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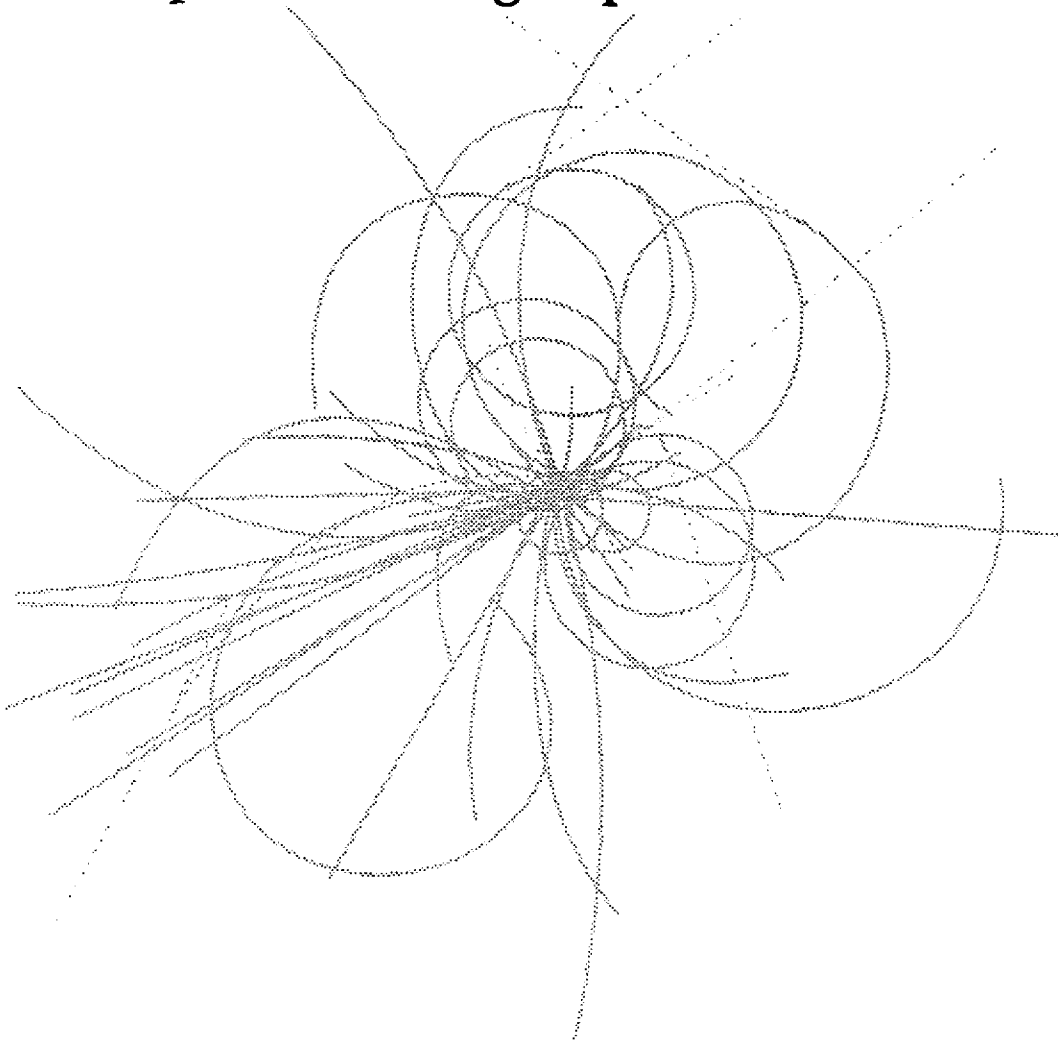

Elementary Design and Scaling Considerations of Storage Ring Colliders

A. Chao

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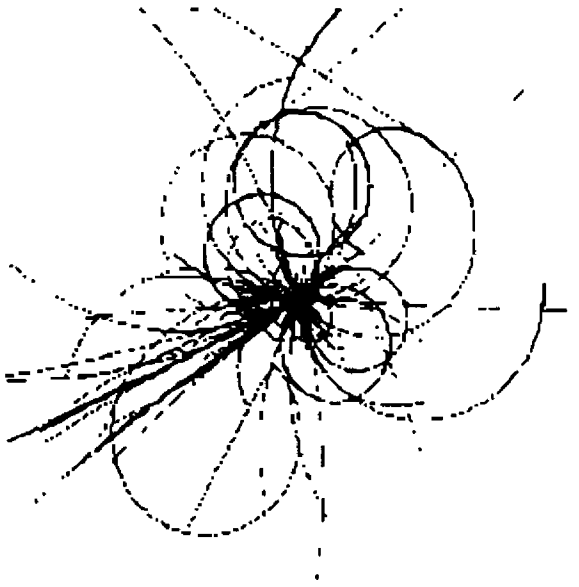
THE SUPERCONDUCTING SUPER COLLIDER LABORATORY



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ELEMENTARY DESIGN AND SCALING CONSIDERATIONS OF STORAGE RING COLLIDERS*

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ABSTRACT

This article is a compilation of some considerations encountered in the design of storage ring colliders. It consists of two chapters. The first chapter describes an approach to the collider design from a particular point of view at an elementary level. The second chapter discusses a few semi-empirical scaling properties in the collider parameters.

