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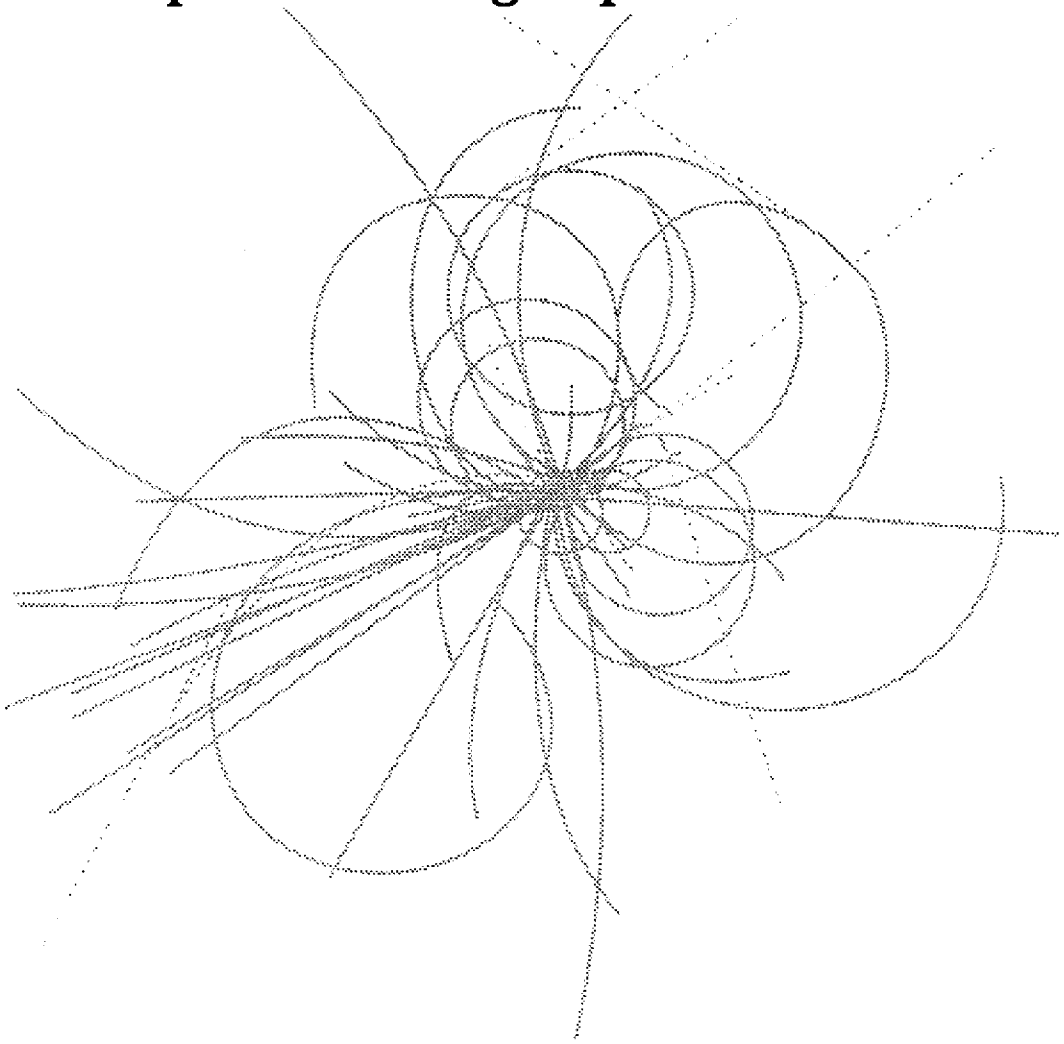

Treatment of the Long Range Coherent Beam-Beam With the Vlasov Equation

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October 1985

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TREATMENT OF THE LONG RANGE COHERENT BEAM-BEAM WITH THE VLASOV EQUATION*

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October 1985

*An extension of a calculation by A. W. Chao and R. D. Ruth
**Operated by Universities Research Association under contract with the Department of Energy

1. Introduction.

In a previous SSC-report (SSC-32), we explained how one could get some analytical results in the study of dipole beam-beam collisions. We examined three cases. We solved exactly for the stability condition in the presence of head-on collisions (h.o.). Then we studied the change in these conditions as we add long-range collisions(l.r.). We obtained analytical results which agreed very well with the exact computer solution.

The purpose of this paper is to extend to the l.r. collisions a calculation due to Chao and Ruth. This calculation reported in SLAC-PUB-3400/AP-37 (SLAC-37) used the Vlasov equation in the study of h.o. collisions. They looked at the effect of modes up to octupole order. In their formalism the ribbon beams are represented by two distribution functions Ψ and Φ.

The kick received by a relativistic particle in the Ψ beam due to the Φ ribbon beam is given by:

$$\Delta p_x = \frac{2\pi N r}{L_x \gamma} \int \Phi(y,p_y;s=0)\, H_1(x-y)\, d\Gamma_y \qquad (1.1)$$

Here r is the classical radius of the particles, L_x is the horizontal width of the beams , γ is the Lorentz factor and

$$H_1(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases} \qquad (1.2)$$

One can then write down a Vlasov equation for Φ and Ψ.

$$\partial_s \Phi + p\, \partial_x \Phi - \partial_p \Phi \left\{ Kx - A\delta(s) \int H_1(x-y) \Psi\, d\Gamma_y \right\} = 0 \qquad (1.3)$$

$$A = \frac{2\pi N r}{L_x \gamma}.$$

A similar equation can be written for Ψ.

