y \ . \tag{6.5c}$$

The matrix $M_{\ell k}$ was evaluated by Chao and Ruth.[1] Both N and M are traceless and their square is zero. This insures that to first order the determinant of the transfer map is one. To preserve this property to all order, we factor the transfer map as we did for the dipole mode:

$$\gamma^+ = (I - \frac{A\sqrt{\beta/\epsilon_x}}{2\pi^2} M\Delta)\ (I - \frac{AL_x q(\lambda)}{2\pi^3 \quad \alpha\sqrt{\beta\epsilon_x}} N\Delta)\gamma^- \tag{6.6}$$

$$= M_0 M^{\lambda}_{LR}\gamma^-$$

$$\gamma = (f\ ,\ g\ ) = (f_1,\ f_{-1},\ g_1,\ g_{-1},\ \ldots, f_\ell,\ f_{-\ell}, g_\ell,\ g_{-\ell}\ldots)$$

Here again, the matrix Δ interchanges the components of γ. Notice the structure is identical to the dipole case (3.15).

Finally, the phase advance matrix $R(\nu)$ is given by:

$$R(\nu)\ \gamma_\ell = e^{-i2\pi\nu\ell}\gamma_\ell \ . \tag{6.7}$$

Using the group theoritical tools of section 2, we can write the map for the full N-sites ring:

$$C^{\lambda}_{\mu} = \omega_\mu\ R(\nu)\ M_0 M^{\lambda}_{LR}. \tag{6.8}$$

The dimension of C^{λ}_{μ} will be $4\ell_{max}$ where ℓ_{max} is the highest mode considered. Of course, we must examine $\frac{N}{2}$ values of μ and a continuum of values for λ . But, guided by the dipole mode studies, we will present in the next section the tricks which makes this problem finite.

VII. Determination of Stability Curve

The final curves will exhibit the stability limit in terms of ξ_0 and ν_0. To compare with the dipole studies done previously, we will compute ξ and ν in terms of ξ_0 and ν_0. Also we will compute the long range with the same coefficient p as used in the dipole studies (Fig. 6 and 7).

In Appendix V, it is shown that p from the Vlasov equation is predicted to be:

$$p = -\frac{L_x\sqrt{\beta_* \epsilon_x}}{4\alpha^2} . \tag{7.1}$$

The tune shift parameter ξ_0, is given by:

$$\xi_0 = \frac{A\sqrt{\beta_*}}{\pi^2\sqrt{\epsilon_x}} \tag{7.2}$$

In terms of these quantities, we get for ν_0 and β_*:

$$\nu_0 = \nu + \left(\frac{16}{3} + \frac{4\pi Jp}{\beta_*^2}\right) \xi_0 \tag{7.3a}$$

$$\beta_* = \frac{\beta \sin(2\pi\nu)}{\sin(2\pi\nu_0)} . \tag{7.3b}$$

This tune shift is computed at $J_x = \epsilon_x/2$ and is a first order result in the beam-beam parameters. It is obtainable from V + W through the formula:

$$2\pi(\nu-\nu_0) = \frac{\partial}{\partial J_x} \frac{1}{2\pi} \int_0^{2\pi} (v(x') + w(x))\, d\phi_x \Bigg|_{J_x = \epsilon_x/2} \tag{7.4a}$$

$$v(x') + w(x) = \int_{0^-}^{0^+} (V(x';s) + W(x;s))\, ds . \tag{7.4b}$$

Finally, by manipulating $N_{\ell k}$, it is possible to write relations obeyed by this matrix:

$$N_{\ell k} = - \exp(\frac{i\pi}{2} (k-\ell)) \mu_{\ell k} \tag{7.5a}$$

$$\begin{aligned} \mu^* &= -\mu \\ \mu_{\ell -k} &= \mu_{\ell k} \Rightarrow \mu^2 = 0 \Rightarrow N^2 = 0. \\ \mu_{-\ell k} &= -\mu_{\ell k} \end{aligned} \tag{7.5b}$$

One can also get a messy generating function for $\mu_{\ell k}$:

$$\mu(x,y) = \sum_{\ell,k=-\infty}^{\infty} x^{\ell} y^{k} \mu_{\ell k} \tag{7.6a}$$

$$\mu(x,y) = - \delta\pi^2 i(x-x^{-1}) \sum_{n=0}^{\infty} (\frac{\delta}{2})^n (x+x^{-1}+y+y^{-1})^{n+1} \tag{7.6b}$$

$$\delta = \sqrt{\frac{\varepsilon_x/\beta_*}{\alpha}} \quad . \tag{7.6c}$$

The parameter δ measures the ratio of the beam size to the beams separation during long range collision. For our model to be consistent, δ has to be much less than one.