I. Elementary Design Considerations

In this chapter, the design of a storage ring collider is described from a particular point of view. We begin with the luminosity requirement on the collider and proceed to describe the various design features needed to provide the luminosity. Design features chosen to be discussed follow from the view point chosen. No attempt has been made to cover all important features in a collider design.

For illustration, a numerical example is carried along with the discussion. To avoid possible confusion the general equations are arranged in a numerical sequence. Parameters used in the numerical example are arranged in an alphabetical sequence.

Luminosity

The end product of a storage ring collider can be summarized by three parameters: the type of particles, the particle energy E and the luminosity $\mathcal{L}$. The type of particles is most likely electron or proton. In the following discussions, we assume that the particle type and particle energy are given and begin by a discussion on luminosity.

Consider a certain type of high energy physics events of interest with cross-section Σ . The counting rate $\mathcal{R}$ of these events in a collider is proportional to Σ . The proportionality constant is called the luminosity, i.e.

$$\mathcal{R} = \mathcal{L}\Sigma \tag{1}$$

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Consider two beam bunches with N particles each colliding head-on. Let the beams have a uniform transverse distribution with area A, as shown in Fig.1. Let f be the frequency for collisions occurring at the collision point under consideration. Luminosity is given by

$$\mathcal{L} = N^2 f/A \qquad (2)$$

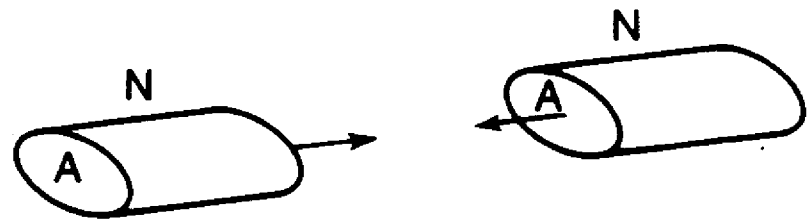

Figure 1. Two colliding bunches.

The collision frequency is related to the revolution frequency f_{rev}

$$f = f_{rev} B, \qquad (3)$$

where B is the number of bunches in each beam. In case the two beams have a round gaussian distribution in the transverse dimensions with rms size σ, the effective beam area is[1]

$$A = 4\pi\,\sigma^2 \qquad (4)$$

It is sometimes convenient to remove the factor N^2 from the luminosity expression (2). The remaining quantity is called the specific luminosity. It depends only on the overlapping geometry of the two transverse distributions and is independent of beam intensities, type of events, particle types and beam energy.

As a numerical example, consider an event type with Σ = 1 picobarn = $10^{-36}\mathrm{cm}^2$. Suppose a counting rate of $\mathcal{R}$ = 1/day is required on this particular event type, the needed luminosity is

$$\mathcal{L} = \mathcal{R}/\Sigma = 10^{31}\mathrm{cm}^{-2}\mathrm{sec}^{-1}. \qquad (5)$$

To achieve this luminosity, a possible set of parameters is

$$N = 10^{11}$$

$$f_{rev} = 10^5\mathrm{sec}^{-1} \qquad (6)$$

$$A = 0.01\ \mathrm{mm}^2$$

$$B = 1.$$

The probability that a particle actually collide with a particle in the on-coming beam is $P = N\Sigma_{tot}/A$, where Σ_{tot} is the total cross-section of collision. Assuming $\Sigma_{tot} = 100$ mb = $10^{-25}cm^2$, the probability of collision is found to be 10^{-10} per crossing. The lifetime of the beam due to actual collisions is thus 10^{10} crossings, which corresponds to 10^5 sec, or about 1 day, since we assumed a revolution period of 10^{-5} sec.

Beam-Beam Effects

We found that the probability of actual collision is extremely small per crossing, meaning the bunch is basically transparent as far as particle crossing is concerned. However, particle motion across the collision points is by no means unperturbed. The perturbation comes from the elastic scattering by the collective Coulomb field associated with the on-coming bunch. This perturbation is referred to as the beam-beam interaction.[2]

The beam-beam interaction constitutes one of the main limiting effects on the luminosity. To achieve a high luminosity, one needs a high beam intensity and a small beam area. These requirements must be made so that the beam-beam effect is not made untolerably strong.

First let the transverse beam distribution be a uniform disc with radius a. Consider a test particle in beam 1 that passes through beam 2 with a transverse displacement x from center, as shown in Fig.2. In the ultra-relativistic limit, the electric field seen by the test particle points in the radial direction perpendicular to its direction of motion. Applying the Gauss' law yields

$$E_r = (2Ne/\ell)\,(x/a^2) \quad \text{if } |x| < a \quad \text{uniform disc} \qquad (5)$$

$$(2Ne/\ell)\,(a/x) \quad \text{if } |x| > a$$

where ℓ is the length of the beam bunch.

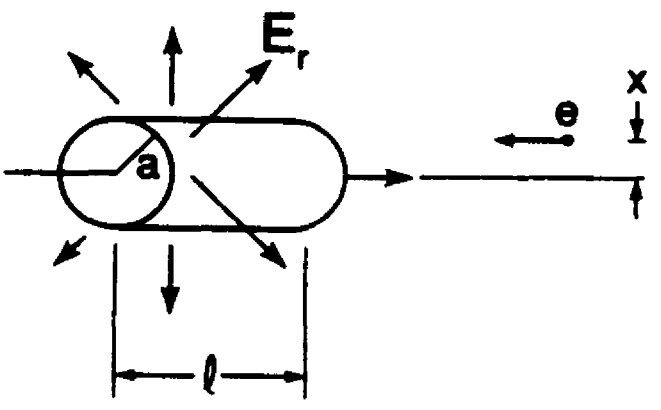

Figure 2. The beam-beam encounter seen by a test charge.