2. The long range Vlasov equation.

The expression for the head-on kick assumed a ribbon beam. For the long range kick a localised distribution is assumed. The kick is then given by:

$$\Delta p_x = \int \Psi(y,p_y;s)\, \chi_s(x-y)\, d\Gamma_y \qquad (2.1)$$

$$\chi_s = AL_x\, ((d(s)+x-y)^{-1} - d(s)^{-1})/\pi \qquad (2.2)$$

$d(s)=\alpha s$; α=crossing angle ; s=path length coordinate.

Here we imagine a machine filled with mN bunches in each beam. The machine has N equally space generalized interaction points (or G.I.P. see SSC-32). The state of the machine is contained in a mN vector Γ.

$$\Gamma = (\Phi^0, \Psi^0, \Phi^1, \Psi^1, \ldots\ldots, \Phi^{N-1}, \Psi^{N-1}) \qquad (2.3)$$

The m-vectors Φ^j and Ψ^j describe the state of a particular G.I.P.

$$\Phi^j = (\Phi^j_o, \ldots\ldots, \Phi^j_{m-1}) \qquad (2.4a)$$

$$\Psi^j = (\Psi^j_o, \ldots\ldots, \Psi^j_{m-1}) \qquad (2.4b)$$

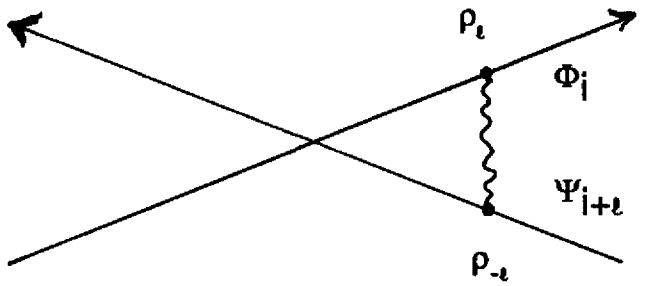
The problem is quite complicated. However since we will eventually linearize the Vlasov equation, the machinery of group theory can be used. This has been described in SSC-32. As a result we need to focus our attention on only one G.I.P. From now on we omit the superscript j in equation (2.4). For our problem the Vlasov equation becomes:

$$\partial_s \Phi_i + p\, \partial_x \Phi_i - \partial_p \Phi_i \left\{ Kx - \sum_{|\ell| \le J} \delta(s-\ell\lambda) \int \chi_s(x-y) \Psi_{i+\ell}(y,p_y;-s)\, d\Gamma_y \right\} = 0 \qquad (2.5)$$

Here J is the number of l.r. each side of the h.o. collision. The parameter λ is L/2 where L is the interbunch spacing. As in SSC-32, we inforce the Von Karman boundary condition.

There are 2m equations like (2.5). This is obviously a messy problem. However if we neglect cross terms within a G.I.P., it is then possible to

express all the quantities in (2.5) in terms of a local δ-function at s=0.



Configuration of a G.I.P.

If we denote by ρ_ℓ the one particle map which connects the I.P. to the site of the ℓth I.r. collision, it is then possible to rewrite (2.5) as follows:

$$\partial_s\Phi_i + p\partial_x\Phi_i - \partial_p\Phi_i\, Kx + \delta(s)\partial_z\Phi_i \cdot \sum_{|\ell|\le J} \partial_p(\rho_\ell^{-1}z) \int \chi_s(\rho_\ell x - \rho_{-\ell} y)\Psi_{i+\ell}(s)d\Gamma_y = 0 \quad (2.6)$$

$$z=(x,p)$$

If we assume that ρ_ℓ is a drift, we obtain:

$$\rho_\ell z=(x+\ell\lambda p,p) \quad (2.7)$$

In SSC-32, we kept terms of order $\ell\lambda$ only. This simplifies the mathematics. For this pseudo-drift, (2.7) is replaced by the symplectic map :

$$\ell\neq 0\ ; \qquad \rho_\ell z=(\ell\lambda p, p-x/\ell\lambda). \quad (2.8)$$

Using (2.8), equation (2.6) reduces to:

$$\partial_s\Phi_i + p\partial_x\Phi_i - \partial_p\Phi_i\, Kx + \delta(s)\partial_x\Phi_i \sum_{|\ell|\neq 0} \ell\lambda \int \chi_{\ell\lambda}(\ell\lambda(p+p_y))\Psi_{i+\ell}\, d\Gamma_y \quad (2.9)$$

$$+ \partial_p\Phi_i\, \delta(s) \int \chi_o(x-y)\, \Psi_i\, d\Gamma_y = 0$$

This equation simplifies if one notices that $s\chi_s$ is not a function of s. We denote this function by the letter $\mathfrak{A}$:

$$\mathfrak{A}(p+p_y)=(AL_x/\pi\alpha)\sum_{n=1}^{\infty}(-1/\alpha)^n(p+p_y)^n=(AL_x/\pi\alpha)G \tag{2.10}$$

In this paper, we will not treat the telescopic model of SSC-32. Actually this model leads to an even simpler Vlasov equation.
In the next section we linearize equation (2.9) around a fixed distribution.