Armed with all these results, we can substitute them in C_{μ}^{λ}, and find its eigenvalues $\Lambda_{\mu}^{i}(\lambda)$. Inspired by the dipole study we try to extremize $\Lambda_{\mu}^{i}(\lambda)$:

$$\frac{d}{d\lambda} \Lambda_{\mu}^{i}(\lambda) = \frac{d\Lambda_{\mu}^{i}(\lambda)}{dq} \quad \frac{dq}{d\lambda} = 0 \tag{7.7c}$$

$$i = 1, 4\ell_{max}$$

Again, $\frac{dq}{d\lambda}$ is zero for $\lambda = 0$ and $\lambda = \lambda_{min}$. Assuming that the most unstable mode occurs at one of these two values, the search of the stability limit is now a finite problem.

In Fig. 8 and 9, we reproduce the curves for the cases of Fig. 6 and 7 respectively. The highest multipole mode included is the sextupole mode, $\ell=3$. The reader will notice the degradation of the 10 μrad case of Fig. 9. Figure 7 shows a much higher stability limit. This is due to the size of δ which has the value 0.7 at 10μrad. If only the linear terms in δ are kept, the stability curve for 10μrad looks again like Fig. 7 as indicated by Fig. 10.

Conclusion

By a series of tricks, we succeeded in adapting the matrix techniques of Chao and Ruth to the complex problem of the long range interaction. Our calculations suffer from the same inadequacies as theirs. In particular, it is imperative to include tune spread into the calculations, even at the expense of further approximations.

Acknowledgments

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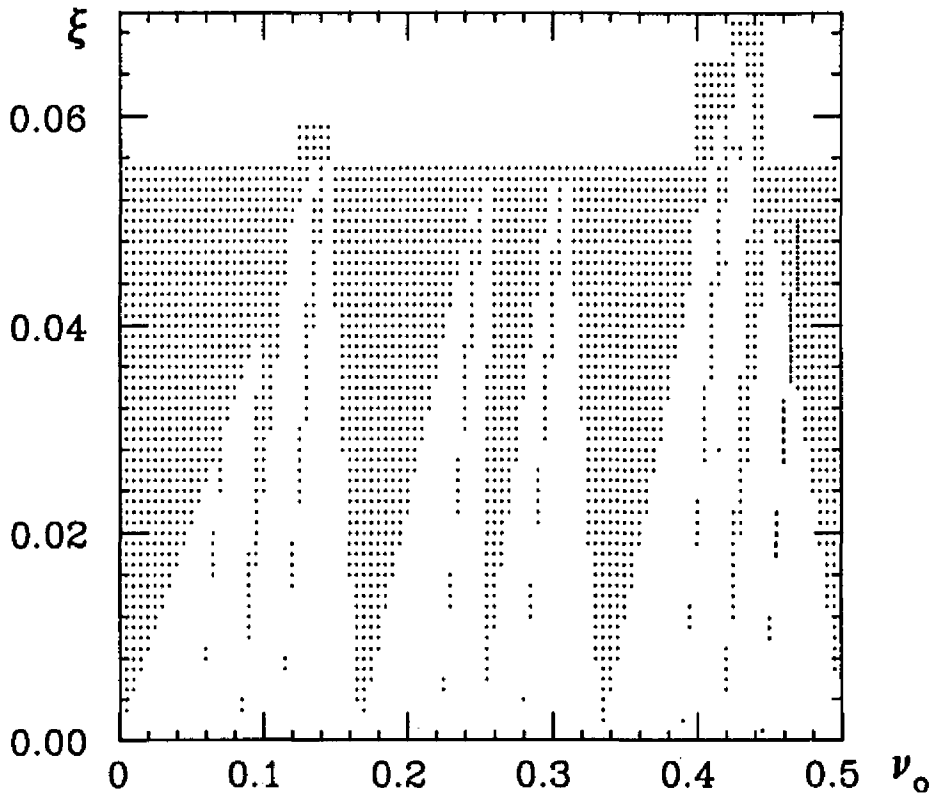

Fig. 8. The configuration of N=3, J=4 and α=50μrad is studies with the Blasov technique. Black dots are unstable points. Here, we used the pseudo-drift model.