In addition to the electric field, there is a magnetic field B_θ of the same strength (in cgs units) as the electric field. The Lorentz forces due to the electric and the magnetic fields add to give a force twice that due to the electric field alone. Continuin our numerical example with $N=10^{11}$ and $A = \pi a^2 = 0.01 mm^2$ and take ℓ = 10cm, the electric field is found to be 50MV/m and the magnetic field is 1.5KG evaluated at the edge of the beam, x = a, where the fields are maximum.

In case of a round gaussian distribution, the electric field is

$$E_r = (2Ne/\ell)[1-\exp(-x^2/2\sigma^2)]/x \quad \text{round gaussian} \qquad (6$$

The beam-beam interaction imposes limitation on luminosity not because it is extraordinarily strong but because it is extraordinarily nonlinear. The linear part of the force acts like a quadrupole magnet, whose effects can be compensated by adjusting the strengths and arrangements of the neighboring quadrupoles. In particular, for the case of gaussian distribution, the linear force gives rise to a beam-beam tune shift given by[2]

$$\xi = \frac{N r_0 \beta^*}{4\pi \sigma^2 \gamma} \qquad (7$$

where β^* is the β-function at the interaction point, γ is the relativistic factor, r_0 is the classical radius of the particle type under consideration. Unlike the linear part, the nonlinear part of the beam-beam force is not so easy to deal with. Fig.3 shows the beam-beam force as a function of x for gaussian and uniform disc beams. The nonlinearity starts around $x > \sigma$ for the gaussian case. For the uniform disc case, the force inside the bea distribution is strictly linear. No particle feels the nonlinearit since the nonlinear region is not populated. If beams can be prepared with strictly uniform disc distributions at the collision points, it follows that there will be no beam-beam problem. However, there are two potential obstacles to this idea that we wil only mention here. One is obviously the practical difficulty to provide a uniform disc beam. The other is that there is another type of beam-beam interaction -- the type that involves coherent motion of the bunches -- that perturbs beam motion; uniform disc beams are not exempted from the coherent beam-beam effects.

To compare the nonlinearity of the beam-beam force with that of a multipole magnet field error, note that the beam-beam force deviates from linearity at a transverse distance of the order of the beam size at the interaction point (of the order of 0.1 mm) while a magnet field nonlinearity has the characteristic distance of the magnet coil or gap size (of the order of centimeters).

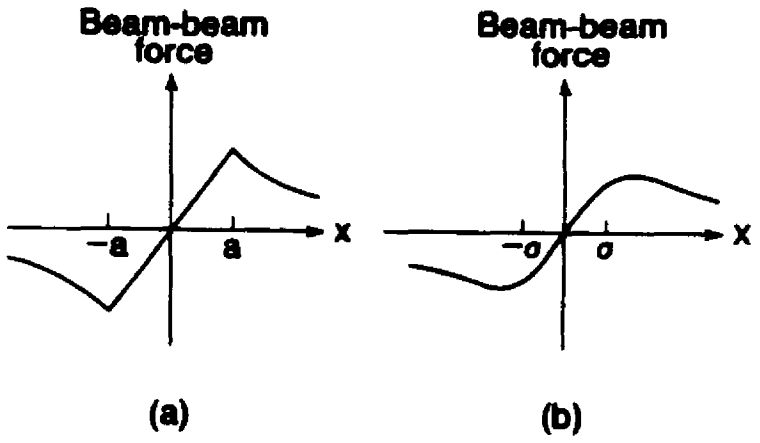

Figure 3. Beam-beam force as a function of the transverse displacement of the test particle for (a) a uniform disc beam and (b) a round gaussian beam.

It turns out that the strength of the beam-beam nonlinearity is specified by the same quantity that specifies the linear part of the beam-beam force, namely the beam-beam tune shift ξ. To control the beam-beam nonlinear effects, it is therefore necessary to limit ξ. One of the fundamental constants in the design of a storage ring collider is in fact the maximum ξ that allows stable motion of particles in the presence of the nonlinear beam-beam perturbation. For the purpose of our discussions, we assume without elaboration that the beam-beam limit is reached at

$$\xi_{\text{limit}} = 0.005 \text{ for protons}$$

$$0.05 \text{ for electrons} \qquad (8)$$

By comparing the expressions (2) and (7), the condition imposed by the beam-beam limit puts a limit on gaining luminosity by increasing N and decreasing A. However, there is still the factor β^* in expression (7) free to be used for optimization.