3. The linearized equation.

Let us assume the existence of a stationary solution f:

$$\Phi_i=f+\Delta\Phi_i\ ;\ \Psi_i=f+\Delta\Psi_i\ . \tag{3.1}$$

We substitute (3.1) into (2.9) and we linearize the equation:

$$\partial_s f+[f,H_o+V+W]=0 \tag{3.2a}$$

$$H_o=p^2/2+Kx^2/2 \tag{3.2b}$$

$$V(p;s)=-2J\delta(s)\int_0^p d\mu\int \mathfrak{A}(\mu+p_y)\, f\, d\mu\, d\Gamma_y \tag{3.2c}$$

$$W(x;s)=-\delta(s)\int_0^p d\mu\int \chi_o(x-y)\, f\, d\mu\, d\Gamma_y \tag{3.2d}$$

$$\partial_s\,\Delta\Phi_i+[\,\Delta\Phi_i,H_o+V+W]\ -\delta(s)\partial_x f\sum_{\ell>0}\int \mathfrak{A}(p+p_y)(\,\Delta\Psi_{i+\ell}+\Delta\Psi_{i-\ell})\,d\Gamma_y$$

$$+\,\partial_p f\,\delta(s)\int\chi_o(x-y)\,\Delta\Psi_i\,d\Gamma_y=0 \tag{3.2e}$$

Equations (3.2a) and (3.2e) constitute the basis of Chao and Ruth's treatment. The only difference here is the inclusion of l.r. forces. To proceed further, we take advantage of the Von Karman condition by a Fourier transform of equation (3.2e). Here again we follow the procedure described in ugly details in SSC-32:

$$D\Phi_\lambda = \sum_{j=0}^{m-1} \exp(i2\pi j\lambda/m)\, \Delta\Phi_j \tag{3.3a}$$

$$\partial_s D\Phi_\lambda + [D\Phi_\lambda, H_o+V+W] - \delta(s)\partial_x f\, Q(\lambda) \int \mathfrak{A}(p+p_y)\, D\Psi_\lambda\, d\Gamma_y \tag{3.3b}$$

$$+ \partial_p f\, \delta(s) \int \chi_o(x-y)\, D\Psi_\lambda\, d\Gamma_y = 0$$

$$Q(\lambda) = \sum_{\ell=1}^{J} \cos(2\pi\ell\lambda/m) \tag{3.3c}$$

In the next section, we transform (3.3b) into a mapping problem.

4. <u>The transfer map technique.</u>

It is possible to integrate (3.3b) around s=0 taking advantage of the delta function. Notice that f is a stationary solution if it is a function of the Courant-Snyder invariant of H_o+V+W. Furthermore we assume that this Hamiltonian is still quadratic. This is of course incorrect! Also we will assume that the Twiss parameter α is zero at the I.P.. This is correct if the initial H_o has a vanishing α at the I.P.. With all these remarks and assumptions, we can rewrite (3.3b) in the action-angle variables (ϕ, J):

$$\partial_s D\Phi_\lambda + \partial_\phi D\Phi_\lambda/\beta - \delta(s)\, \partial_J f\, \sqrt{(2J\beta)} \sin\phi \int \chi_o(\sqrt{(2J\beta)}(\cos\phi - \cos\phi_y))\, D\Psi_\lambda\, d\Gamma_y$$

$$- \delta(s)\, \partial_J f\, \sqrt{(2J/\beta)} \cos\phi\, Q(\lambda) \int \mathfrak{A}(-\sqrt{(2J/\beta)}(\sin\phi + \sin\phi_y))\, D\Psi_\lambda\, d\Gamma_y = 0 \tag{4.1}$$

We can integrate (4.1) around s=0:

$$D\Phi_\lambda^+ = D\Phi_\lambda^- + \partial_J f \sqrt{(2J\beta)} \sin\phi \int \chi_o(\sqrt{(2J\beta)}(\cos\phi - \cos\phi_y))\, D\Psi_\lambda^-\, d\Gamma_y$$

$$+\partial_J f \sqrt{(2J/\beta)} \cos\phi\, Q(\lambda) \int \mathfrak{A}(-\sqrt{(2J/\beta)}(\sin\phi + \sin\phi_y))\, D\Psi_\lambda^-\, d\Gamma_y \qquad (4.2)$$

At this stage, we assume a water-bag distribution for f. This forces all the perturbations to be at the edge of the beam:

$$f(J) = H(\varepsilon/2 - J)/\pi\varepsilon \qquad (4.3a)$$

$$\partial_J f = \delta(J - \varepsilon/2)/\pi\varepsilon \qquad (4.3b)$$

With this choice (already assumed in (4.2)), one can rewrite $D\Phi_\lambda$ and $D\Psi_\lambda$ in terms of multipole modes:

$$D\Phi_\lambda = \delta(J-\varepsilon/2) \sum_{\ell=-\infty}^{\ell=\infty} f_\ell(s)\, e^{i\ell\phi} \qquad D\Psi_\lambda = \delta(J-\varepsilon/2) \sum_{\ell=-\infty}^{\ell=\infty} g_\ell(s)\, e^{i\ell\phi} \qquad (4.4)$$

With the help of (4.4), equation (4.2) is rewriten :

$$f_\ell^+ = f_\ell^- - (AL_x/2\pi^3\alpha\sqrt{(\beta\varepsilon)})\, Q \sum N_{\ell k} g_k^- - (A\sqrt{(\beta/\varepsilon)}/2\pi^2) \sum M_{\ell k} g_k^- \qquad (4.5a)$$

$$N_{\ell k} = \int_0^{2\pi}\int_0^{2\pi} e^{-i\ell\phi} \cos\phi\, d\phi\, G(-\sqrt{(\varepsilon/\beta)}(\sin\phi + \sin\phi_y)) e^{ik\phi_y}\, d\phi_y \qquad (4.5b)$$

$$M_{\ell k} = \int_0^{2\pi}\int_0^{2\pi} e^{-i\ell\phi} \sin\phi\, d\phi\, H_1(\cos\phi - \cos\phi_y)\, e^{ik\phi_y}\, d\phi_y \qquad (4.5c)$$