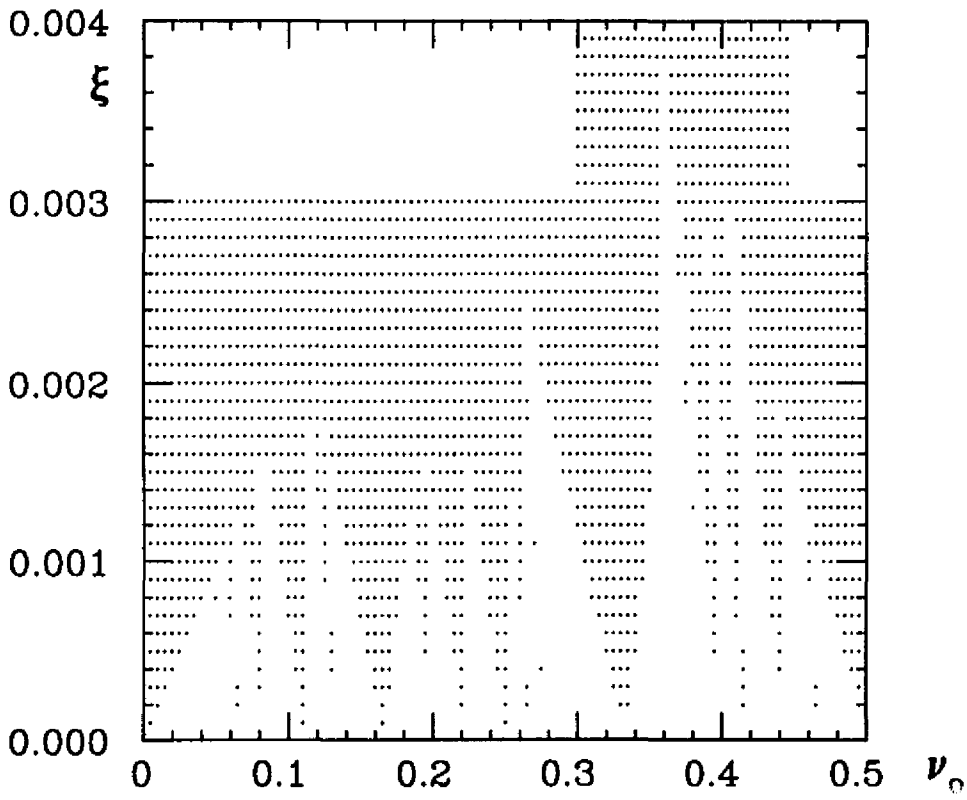

Fig. 9. This figure shows the N=3, J=4, and α=10μrad configuration. A strong degradation has occurred due to the large size of δ(δ≈0.7).