To proceed, we need to consider the condition imposed by the beam emittance. For a round beam, the beam emittance is given by

$$\varepsilon = \sigma^2/\beta^* \qquad (9)$$

For protons, ε is inversely proportional to γ due to the adiabatic damping. For electrons, ε is proportional to γ^2 due to quantum emission effects.[1] For a given beam energy, we assume for now that the beam emittance is given. The luminosity and the beam-beam tune shift are then rewritten as

$$\mathcal{L} = \frac{N^2 f}{4\pi \epsilon \beta^*} \qquad (1$$

$$\xi = \frac{N r_0}{4\pi \epsilon \gamma}$$

The only free variable that appears in ξ is the number of particles per bunch N. Take a proton storage ring for example, we may have a normalized emittance $\epsilon\gamma = 2\times10^{-6}$m-rad, the beam-beam limit is reached at $N = 10^{11}$, i.e.

$$\xi = 0.005 \qquad ($$

With N given, luminosity can still be improved by increasing th number of bunches per beam B and by decreasing β^*. A small β^* is beneficial because (a) it makes the beam area A small which in turn makes higher luminosity and (b) it makes particle motion less sensitive to the nonlinear beam-beam perturbation. In the following, we will set B = 1 and concentrate on the choice of β^*.

<u>Low-β^* Insertion</u>

We have assumed in our numerical example a revolution frequency of $f_{rev} = 10^5 sec^{-1}$. This means the storage ring has a circumference of 3Km or the ring radius is about 500m. For a superconducting proton storage ring, this means a beam energy of 50 GeV using a scaling property to be discussed in the next chapter, i.e.

$$R = 500 \text{ m}$$

$$E = 500 \text{ GeV}. \qquad ($$

The value of γ is therefore about 500 and the beam emittance is

$$\epsilon = 2 \times 10^{-6} \text{ m-rad}/500 = 4 \times 10^{-9} \text{m-rad}. \qquad ($$

Substituting in Eq.(10), we find that the design goal of $\mathcal{L} = 10^{31} cm^{-2} sec^{-1}$ can be reached if

$$\beta^* = 0.2 \text{ m}. \qquad ($$

This value of β-function at the interaction point is to be compared with the average β-function in the storage ring. For a storage rin with 500m radius, as will be explained in the next chapter, the average β-function is approximately given by

$$\langle\beta\rangle = 22 \text{ m}. \qquad ($$

We thus conclude that there needs to be a special lattice insertion consisting of a sequence of quadrupole magnets to focus the β-function from an average value of 22 m down to 0.2 m at the interaction point. This special insertion -- the low-β* insertion -- is an important invention that has strongly enhanced the luminosity of storage ring colliders.[3, 4] The price to pay is that it also makes the optics of the storage ring quite strained, as will be discussed next.

Optical Aberrations Due To Small β*

There are several limitations in reducing β* indefinitely in order to gain luminosity. Practical limitation on the strength of the low-β* insertion quadrupole magnets is one example. Another limitation is that β* should not be smaller than a length of the order of the bunch length.[5, 6] These limitations, however, will not be discussed below. Instead, we discuss a third limitation, i.e. the aberration of the storage ring optics caused by the low-β* insertions.

At the start, a storage ring is composed of bending magnets and quadrupole magnets -- bending magnets to guide the trajectory of particles and quadrupoles to provide the focussing. The β-function around the storage ring is sketched in Fig.4. The insertion has produced a small β* at the interaction point, but it also produces a large β-function, which we assume to be $\hat{\beta}$ = 500m, at the insertion quadrupole magnets.

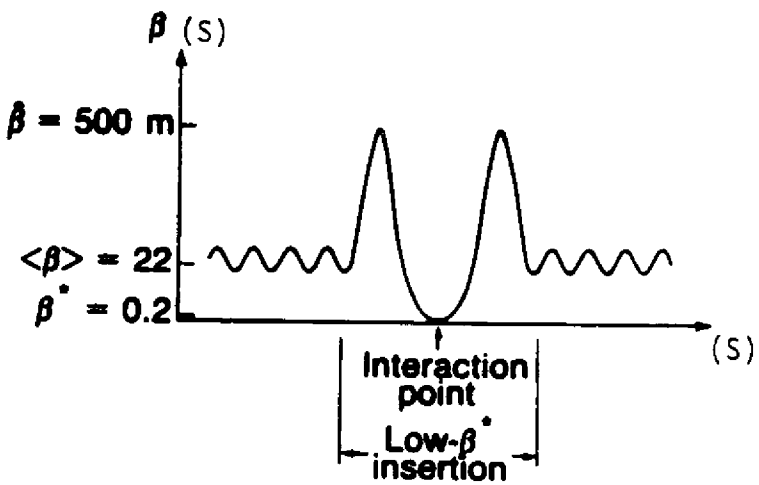

Figure 4. Sketch of the β-function in a storage ring collider.

If the particles in the beam do not have any energy spread, a storage ring consisting of only bending and quadrupole magnets will satisfactorily produce the behavior shown in Fig. 4 and there will be no optical restriction on indefinitely reducing β*. The motion of particles maybe wild going through the variation of β-function but the motion is linear and is perfectly stable.

The difficulty arises when the beam has a finite energy spread. The low β^*, and thus the large $\hat{\beta}$ at the strong insertion quadrupoles, has associated with it a strong chromatic optical aberration. To see that, consider the effect of a quadrupole magnet on the motion of an off-momentum particle with energy error $\delta = \Delta E/E$. The kick angle is given by

$$\begin{aligned} \Delta x' &= K\, x \\ &= K_0\, x \,/\, (1+\delta) \\ &= K_0\, x\, (1 - \delta + \delta^2 - \ldots\ ) \end{aligned} \tag{11}$$

where K_0 is the quadrupole gradient seen by an on-momentum particle. The factor $1/(1+\delta)$ represents the rigidity in the kick to the off-momentum particle under consideration. This rigidity factor is expanded in Eq.(11) into a power series to show its nonlinear behavior in δ. It is very nonlinear even it may not look like so.