The matrices N and M have many properties. Their traces and squares are zero. By factoring (4.5a) we can take advantage of these properties. The reader can check that the map will have a unit determinant. This is

consistant since we worked out the Vlasov operator only to first order in the beam-beam collision. The factorized map is given by :

$$\mathbf{v}^{+} = \left(I - (A\sqrt{(\beta/\varepsilon)}/2\pi^2)M\Delta\right)\left(I - (AL_x/2\pi^3\alpha\sqrt{(\beta/\varepsilon)})N\Delta\right)\mathbf{v}^{-} \quad (4.6a)$$

$$= V\,W_\lambda\,\mathbf{v}^{-}$$

$$v_\ell = (f_\ell, g_\ell) \quad (4.6b)$$

The matrix Δ mixes the two beams. It is fully described in SSC-32. Finally we introduce the phase advance μ between the interaction points (I.P.).

$$R_{(\mu)}\,\mathbf{v}^{+} = R_{(\mu)}\,V\,W_\lambda\,\mathbf{v}^{-} \quad (4.7a)$$

$$R_{(\mu)}\,v_\ell = \exp(-i\mu\ell)v_\ell \quad (4.7b)$$

At the the beginning of this paper we decided to focus on one G.I.P. only. This is permissible because of the symmetry of the ring. Using the group theoritic formulation of SSC-32, we introduce the kth irreducible representation of the ring. In this representation, the transfer map for $1/N^{th}$ is given by:

$$\mathbf{v}^{t+1}_{\lambda k} = T_{\lambda k}\,\mathbf{v}^{t}_{\lambda k} = \Omega_k R_{(\mu)} V W_\lambda\,\mathbf{v}^{t}_{\lambda k} \quad (4.8)$$

The letter t labels the tth collision in the (λ,k) representation. The matrix Ω rotates the two beams **f** and **g** into one another by an angle $2\pi k/N$.
In the next section we determine how one finds the stability curve for the map T.

5. <u>The determination of the stability curve.</u>

By comparison to the dipole studies of SSC-32, we define a tune shift

parameter ξ_o :

$$\xi_o = A/\pi^2\sqrt{\varepsilon} \tag{5.1}$$

Our goal is to find for what value of ξ_o the matrix T becomes unstable. We assume that this happens when one of its eigenvalues is unimodular. Implicitly, we define $\xi_{\lambda k}$ to be the minimum ξ_o which satifies the relation:

$$\det(\mathbf{T}_{\lambda k} \pm \mathbf{I}) = 0 \tag{5.2}$$

Clearly, we want the most unstable mode. This implies that the smallest value of $\xi_{\lambda k}$ must be chosen.The index k is discrete. It is related to the number of I.P. The index λ is practically continuous since it slices the unit circle in "m" sectors (m>2000 for the SSC). Hence we are allowed to differentiate $\xi_{\lambda k}$ with respect to λ:

$$\partial_\lambda \xi_{\lambda k} = \partial_\lambda Q \quad \partial_Q \xi_{\lambda k} = 0 \tag{5.3}$$

One can show in the dipole treatment that (5.3) is realized when $\partial_\lambda Q$ is zero. As shown in SSC-32, this has two solutions:

$$\partial_\lambda Q = 0 \Leftrightarrow (\lambda = 0 \text{ or } \lambda = \lambda_{max}) \tag{5.4}$$

The quantity λ_{max} is only a function of J (# of l.r.) in the pseudo-drift and the telescopic model. This simplifies the search for the stability curve. In the case of the SSC with six sites, we scan through the following values:

$$(\lambda,k) = (0,0)\ ;\ (0,1)\ ;\ (\lambda_{max},0)\ ;\ (\lambda_{max},1). \tag{5.5}$$

Notice that the dimensionality of $\mathbf{T}_{\lambda k}$ is $4\ell_{max}$, where ℓ_{max} is the maximum multipole mode under consideration. The case $\ell=0$ is not considered because it amounts to a change in f(J).

A final word about the phase advance μ and the β function. These are evaluated for the perturbed Hamiltonian H_0+V+W.

At the edge of the water-bag μ_0 is shifted to first order in ξ_0 by:

$$\Delta\mu=(-16/3+J\pi\sqrt{\varepsilon}L_X/\alpha^2)\xi_0 \tag{5.6}$$

For β we use the linear formula:

$$\beta= \sin(\mu_0)/\sin(\mu_0+\Delta\mu) \tag{5.7}$$

In the next section we display the numerical results.

6. The stability curves with long range forces.

The curves we will display here have to be compared on one hand with SSC-32, and SLAC-37 on the other hand. One can show that that the factor P (see SSC-32) relating the h.o. to the l.r. is given here by the formula:

$$P=-L_X\sqrt{\varepsilon}/4\alpha^2 \tag{6.1}$$

We chose L_X in such a way that we reproduce approximatly the P values of SSC-32. In that note we had at 50μrad a P of -0.0396. Here we took the following numerical values:

$$L_X=8\sqrt{\varepsilon} \quad \sqrt{(\varepsilon\beta^*)}=7\ 10^{-6} \quad \beta^*=1\text{m} \Rightarrow P=-0.0392 \tag{6.2}$$

The figures are displayed for the following values of the parameters:

figure#	J	ℓ_{max}	α (μrad)
1	4	2	50
2	4	3	50
3	12	2	50
4	12	3	50
5	12	3(only dipole for the l.r.)	50
6	4	2	10

figure#	J	ℓ_{max}	α (μrad)
7	4	3	10
8	4	3(The l.r. contains the first order in $\sqrt{(\varepsilon/\beta)}/\alpha$)	10

In the first five figures we notice that only the dipole mode is influenced by the l.r. forces. Indeed the l.r. matrix N is a function of $\delta = \sqrt{(\varepsilon/\beta)}/\alpha$. This quantity is about 0.14 at 50μrad. The higher modes will contain higher powers of δ. There are virtually no differences between fig.4 and fig.5 proving the point just mentioned.