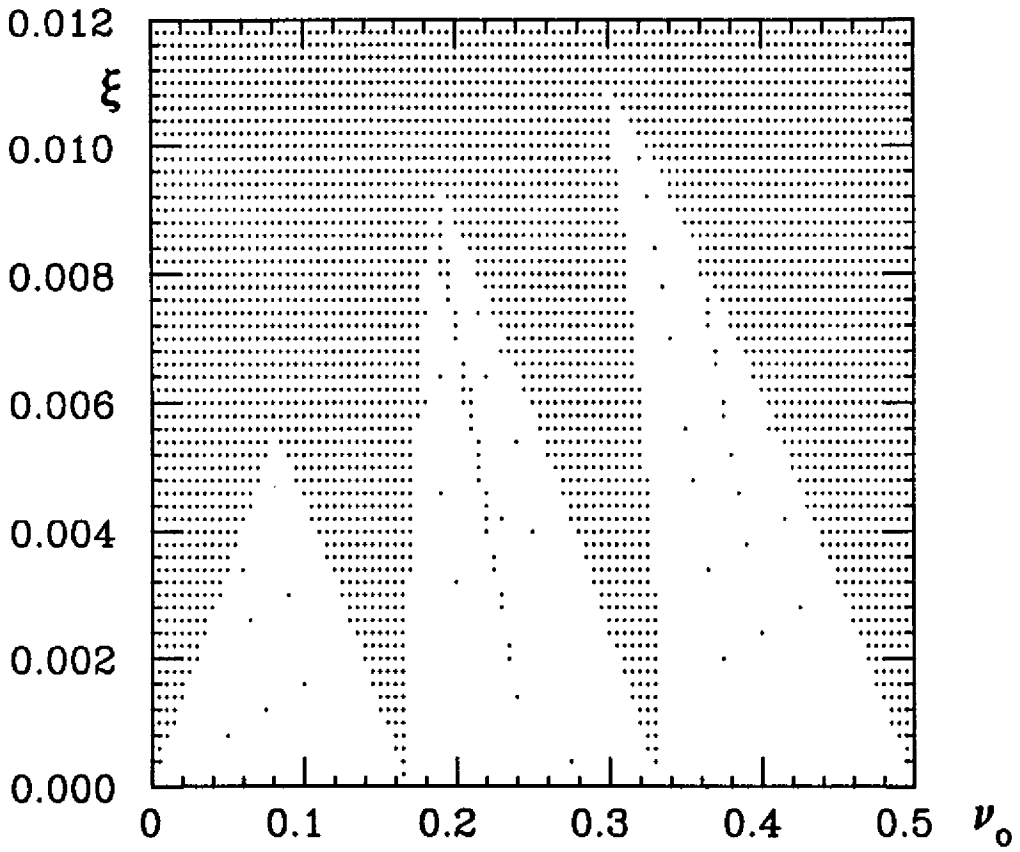

Fig. 10. Results more or less consistent with the dipole study of figure 7 are regained if only the terms linear in δ are kept.

References

1. A.W. Chao and R.D. Ruth, Particle Accelerators, 16, Number 4 (1985), p. 201.

2. See for example, Hammermesh, Group Theory and its Application to Physical Problems, Addison-Wesley Publishing Company.

3. A.W. Chao, A Summary of Some Beam-Beam Models, Proceeding of the Non-Linear the Beam-Beam Interaction Conference, AIP Conference Proceedings No. 57, p. 42.

4. M.A. Furman and A.W. Chao, Effects of Long Range Beam-Beam Interaction on the Stability of Coherent Dipole Motion, Proceedings of the 1985 Particle Accelerator Conference, IEEE Transactions on Nuclear Sciences, VOL NS-32, October 1985, p. 2297.

5. A Summary of Lie Algebraic Methods can be found in Dragt's lecture on Non-linear Orbit Dynamics, Fermilab Summer School, 1981, AIP Proceedings, No. 87, p. 147.

6. We factor the map in order to preserve the sympletic nature of the total map. Factorization is used again in the treatment of higher order modes, equation (6.6).

7. See reference 5, p. 168.

8. This assumption is hard to justify in the presence of a complicated long range operator. Indeed, we have checked that some individual modes become unstable through a Krein collision on the unit circle. During a Krein collision, the eigenvalues meet on the unit circle away from the real axis. However, these modes were more stable than the modes becoming unstable at ±1.

9. It is our belief that the coefficient A in reference 1 is to large by a factor of 2. In any case, it does not change the results of this paper or those of reference 1.

10. In equation (5.4b), we subtracted the d.c. term of the long range force.

Appendix I. The Group Theoretical Approach.