One consequence of Eq.(11) is that the betatron tunes for an off-momentum particle will be different from the on-momentum values. In particular, one can define a quantity called chromaticity ν' to be the linear variation of tune with δ, i.e.

$$\nu = \nu_0 + \nu'\delta + \ldots\ldots \tag{12}$$

In a storage ring with only bending and quadrupole magnets, ν' tends to be negative because the focussing effect provided by the quadrupoles is weaker for a higher energy particle. There are of course two chromaticities, one for the horizontal tune and one for the vertical tune.

It is mathematically possible to design a storage ring with only bending and quadrupole magnets and achieve $\nu' = 0$. These designs however tend to give unacceptably strong nonlinearities in higher orders in δ. So far no such design has yet been regarded as practical.

Take now an electron storage ring collider for example. As we will discuss in the next chapter, the rms beam energy spread almost always is of the order of $\sigma_\delta = 10^{-3}$. Furthermore, in electron storage rings, the momentum aperture needed is of the order of $10\ \sigma_\delta$ for the purpose of maintaining a good quantum lifetime,[1] leading to a needed total energy span of about ±1%.

The finite energy spread, together with a finite chromaticity, gives a finite spread in the tunes. Since the tune spread in a storage ring is limited to avoid resonances, we need to impose the condition

$$\left| \nu'\delta \right| \leq 0.03, \tag{13}$$

the absolute value of the chromaticities must not exceed a value of the order of 3 or so.

There is another restriction on the chromaticity due to the head-tail instability effect,[7, 8] which we will not elaborate on. To avoid the head-tail instability of beam motion, it is necessary to have a positive chromaticity, in conflict to the natural tendency. The range of acceptable chromaticity is thus approximately $0 < \nu' \lesssim 3$.

A low-β^* insertion contributes to the chromaticity a term approximately given by

$$\nu' = \frac{1}{2\pi}\sqrt{\frac{\hat{\beta}}{\beta^*}} \quad \text{per insertion} \tag{14}$$

The smaller we make β^*, the higher $\hat{\beta}$ becomes, and the larger the chromaticity contribution from each insertion. In our numerical example, there is a chromaticity contribution of about -10 from each insertion, which is large and negative.

In a storage ring consisting of only bending and quadrupole magnets, therefore, the on-momentum particles enjoy a purely linear motion and no instability limit but the motion of an off-momentum particle will most likely be unstable because of unfortunate tunes. Fig.5(a) shows schematically the stable region in an aperture diagram.

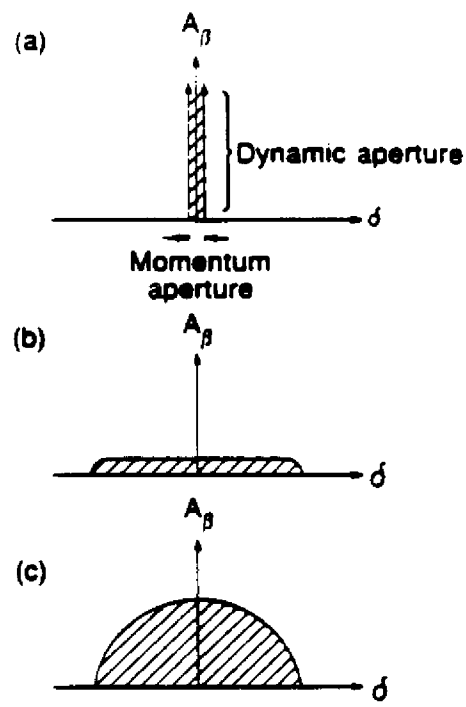

Fig. 5 Aperture diagram for a storage ring that consists of (a) dipole and quadrupole magnets only, (b) dipole, quadrupole and sextupole magnets, and (c) same as (b) but with special sextupole arrangements. A_β is the betatron amplitude, δ is the relative energy error. Shaded regions indicate region of stable motion.

In the aperture diagram, the maximum stable betatron amplitude is called the dynamic aperture. The maximum stable energy width is called the momentum aperture. Fig.5(a) shows an infinite dynamic aperture and a very small momentum aperture in the case when the ring consists of only bending and quadrupole magnets. The momentum aperture is too small to be acceptable.

Sextupoles

The question is then how to control the chromaticities. The answer is to install sextupole magnets in the storage ring. Sextupoles have the property that they act like quadrupoles when the center of the beam passes through them off-centered horizontally. We recall that a particle with $\delta \neq 0$ has its closed orbit displaced horizontally by an amount $\eta\delta$, where η is the horizontal dispersion function. The kick given by a sextupole is therefore

$$\begin{aligned} \Delta x' &= S(x + \eta\delta)^2 \\ &= S\,x^2 + 2S\,\eta\,x\,\delta + S\,\eta^2\,\delta^2 \end{aligned} \tag{15}$$

There are three terms in Eq.(15): one good, one bad and one irrelevant. The good term is the middle one which is linear in x, providing a quadrupole type of action and contributing an additional term to the chromaticity. If the sextupole polarities are such that its equivalent quadrupole field is defocussing for particles with $\delta > 0$ and focussing for $\delta < 0$, it is possible to make the net chromaticity small and positive. The momentum aperture is thus greatly increased. To control the two chromaticities, there need to be at least 2 sextupole families. The third term is Eq.(15) gives rise to a dispersion second order in δ. We do not consider its effect here.