The situation is dramatically different at 10μrad. There δ is five times larger ! The results are in qualitative disagreement with SSC-32. In that note we included only one power of δ. In fig.8, we truncated the l.r. matrix N at the first power in δ. The stability diagram went up by a factor of three, in quasi-agreement with SSC-32. The reader will notice the strong l.r. coupling in fig.6 and 7.

The moral is that as the beams approach each other, the situation deteriorates faster than what is predicted by a linear theory in δ ($\delta \approx$size of the beams/separation of the beams).

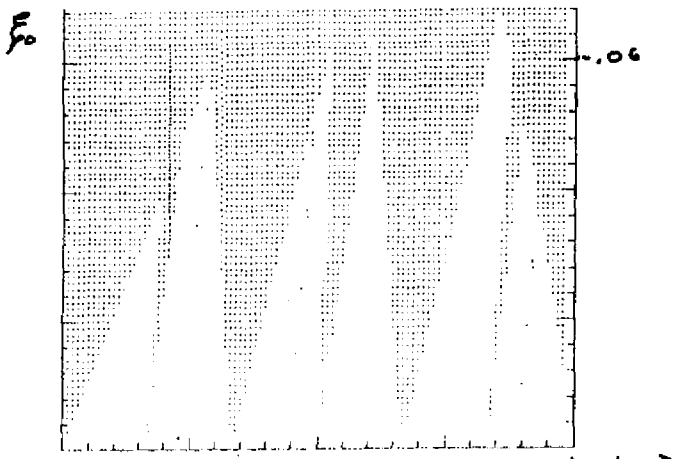


fig 1

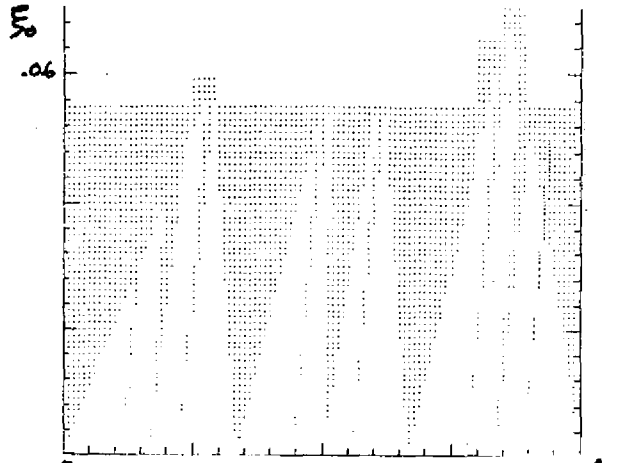
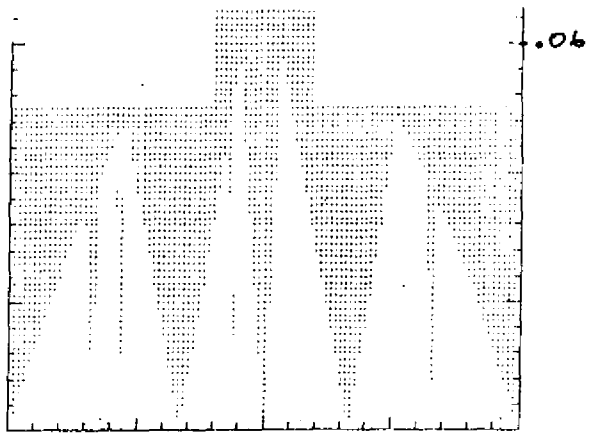


fig 2



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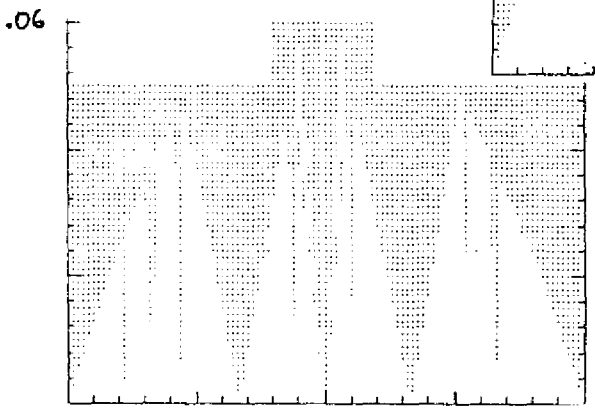
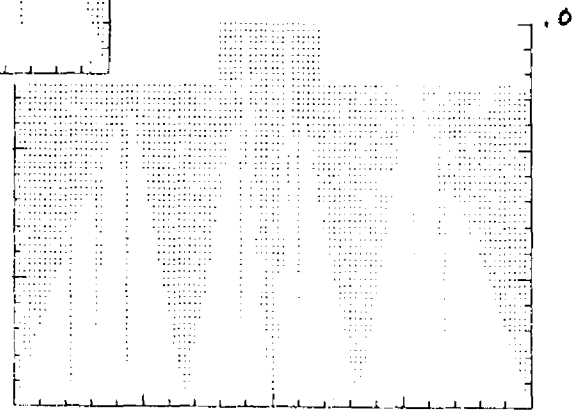