The structure of the group generated by Ω and Δ is quite simple. It is easy to show that this group has 2N elements. In fact, the group is composed of the following set of elements:

$$G = \{x | x = \Omega^k,\ x = \Delta\Omega^k;\ k = 0,\ N - 1\} \tag{A1}$$

All other products of Ω and Δ must reduce to one of these 2N elements. In the study of the operator $C(= \Omega A + \Omega\Delta B)$, it is convenient to find all the irreducible representations of G. According to the theory of finite groups, the number of distinct irreducible representations is equal to the number of equivalence classes of the relation P defined as follows:

$$xPy \Leftrightarrow \exists\ a \epsilon G \text{ such that } axa^{-1} = y. \tag{A2}$$

Those classes partitioned G. They are given by:

$$N_{odd}: G = \{I\} \cup \{\Delta\Omega^k;\ k = 0,\ N-1\} \bigcup_{k=1,\ \frac{N-1}{2}} \{\Omega^k,\ \Omega^{-k}\}$$

$$N_{even}:\ G = \{I\} \cup \{\Delta\Omega^k;\ k = 0,\ \tfrac{N}{2}-1\} \cup \{\Delta\Omega^{k+1};\ k = 0,\ \tfrac{N}{2}-1\} \tag{A3}$$

$$\bigcup_{k=1,\ \frac{N-2}{2}} \{\Omega^k,\ \Omega^{-k}\} \cup \{\Omega^{\frac{N}{2}}\}\ .$$

Counting the number of classes, we conclude that there are (N+3)/2 classes for odd N and (N+6)/2 classes for even N.

As mentioned in section 2, the dimensionality of the irreducible representations of G does not exceed two. This is in contrast with Ω, Δ and their various combinations which in our problem are 2N by 2N matrices. In the

next tables, we list the irreducible representations as well as the frequency of their occurrence in the original 2N by 2N representation:

N_{odd}	I	Ω	Δ	Frequency	Total contribution to the dimensionality
0-mode	1	1	1	1	1
π-mode	1	1	-1	1	1
μ-mode	ω_0	ω_μ	Δ_μ	2	2(N-1)

N_{even}	I	Ω	Δ	Frequency	Total contribution to the dimensionality
0-mode	1	1	1	1	1
π-mode	1	1	-1	1	1
0^--mode	1	-1	1	1	1
π^--mode	1	-1	-1	1	1
μ-mode	ω_0	ω_μ	Δ_μ	2	2(N-2)

$$\omega_\mu = \begin{pmatrix} \cos(\frac{2\pi\mu}{N}) & \sin(\frac{2\pi\mu}{N}) \\ -\sin(\frac{2\pi\mu}{N}) & \cos(\frac{2\pi\mu}{N}) \end{pmatrix}$$

$$\Delta = \delta_\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A4}$$

The last column adds up to 2N as required. The results can be checked by standard tools of group theory. In particular, the two following relations are useful:

$$\sum_{x \in G} Tr^\nu(x)\ Tr^\mu(x)^* = 2N\ \delta_{\nu\mu} \tag{A5}$$

The frequency of the νth irreducible representation is given by:

$$f = \frac{1}{2N} \sum_{x \varepsilon G} Tr(x)\ Tr^{\nu}(x)^{*} \ . \tag{A6}$$

In these sums, $Tr^{\nu}(x)$ denotes the trace of x in the νth irreducible representation. In (A6), Tr(x) is the trace in the reducible representation under study. Finally, we conclude with a word about the relocation operator Ω. The readers familiar with group theory will notice that, by the very nature of Ω, the original 2N by 2N representation is the so-called regular representation of a finite group. It obeys the relation:

$$Tr(x) = \begin{cases} 0 & \text{for } x \neq I \\ 2N & \text{for } x \neq I \end{cases} \tag{A7}$$

Appendix II. The finite transform approach.

The group theoretical results of Appendix I can be rederived with the help of the transformation introduced on line (3.10). To use this technique we must rewrite (2.5) in component form:

$$\bar{\Gamma} = (\Omega A + \Omega \Delta B)\ \Gamma \tag{B1}$$

is replaced by

$$\bar{\varphi}^{i+1} = A\varphi^{i} + B\psi^{i}$$

$$\bar{\psi}^{i-1} = B\varphi^{i} + A\psi^{i} \quad .$$

Here, the superscript "i" is a modulo N number.