The bad term is the first term in Eq. (15). It produces a serious side effect due to its nonlinear nature in x. As a result, although we have compensated a large fraction of the chromatic aberrations, we have introduced new nonlinearities in the x variable, which substantially suppress the dynamic aperture. The situation is sketched in Fig.5 (b). The achieved stable region is still not acceptable.

Sextupole Schemes

It is possible to improve the situation substantially by cleverly choosing the locations and strengths of the sextupoles. The idea is to make the sextupole nonlinearities to cancel among themselves to a large extent. There are a few schemes to do that but we will describe one that is particularly simple, namely the achromat scheme.[9]

The spirit of the achromat scheme can perhaps be summarized by the simple principle of always forming pairs. In other words, if a sextupole is needed in the lattice, do not insert one sextupole;

insert a pair instead. A properly arranged sextupole pair has much less nonlinear content than a single sextupole alone.

To make a sextupole pair, the two sextupoles of equal strength are spaced by a -1 transformation in the betatron motion, as shown in Fig.6. It is easy to show that if a particle enters the first sextupole with coordinate and slope of (x_0, x_0'), the coordinate and slope of the particle as it exits the second sextupole is $(-x_0, -x_0')$, independent of the existence of the sextupoles. The nonlinear effects of the sextupoles thus cancel each other as far as the betatron motion (the bad term) is concerned. On the other hand, the focussing effect for off-momentum particles (the good term) can be arranged to be additive between the two sextupoles, yielding the needed chromaticity control.

Fig.5(c) shows the aperture diagram when sextupoles are arranged to minimize their betatron nonlinear effects. The arrangement does not affect the momentum aperture much but it increases the dynamic aperture substantially. Hopefully the operation region needed by the beam is inside the stable region finally achieved.

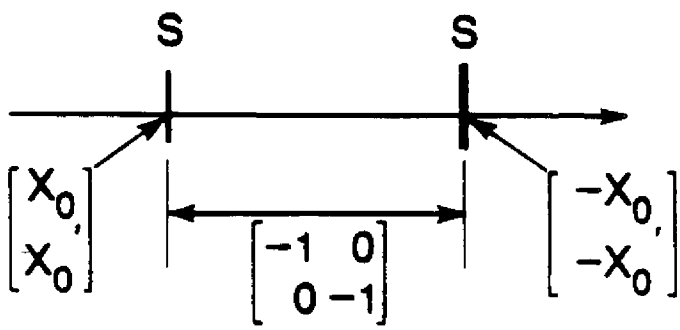

Figure 6. A sextupole pair used in the achromat scheme.

<u>Recap</u>

We started the discussion with the end product, in particular the luminosity, requirements. Particle dynamics was introduced by asking the question what does a particle experience as it collides with the on-coming beam. The beam-beam interaction was then discussed. It was pointed out that the beam-beam interaction is a highly nonlinear effect. To limit its damage to the beam stability, a low-β^* insertion is needed around each collision point. It was then found that these low-β^* insertions have severe side effects; they cause strong chromatic aberrations which impose on the rest of the lattice design. To compensate for the chromatic aberrations, sextupoles are needed. The sextupoles in turn have their unfortunate side effects; to minimize those effects, it is necessary to arrange the sextupoles according to some clever scheme.

The design of a storage ring collider obviously does not stop here. The discussions offer only a possible elementary view, with its over-simplifications, of the design effort from a particular angle of single particle dynamics. To proceed further along this line, it is necessary to discuss another source of optical aberration, i.e. that from the magnet field errors. After that, there is the study of the effects due to the magnet alignment errors. Then there are issues such as how to make closed orbit, dispersion function, β-function and linear coupling corrections. One shortcoming in discussing the design along this direction is that the important subject of collective effects is not discussed.

II. Some Scaling Properties

In the previous chapter, we have carried along a numerical design example of a proton storage ring. We assumed it has a radius of R = 500m. We mentioned in Eqs.(d) and (g) that for a storage ring of this size, the beam energy is going to be 500 GeV, average β-function is 22m. We also mentioned that an electron storage ring collider will have an rms energy spread of $\sigma_\delta = 10^{-3}$, regardless of its energy or size. These results are explained in this chapter by a few general scaling properties of storage ring collider parameters.

We first list three scaling laws together with the "experimenta data" that confirm them. A few corollaries are derived from these laws. We then give the "mathematical proofs" of these laws. No attempt of being rigorous has been made. The three semi-empirical scaling laws relate the design beam energy E, the ring radius R and the betatron tune ν.

<u>The first law</u>: $\nu = \sqrt{R}$.
To apply this law, take the ring radius R in unit of meters and tak the square root to obtain an approximate value for the betatron tune ν. This law applies to both electron and proton storage ring colliders.

<u>The second law</u>: $E = \sqrt{R}$.
This law applies only to electron rings. Again, R is expressed in meters. The result expressed in GeV is approximately equal to the beam energy.

<u>The third law</u>: E = R for superconducting ring, R/4 for conventional ring.
This law applies to proton rings. R and E are expressed in meters and GeV, respectively.