fig 4

fig5



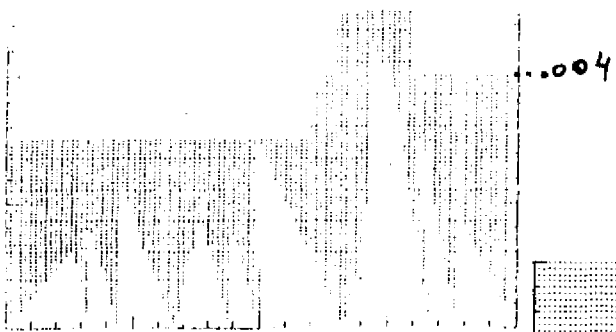


fig 6

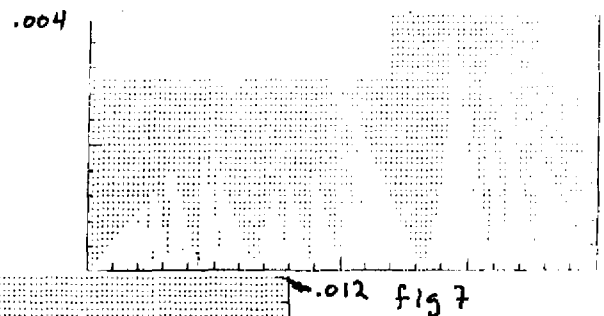


fig 7

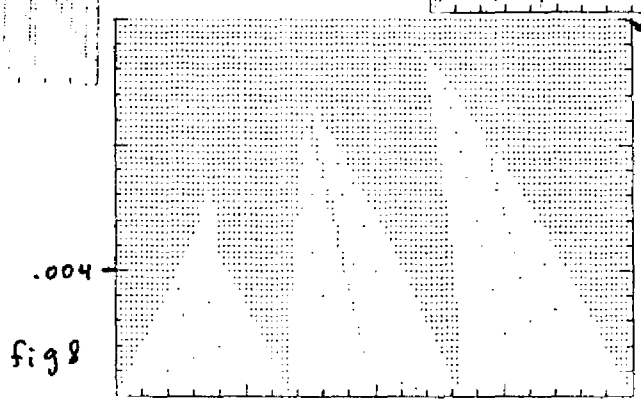


fig 8

Appendix A. Connection between the Vlasov technique and the dipole treatment of SSC-32. The factor P.

We want to derive an expression for $\langle p\rangle^+ - \langle p\rangle^-$ for the h.o. as well as the l.r..

Treating the two types of collisions simultaneously ,we can write a Vlasov equation:

$$\partial_s \Phi + [\Phi, H_o] + \delta(s)\, \partial_p \Phi \int \chi_s\, \Psi\, d\Gamma_y = 0 \qquad \text{(A1)}$$

Multiplying by p and integrating accross the delta function, we get:

$$\langle p\rangle^+ = \langle p\rangle^- - \int p\, \partial_p \Phi\, d\Gamma_x \int \chi_s\, \Psi\, d\Gamma_y \qquad \text{(A2)}$$

We integrate (A2) by parts:

$$\langle p\rangle^+ = \langle p\rangle^- + \int \Phi\, d\Gamma_x \int \chi_s\, \Psi\, d\Gamma_y \qquad \text{(A3)}$$

First, we look at the long range operator. In this case we expand χ_s:

$$\chi_s \approx (AL_x/\pi d^2)\,(y-x) \qquad \text{(A4)}$$

Substitution in (A3) gives immediatly the result for l.r. collisions:

$$\langle p\rangle^+ = \langle p\rangle^- + (AL_x/\pi d^2)\,(\langle y\rangle^- - \langle x\rangle^-) \qquad \text{(A5)}$$

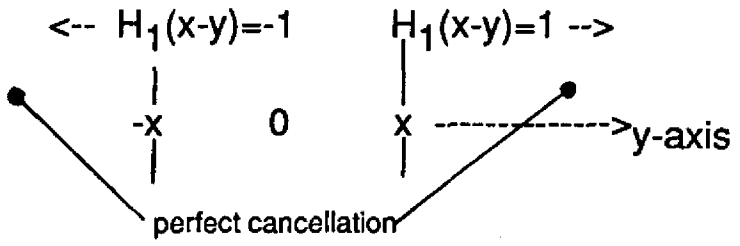
The case of head-on collisions is harder because the operator is not analytic in x-y. The best way to proceed is to linearize the distributions in the integral of (A3):

$$I = A \iint \{\, f(x,p)\, \Delta\Psi(y,p_y) + f(y,p_y)\, \Delta\Phi(x,p) \,\}\, H_1(x-y)\, d\Gamma_x\, d\Gamma_y \qquad \text{(A6)}$$

Calling ρ_o the space density resulting from f, we can derive the relation:

$$\int_{-\infty}^{+\infty} \rho_o(y)\, H_1(x-y)\, dy \approx 2\, \rho_o(0)\, x \qquad \text{(A9)}$$

This mysterious relation is easely understood if we examine the domain of integration of H_1:



Using the symmetry of ρ_o and the antisymmetry of H_1, we can deduce (A9) from the above figure for a slowly varying f. Applying (A9) to the integral (A6), we get the final result:

$$\langle p\rangle^+ = \langle p\rangle^- + 2\, \rho_o(0)\, A(\langle x\rangle^- - \langle y\rangle^-) \qquad \text{(A10)}$$

From this we obtain the factor P of SSC-32, which is simply the ratio of the l.r. over the h.o. times the square of the distance separating them:

$$P = L_x / 2\pi\alpha^2 \rho_o \qquad \text{(A11)}$$

We can substitute the water-bag for ρ_o:

$$\rho_o(x) = (2/\pi\varepsilon)\, \sqrt{(\varepsilon - x^2)} \qquad \text{(A12)}$$

$$P = L_x\, \varepsilon / 4\alpha^2 .$$

Appendix B.The long range matrix $\mathbf{N}$.