$$i \in Z/NZ \quad . \tag{B2}$$

Using (3.10), with m replaced by N, we transform (B1) with the result:

$$\begin{aligned} \cos(\tfrac{2\pi\mu}{N})\ \bar{\varphi}^{\mu} - \sin(\tfrac{2\pi\mu}{N})\ \bar{\varphi}^{-\mu} &= A\varphi^{\mu} + B\psi^{\mu} \\ \cos(\tfrac{2\pi\mu}{N})\ \bar{\psi}^{\mu} + \sin(\tfrac{2\pi\mu}{N})\ \bar{\psi}^{-\mu} &= B\varphi^{\mu} + A\psi^{\mu} \end{aligned} \tag{B3}$$

Equation (B3) involves four components. These can be decoupled further by a transformation to the new vector $w^{\mu}_{\pm}$:

$$\begin{aligned} w^{\mu}_{\pm} &= (\xi^{\mu}_{\pm}\ ,\ \zeta^{\mu}_{\pm}) \\ \xi^{\mu}_{\pm} &= (\varphi^{\mu} \pm \psi^{-\mu})/\sqrt{2} \\ \zeta^{\mu}_{\pm} &= (\varphi^{-\mu} \pm \psi^{\mu})/\sqrt{2} \end{aligned} \tag{B4}$$

In term of $w^{\mu}_{\pm}$, (B3) takes the form:

$$\begin{aligned} \omega_{-\mu}\ \bar{w}^{\mu}_{\pm} &= (Ae \pm B\delta)\ w^{\mu}_{\pm} \\ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \delta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ 0 \le \mu \le \tfrac{N}{2}. \end{aligned} \tag{B5}$$

To get (B5) into a form identical to the group theoretical description of Appendix I, we transform $w^{\mu}_{\pm}$ into two new vectors V^{μ}_{1} and V^{μ}_{2}:

$$V^{\mu}_{1} = w^{\mu}_{+}$$

$$V^{\mu}_{2} = \omega_{\frac{N}{2}} w^{\mu}_{-} \tag{B6}$$

$$\overline{V}^{\mu}_{i} = (\omega_{\mu} A + \omega_{\mu}\delta B)V^{\mu}_{i}, \quad \begin{matrix} i = 1,2 \\ 0 \leq \mu \leq \frac{N}{2} \end{matrix} .$$

The connection to the group theoretical approach can be made more exact. For $\mu \neq 0 \neq \frac{N}{2}$, we regained the two dimensional representations referred to as the μ-modes in Appendix I. They indeed appear twice (i.e. V_1 and V_2). Finally for even and odd N, we get two modes at $\mu = 0$ (0-mode, π-mode). For even N, $\frac{N}{2}$ is an integer. We get the 0^--mode and π^--mode at $\mu = \frac{N}{2}$.

Appendix III. The collision operator in the dipole mode.

This appendix as well as the subsequent one, will treat the problem of rewriting a non-local process in terms of a local one. Our final result in Appendix III will be equation (3.4). To start, we write a kick in terms of its Lie operator. We assume that the collisions take place at some location s, where s = 0 is the head on site. At a given s, the kick has the form:

$$\begin{aligned} x_+(s) &= x(s) \\ x'_+(s) &= x'(s) + \epsilon_s(x(s) - y(-s)). \end{aligned} \tag{C1}$$

In Lie algebraic language, (C1), is just represented by the operator:

$$H^s_{ij} = \frac{\epsilon_s}{2}\,(x(s) - y(-s))^2. \tag{C2}$$

Our goal is to write an operator which is first order in ϵ_s and is expressed in terms of the s = 0 values of the bunches coordinates. First of all, we notice that at s the ith bunch can only collide with the $(i+s)^{th}$ bunch. Here we assume that collisions happen at s = 0, ± 1, ...$\pm J$.