Before discussions of these scaling laws, Figs.7 and 8 are the experimental data confirming them. Fig.7 is a plot of the horizontal betatron tune versus the ring radius for several electror and proton storage rings. The dashed line represents the predictio

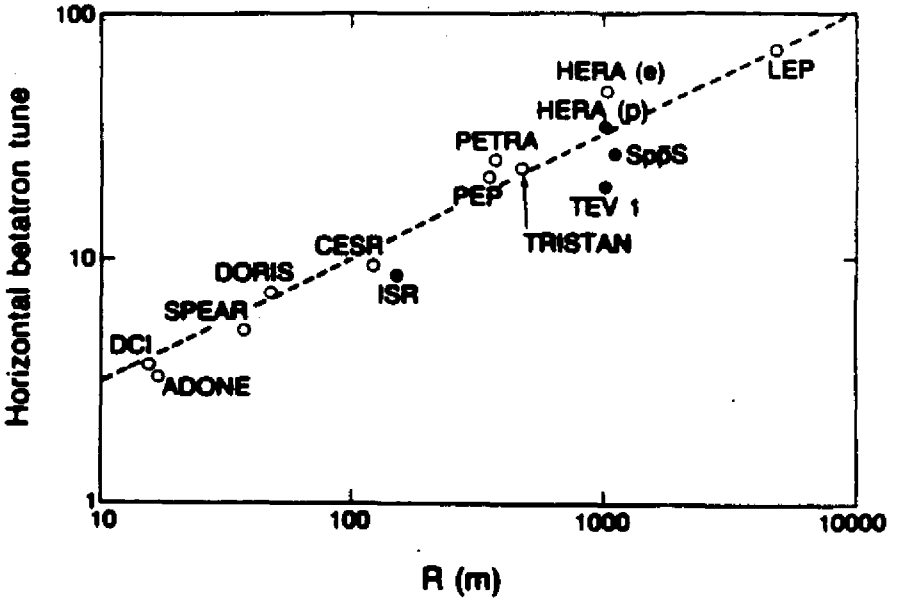

Figure 7. Data showing scaling of betatron tune with storage ring radius.

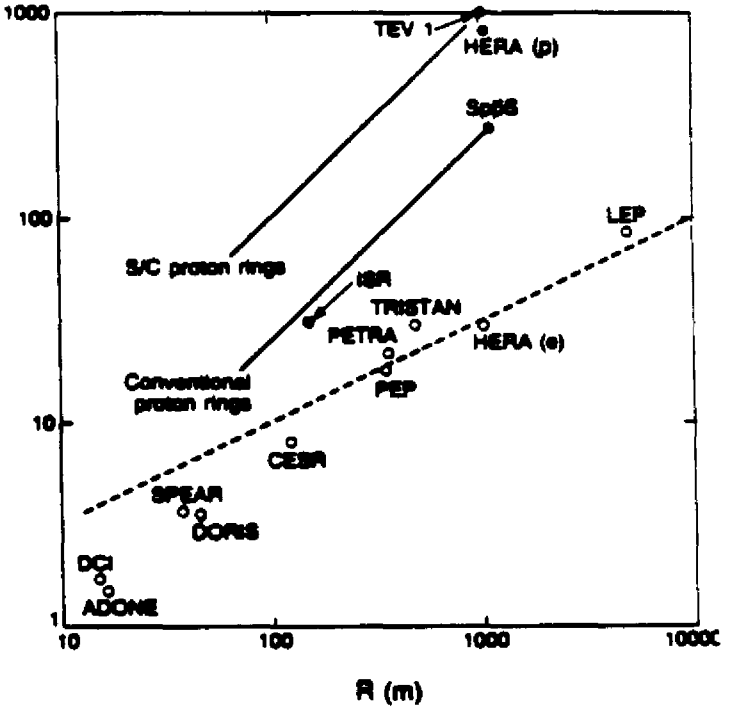

Figure 8. Data showing scaling of beam energy with storage ring radius.

according to the first scaling law. Fig.8 is a plot of the beam energy versus ring radius. The dashed line represents the second scaling law. The solid lines represent the third scaling law. We see that the experimental results fit the first and the second laws quite well. As to the third law, not yet enough data points are available.

Corollaries

There are a few corollaries of the first law. First, we learn from elementary lattice theory that the average β-function $\langle\beta\rangle$ is given by R/ν; therefore applying first law gives

$$\text{corollary:} \quad \langle\beta\rangle = \sqrt{R} \tag{16}$$

where $\langle\beta\rangle$ is in meters. Second, the average dispersion function $\langle\eta\rangle$ is given by R/ν^2, therefore

$$\text{corollary:} \quad \langle\eta\rangle = 1\text{m}, \tag{17}$$

which says all collider rings have average dispersion function of 1m, regardless of whether it is electron or proton ring, or its size. A third corollary applies to the momentum compaction factor α. It is approximately equal to $1/\nu^2$. Therefore,

$$\text{corollary:} \quad \alpha = 1/R \tag{18}$$

where R is in meters, α is dimensionless.

Our numerical example is meant to be a superconducting proton ring. It has R = 500m. Applying the corollary (16), it has an average β-function of $\sqrt{500}$ = 22m. Applying the 3rd scaling law, the ring energy is 500 GeV. As a more realistic example, the electron storage ring PEP has R = 350m. The scaling laws predict a betatron tune of $\nu = \sqrt{350} = 19$, a beam energy of $\sqrt{350}$ = GeV, and an average β-function of $\sqrt{350}$ = 19m. These values agree quite well with the PEP data. It is also interesting to see that ν, E in GeV, and $\langle\beta\rangle$ in meters are approximately equal for electron storage rings.