In this paper we introduced a new matrix for the long range interaction. In general (when $\alpha_{twiss}\neq 0$) this matrix is easely evaluated in the following set of action-angle variables:

$$x=\sqrt{(2J/\gamma)}\,(\cos\psi+\alpha\sin\psi) \qquad \text{(B1)}$$

$$p=\sqrt{(2J\gamma)}\,\sin\psi \qquad \gamma\text{=twiss parameter}$$

This set is related to the more conventional one used by Chao and Ruth by the relations:

$$\psi=\phi+\tan^{-1}\alpha \qquad \text{(B2)}$$

$$J=J_\phi$$

In this set , the matrix $\mathbf{N}$ takes a simple form even when α_{twiss} is not zero. In fact, is is given by (4.5b) provided that β be replaced by $1/\gamma$.

$$\mathbf{N}_{\ell k}=\int_0^{2\pi}\!\!\int_0^{2\pi} e^{-i\ell\phi}\cos\phi\, d\phi\, G(-\sqrt{(\epsilon\gamma)}(\sin\phi+\sin\phi_y))e^{ik\phi_y}\, d\phi_y \qquad \text{(B3)}$$

We re-express $\mathbf{N}$ by changing the variables of integration:

$$\phi=\psi+\pi/2 \qquad \text{(B4b)}$$

$$\phi_y=\psi_y+\pi/2$$

$$\mathbf{N}_{\ell k}=-\exp(i(\pi/2)(k-\ell))\,\mu_{\ell k} \qquad \text{(B4b)}$$

$$\mu_{\ell k} = \int_0^{2\pi}\int_0^{2\pi} e^{-i\ell\psi} \sin\psi d\psi \, G(-\sqrt{(\varepsilon\gamma)}(\cos\psi + \cos\psi_y)) e^{ik\psi_y} \, d\psi_y \quad \text{(B4c)}$$

The matrix μ has the same properties as the matrix M of Chao and Ruth (equation 4.5c):

μ is pure imaginary (B5)

$$(\mu_{\ell -k} = \mu_{\ell k} \text{ and } \mu_{-\ell k} = \mu_{\ell k} \Rightarrow \mu^2 = 0)$$

From these relations we deduce the properties of N:

$$N_{\ell k}^{*} = N_{-\ell -k} \quad \text{(B6a)}$$

$$N_{\ell -k} = (-1)^{k} N_{\ell k} \quad \text{(B6b)}$$

$$N_{-\ell k} = (-1)^{\ell+1} N_{\ell k} \quad \text{(B6c)}$$

$$\Rightarrow N = 0. \quad \text{(B6d)}$$

By manipulating (B4c), we can derive a recurrence relation for $\mu_{\ell k}$:

$$(\mu_{\ell k+1} + \mu_{\ell k-1} + \mu_{\ell+1\, k} + \mu_{\ell-1\, k}) = (2/\delta)\, \mu_{\ell k} + 2\pi^2 i\, F_{\ell k} \quad \text{(B7a)}$$

$$\delta = \sqrt{(\varepsilon\gamma)}/\alpha \quad \text{(B7b)}$$

$$F_{\ell k} = (\delta_{\ell,1} - \delta_{\ell,-1})(\delta_{k,-1} - \delta_{k,1}) + \delta_{k,0}(\delta_{\ell,2} - \delta_{\ell,-2}) \qquad \text{(B7c)}$$

$\delta_{i,j}$ is the Kronecker delta.

This can be formally solved by introducing a messy generating furnction μ_{xy}:

$$\mu_{xy} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{\ell} y^{k} \mu_{\ell k} \qquad \text{(B8a)}$$

$$\mu_{xy} = -\delta\pi^2 i\,(x - x^{-1}) \sum_{n=0}^{\infty} (\delta/2)^n\,(x + x^{-1} + y + y^{-1})^{n+1} \qquad \text{(B8b)}$$

The quantity δ is usually small compared to one for our approximations to be valid. In our studies, we kept powers as high as n=5. This gives us sextupole long range contributions. It is easy to show using (B8a) that the leading multipole contribution of order ℓ is given by :

$$N_{\ell\ell} = i\pi^2\delta\,(\delta/2)^{2\ell-2}\,(2\ell-1)!/(\ell-1)!\,\ell! \qquad \text{(B9)}$$

Appendix C. Derivation of the shifted to the I.P. Vlasov equation.