The transformation representing this collision, can be written as follows:

$$M_{ii+s} = \exp(:H^s_{ii+s}:) \tag{C3}$$

This operator acts on the vector $(\varphi_i(s), \psi_{i+s}(-s))$. Neglecting the existence of other collisions, we can write a relation connecting this vector to its s = 0 value:

$$\begin{aligned} \varphi_i(s) &= \rho_s\varphi_i(0) \\ \psi_{i+s}(-s) &= \rho_{-s}\psi_{i+s}(0) \end{aligned} \tag{C4}$$

The operator ρ_s is the single particle map bringing a particle from s = 0 to s. It is, of course, a canonical transformation. Using (C4), we can write the operator H^s_{ii+s} at s = 0:

$$H^0_{ii+s} = \frac{\epsilon_s}{2} (\rho_s \, x_i(0) - \rho_{-s} \, y_{i+s}(0))^2 \tag{C5}$$

Since all the cross terms are neglected, the total operator will just be the sum of all the H^0_{ii+s} operators.

$$H = \sum_{i=1}^{m} H^0_{ii} + \sum_{\substack{s=-J \\ s\neq 0}}^{J} H^0_{jj+s} \tag{C6}$$

One interesting question is why we can neglect cross-terms within an interaction region. The answer can be understood in terms of a resonance analysis of the total map. Instabilities occur when the bunches hit a resonance. This resonance is driven by successive application of the map C. Therefore a correct treatment of the power of C is important. This was achieved without approximations in the appendices I and II. The reader will notice that this approximation must be very accurate since Figure 6 and 7 agree very well with an exact simulation.

Appendix IV. Turning the long range Vlasov equation into a local equation.

We start with equation (5.5). First we integrate it around $s_\ell = \frac{\ell L}{2}$:

$$\varphi_i^+(s_\ell) = \varphi_i^-(s_\ell) - \partial_{x'}\varphi_i^-(s_\ell) \int \chi_{s_\ell}(x-y)\ \psi_{i+\ell}^-(y,\ y';-s_\ell)\ dydy' \tag{D1}$$

Neglecting the collisions, φ_i can be progated with the inverse of the single particle operator:

$$\varphi_i(x,\ x';\ s_\ell) = \rho_\ell^{-1}\varphi(x,\ x';\ 0) \tag{D2}$$

Substituting in (D1) this last result:

$$\begin{gathered}\varphi_i^+(Z;\ 0) = \varphi_i^-(Z;\ 0) - \rho_\ell\ \{\partial_{x'}\varphi_i^-(s_\ell)\int\chi_{s_\ell}(x-\tilde{x})\ \psi_{i+\ell}^-(\rho_\ell^{-1}\tilde{Z};\ 0)d\tilde{Z}\} \\ Z = (x,\ x')\end{gathered} \tag{D3}$$

We must manipulate the derivative in (D3):

$$\begin{aligned}\rho_\ell\partial_{x'}\varphi_i^-(s_\ell) &= \rho_\ell\left\{\frac{\partial\varphi_i^-(\rho_\ell^{-1}Z;\ 0)}{\partial(\rho_\ell^{-1}\ Z)}\ .\ \frac{\partial(\rho_\ell^{-1}Z)}{\partial x'}\right\} \\ &= \partial_Z\bar{\varphi}_i(Z;\ 0)\ .\ \partial_{x'}(\bar{\rho}_\ell^{1}\ \ Z)\end{aligned} \tag{D4}$$

In the linear model, ρ_ℓ^{-1} Z is linear function of Z. Therefore the second term in (D4) is just a number, independent of Z. We substitute (D4) into (D3) and change the integration variable:

$$\varphi_i^+(Z;0)=\varphi_i^-(Z;0)-\partial_Z\varphi_i^-(Z;0)\cdot\sum_{\ell=-J}^{J}\partial_{x'}(\rho_\ell^{-1}Z)\int\chi_{s_\ell}(\rho_\ell x-\rho_{-\ell}\bar{x})\psi_{i+\ell}^-(\bar{Z};0)d\bar{Z} \tag{D5}$$

In (D5), we added all the long range collisions. Since we neglect cross-terms, (D5) is the final result. the reader will notice that (D5) can be obtained by integrating (5.6) around s = 0, hence these equations are equivalent.

Appendix V.