The above scaling laws obviously do not have rigorous proofs. However, some understanding of storage ring designs can be obtained by studying the underlying reasons of their validity, even an approximate one. We shall begin with the proof of the third law, which is the most straightforward among the three.

Proof of third law

For a proton ring, the limit is the strength of the bending magnetic field. The higher the beam energy, the larger the ring has to be in proportion. Thus $R \propto E$ and the third scaling law is basically proven. The difference between the conventional and the superconducting rings is due to their different bending strengths.

Proof of second law

This law has two proofs, a wrong one and a right one. The wrong proof associates the scaling with a cost minimization consideration. The storage ring cost is first written as the sum of two terms,

$$\text{cost} = \text{ring cost} + \text{rf cost} \tag{19}$$

where the ring cost refers to the cost of magnets, tunnel, etc. The rf cost refer to the cost of the rf components needed to compensate for the energy loss by an electron due to synchrotron radiation. The ring cost is proportional to R and is basically independent of the beam energy. The rf cost on the other hand is proportional to the synchrotron radiation loss per turn, which is proportional to E^4/R. Thus,

$$\text{cost} = fR + g\,E^4/R. \tag{20}$$

By differentiating the cost expression with respect to R for given beam energy E, the minimum cost is obtained when

$$f - gE^4/R^2 = 0,$$

or

$$R \propto E^2 \text{ or } E \propto \sqrt{R}, \tag{21}$$

thus proving the second law.[10]

There is nothing wrong with the argument itself. In fact, this exercise leads to the important conclusion that the size and cost of large electron storage rings increases quadratically with increasing beam energy, unlike proton rings whose size and cost increase only linearly. As a consequence, super electron storage rings are unfavorable as compared with electron linac colliders.[10,11] The reason this proof is incorrect here is that it applies only when the rf cost becomes a significant part of the total cost. This does not happen until the beam energy is of the order of 30 GeV or so. Yet the scaling law is approximately valid already around a few GeV.

A more basic reason of the second law is from the beam dynamics considerations. We mentioned before that sextupoles are inserted in the ring design to open up a finite momentum aperture for the beam. It is important that the beam energy spread does not exceed the momentum aperture so painstakingly achieved. In an electron ring, the rms energy spread is given by[1]

$$\sigma_\delta^2 = \frac{55}{64\sqrt{3}} \frac{\hbar}{m^3c^5} \frac{E^2}{R} = 10^{-6} \frac{E(\text{Gev})^2}{R(\text{m})} \tag{22}$$

where $\hbar$ is the Planck's constant, m is the electron mass, c is the speed of light.

If we use a high bending field of a conventional magnet and E=R/4 as the third law predicts, for example, the energy spread will increase as beam energy increases until the momentum aperture (taken to be $\pm 10\sigma_\delta = \pm 1\%$ in our discussion) is reached around 4 GeV or so. Beyond 4 GeV, it is necessary to impose the scaling $E = \sqrt{R}$, thus proving the scaling law.

Inserting the scaling into Eq.(22), one obtains, for all electron collider storage rings, independent of the design energy E and the ring size.

$$\text{corollary:} \quad \sigma_\delta = 10^{-3}. \tag{23}$$

Proof of first law

We offer a proof for the electron case only. Basically what happens is that when ν is chosen too large, there will have to be too many quadrupoles, which makes the ring expensive. On the other hand, if the tune is chosen too small, the beam size becomes large. The transverse bore size of the magnets becomes large correspondingly until at a certain point beyond which any further increase in magnet bore size will sharply drive up the magnet cost.

The rms horizontal betatron beam size $\sigma_{x\beta}$ of an electron beam is[1]

$$\sigma_{x\beta}{}^2/\beta = 2\,\sigma^2_\delta\, R/\nu^3 \tag{24}$$

where β is the β-function at the point where beam size is observed. Inserting Eq.(23) for σ_δ and replacing β by its average value R/ν, we obtain the average rms horizontal betatron size of

$$\sigma_{x\beta} = \sqrt{2}\; 10^{-3}\; R/\nu^2 \tag{25}$$

To restrict $\sigma_{x\beta}$ under a realistic value, ν has to keep in pace at least as fast as $\sqrt{R}$ as R increases. To minimize the number of quadrupole magnets, ν is chosen to be equal to $\sqrt{R}$. This explains the first law for electron rings.

Substituting $\nu = \sqrt{R}$ in Eq.(25), we obtain

$$\text{corollary:} \quad \sigma_{x\beta} = \sqrt{2}\ \text{mm}. \tag{26}$$

Equation (26) refers to the case in the absence of coupling between the two betatron dimensions x and y. In case of a total coupling, the betatron sizes are $\sigma_{x\beta} = \sigma_{y\beta} = 1\text{mm}$.

The total horizontal beam size is a quadratic sum of the betatron and the synchrotron sizes. In the absence of coupling, its value is given by

$$\sigma_{x,\,tot} = \sqrt{\sigma^2_{x\beta} + \eta^2\sigma^2_\delta} = \sqrt{3}\ \text{mm} \tag{27}$$

where we have used the property that η = 1m and $\sigma_\delta = 10^{-3}$. Equations (26) and (27) apply to all electron rings independent of ring size and design energy.

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