We start with equation (2.5).

$$\partial_s \Phi_i + p\,\partial_x \Phi_i - \partial_p \Phi_i \left\{ Kx - \sum_{|\ell| \le J} \delta(s - \ell\lambda) \int \chi_s(x-y)\Psi_{i+\ell}(y, p_y; -s)\, d\Gamma_y \right\} = 0 \qquad \text{(C1)}$$

To solve (C1), we integrate around one value of s, s= $\ell\lambda$ for example:

$$\Phi_i^+(\ell\lambda) = \Phi_i^-(\ell\lambda) - \partial_p\Phi_i^-(\ell\lambda) \int \chi_{\ell\lambda}(x-y)\Psi_{i+\ell}(y,p_y;-s)\, d\Gamma_y \qquad \text{(C2)}$$

To the order of H_o, the function $\Phi_i(\ell\lambda)$ is propagated by the inverse of the Hamiltonian map generated by H_o:

$$\Phi_i(x,p;\ell\lambda) = \rho_{-\ell}\Phi_i(x,p;0) \qquad \text{(C3)}$$

Using this result, we can transform (C2):

$$\Phi_i^+(\mathbf{z};0) = \Phi_i^-(\mathbf{z};0) - \rho_\ell \{ \partial_p\Phi_i^-(\ell\lambda) \int \chi_{\ell\lambda}(x-y)\Psi_{i+\ell}(\rho_{-\ell}^{-1}\mathbf{z}';0)\, d\mathbf{z}' \qquad \text{(C4)}$$

$$\mathbf{z}=(x,p)$$

We must manipulate the derivative in (C4):

$$\rho_\ell\, \partial_p\Phi_i^-(\mathbf{z};\ell\lambda) = \rho_\ell \left\{ \left(\partial\Phi_i^-(\rho_\ell^{-1}\mathbf{z};0)/\partial(\rho_\ell^{-1}\mathbf{z}) \right)\cdot\left(\partial_p(\rho_\ell^{-1}\mathbf{z})\right) \right\} \qquad \text{(C5)}$$

$$= \partial_{\mathbf{z}}\Phi_i^-(\mathbf{z};0) \cdot \partial_p(\rho_\ell^{-1}\mathbf{z})$$

The vector $\partial_p(\rho_\ell^{-1}\mathbf{z})$ is not a function of $\mathbf{z}$ if ρ_ℓ^{-1} is linear. We then substitute (C5) into (C4) and we change the integration variable $\mathbf{z}'$ into $\mathbf{z}''(=\rho_{-\ell}^{-1}\mathbf{z}')$:

$$\Phi_i^+(\mathbf{z};0) = \Phi_i^-(\mathbf{z};0) - \partial_{\mathbf{z}}\Phi_i^-(\mathbf{z};0) \cdot \sum_{|\ell|\le J} \partial_p(\rho_\ell^{-1}\mathbf{z}) \int \chi_{\ell\lambda}(\rho_\ell x - \rho_{-\ell} y)\Psi_{i+\ell}(\mathbf{z}'';0)\, d\mathbf{z}'' \qquad \text{(C6)}$$

To first order, all the kicks are additive, therefore we sum them up in (C6).

Now, one notices that (C6) can be obtained by integrating (2.6) around s=0. Hence the problem of solving (C6) or (2.5) are equivalent.

Appendix D. Computation of the tune shift due to long range forces.

The tune shift $\Delta\mu$ can be computed from the potential V+W. The map of the Hamiltonian H_0 can be represented in Lie algebraic form:

$$\mathfrak{M}_0 = \mathfrak{M}\,\mathfrak{M}_\ell \tag{D1}$$

$$\mathfrak{M} = \exp(:-\mu J:) \tag{D2}$$

$$\mathfrak{M}_\ell = \exp\left(:\int_{0-}^{0+}(V+W)\,ds:\right) = \exp(:\nu+\omega:) \tag{D3}$$

Here, :F: is the usual Deprit Lie operator: $:F:=[F,\ ]_{\psi,J}$.

We now Fourier analyze the Lie operator of $\mathfrak{M}_\ell$:

$$\nu+\omega = k_0(J) + \sum_{\ell\neq 0} k_\ell(J)\, e^{i\ell\psi} \tag{D4}$$

To first order in the perturbation, we can rewrite (D1):

$$\mathfrak{M}_0 = \exp(:-\mu J + k_0:)\ \exp\left(:\sum_{\ell\neq 0} k_\ell(J)\, e^{i\ell\psi}:\right) \tag{D5}$$

CLearly, the first factor will contain the leading tune shifts:

$$\mathfrak{M}_0\psi = \psi + \mu - \partial_J k_0 + \dots\ . \tag{D6}$$

Let us concentrate on the long range ω:

$$\partial_J k_O = (1/2\pi)\, \partial_J \int_0^{2\pi} \nu(p)\, d\psi = -(1/2\pi\sqrt{(2J/\gamma)}) \int \sin\psi\, \nu(p)\, d\psi \qquad \text{(D7)}$$

After substituting for the various functions, we obtain:

$$\partial_J k_O = (JA/\varepsilon\pi^2\sqrt{(2J/\gamma)}) \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\varepsilon/2} dJ'\, \mathfrak{A}(-\sqrt{(2J\gamma)}\sin\psi - \sqrt{(2J'\gamma)}\sin\phi))\, \sin\psi$$

(d8)

For simplicity, we assume that only the leading term in δ matters:

$$\partial_J k_O \approx (JAL_X\gamma/\varepsilon\pi^3\alpha^2) \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\varepsilon/2} dJ'\, \sin^2\psi\, d\psi$$

$$= JAL_X\gamma/\pi\alpha^2 = (J\pi\sqrt{\varepsilon}L_X\gamma/\alpha^2)\xi_o \qquad \text{(D9)}$$

If α_{twiss} is zero, γ can be replaced by $1/\beta$.

This result was used in equation (5.6). The other tune shift can be computed by the same method. One needs to replace ν by ω. This is outlined briefly in Chao and Ruth's paper.

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