The purpose of this appendix is to connect the parameter appearing in the Vlasov model to the simple dipole mode picture derived from a gaussian model. In particular, we would like to derive the parameter p connecting the head-on kick to the long range kick. Let us assume that a collision of unspecified type occurs at a location s. This is described by the Vlasov equation.

$$\partial_s\varphi + [\varphi,H_0] + \delta(s)\ \partial_{x'}\varphi \int \chi_s\psi\ dydy' = 0 \qquad \text{(E1)}$$

Multiplying (E1) by x' and integrating across the delta function, we obtain:

$$\langle x'\rangle_+ = \langle x'\rangle_- - \int x'\partial_{x'}\varphi\ dxdx' \int \chi_s\varphi\ dydy' \ . \qquad \text{(E2)}$$

Integrating (E2) by parts, we eliminate the derivative:

$$\langle x'\rangle_+ = \langle x'\rangle_- + \int \varphi\ dxdx' \int \chi_s\ \psi\ dydy' \ . \qquad \text{(E3)}$$

First, let us consider the long range operators, in that case χ_s has the form:

$$\chi_s \simeq \frac{AL_x}{\pi\alpha^2 s^2}\ (y - x)\ . \qquad \text{(E4)}$$

Substitution in E5, leads immediately to the final result:

$$\langle x'\rangle_+ = \langle x'\rangle_- + \frac{AL_x}{\pi\alpha^2 s^2}\ (\langle y\rangle_- - \langle x\rangle_-) \qquad \text{(E5)}$$

The case of the head on collision is more complex because $\chi(x-y)$ is not analytic. We can approximate (E3) by introducing the unperturbed distribution $f(J_x)$ and linearize around it:

$$\begin{aligned}\langle x'\rangle_+ &= \langle x'\rangle_- + A\iint H_1(x-y)\{f(J_x)\Delta\psi(y,y')+f(J_y)\Delta\varphi(x,x')\}dxdx'dydy' \\ &= \langle x'\rangle_- + A\int\rho(x)H_1(x-y)\ \Delta\psi dxdydy' + A\int\rho(y)H_1(x-y)\ \Delta\varphi dydxdx'\end{aligned} \qquad \text{(E6)}$$

Here, $\rho(x)$ is the particle distribution in x resulting from $f(J_x)$. Under the assumption that $\rho(x)$ is even in x, the first integral in (E6) can be simplified using the following relation:

$$\int_{-\infty}^{+\infty} \rho(x)\, H(x-y)\, dx = -\int_{-\infty}^{-y} \rho(x)\, dx + \int_{y}^{+\infty} \rho(x)\, dx - \int_{-y}^{y} \rho(x)\, dx$$

$$= -\int_{-y}^{y} \rho(x)\, dx \qquad \text{(E7)}$$

$$\simeq -2y\rho(0)$$

A similar relation applies to the second integral

$$\int_{-\infty}^{+\infty} \rho(y)\, H(x-y)\, dy = 2x\, \rho(0) \qquad \text{(E8)}$$

Substituting of (E7) and (E8) into (E6) leads to the final result:

$$\langle x' \rangle_+ = \langle x' \rangle_- + 2A\, \rho(0)(\langle x \rangle_- - \langle y \rangle_-). \qquad \text{(E9)}$$

From (E5) and (E9), we can compute p:

$$p = \frac{-L_x}{2\pi\alpha^2\rho(0)} \qquad \text{(E10)}$$

In the water bag model, $\rho(x)$ is given by the formula:

$$\rho(x) = \frac{2}{\pi\varepsilon_x\sqrt{\beta_*}} \sqrt{\varepsilon_x - \frac{x^2}{\beta_*}}$$

$$\rho(0) = \frac{2}{\pi\sqrt{\varepsilon_x\beta_*}} \qquad \text{(E11)}$$

Hence, we obtain:

$$p_{waterbag} = -\frac{L_x\sqrt{\varepsilon_x\beta_*}}{4\alpha^2} \qquad \text{(E12)}$$

Finally, we can compute ξ_0, with the help of (E9) and (4.3d):

$$4\pi\ \xi_0 = 2A\rho(0)\ \beta_*$$

$$\xi_0 = \frac{A\sqrt{\beta_*}}{\pi^2\sqrt{\varepsilon_x}} \tag{E13}$$

In the numerical studies in this paper, we always used ξ_0 defined by (4.3d) and p defined by (3.2) as the numerical inputs. All other quantities in the Vlasov treatment were re-expressed in terms of ξ_0 and p using the formulae of this Appendix.