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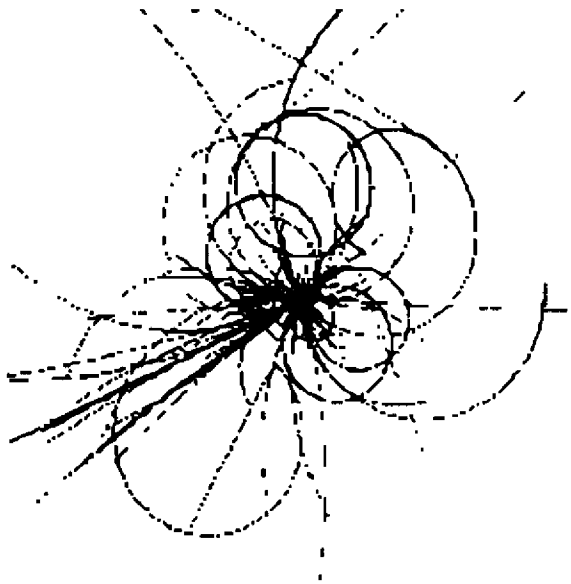

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EQUIVALENCE OF MICHELOTTI'S NORMAL FORM AND THE MAP NORMAL FORM AS USED BY THE MARYLIE CODE

Étienne Forest
SSC Central Design Group

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Introduction

The purpose of this paper is to establish the connection between Michelotti's normal form and our approach. We will show that they are equivalent. Our demonstration will be carried only to order ε in the perturbation. To carry it further is just a matter of "bookkeeping" and is not so essential.

First of all we start with Hamilton's equations in Floquet variables as Michelotti does.

I. The Hamiltonian of the System in Linear Floquet Variables

If the system described represents a circular machine it must have periodicity in the path length s. Without loss of generality we can write for the Hamiltonian K:

$$K(\zeta;s) = K(\zeta;s+1) \tag{1.1a}$$

$$K = K_2 + \varepsilon V \tag{1.1b}$$

$$\zeta = \text{set of canonical variables.} \tag{1.1c}$$

Michelotti's algorithm assumes that all the planes undergo oscillations. It could represent the transverse phase space without synchrotron oscillations and on momentum ($\delta p = p_t = 0$). It could also represent a ring with a rf-cavity turned on. This translates in the existence of Floquet variables. To get the Floquet variables, we solve the linear problem for one period. This is represented by the equation:

$$\frac{d}{ds}M_2(\zeta_o;s) = :-K_2(\zeta_o;s):M_2(\zeta_o;s) \tag{1.2a}$$

$$\zeta(s) = M_2(\zeta_0;s)\zeta_0 \tag{1.2b}$$

By definition $M_2(\zeta_0;s)$ brings the initial ζ_0 to its final linear value $\zeta(s)$. Let us denote $M_2(\zeta_0;1)$ by $\bar{M}_2$. The assumption of oscillations in all planes is mathematically equivalent to the following statement:

$$\exists A_2 \in \left\{\begin{matrix}\text{Linear Symplectic}\\ \text{Maps}\end{matrix}\right\};\ A_2(\zeta_0)\ \bar{M}_2 A_2^{-1}(\zeta_0) = \Lambda_2(\zeta_0)$$

where

$$\Lambda_2 = \exp\left(:\sum_i -\nu_i\pi(X_{i,o}^2 + p_{i,o}^2):\right) \quad . \tag{1.3}$$

Now, we make the following change of variables:

$$\xi(s) = F_2^{-1}(\zeta;s)\,\zeta(s) \tag{1.4a}$$

$$F_2(\zeta;s) = \Lambda_2^{-s}(\zeta)\ A_2(\zeta)M_2(\zeta;s) \tag{1.4b}$$

One can show (see appendix) that ξ undergoes pure linear oscillations when ϵ is set to zero. In fact, K is changed into H according to the formula:

$$H(\xi;s) = \lambda_2 + \epsilon W \tag{1.5a}$$

$$\lambda_2 = \pi \sum_i \nu_i\ (X_i^2 + p_i^2) \tag{1.5b}$$

$$W = F_2\ (\xi;s)\ V(\xi;s) \tag{1.5a}$$

This new Hamiltonian is the starting point of Michelotti's treatment.

II. Computation of the Map for One Turn to Order ϵ

According to the Lie algebraic treatment, the map for one turn can be factorized:

$$M = \ldots \exp(:\epsilon f:)\exp(: - \lambda_2:) \tag{2.1}$$

The function f is related to W through the equation:

$$f(\xi_o) = -\int_0^1 \exp(:-s\lambda_2:)\ W(\xi_o;s)ds \tag{2.2}$$

To proceed further we decompose f in eigenfunctions of $:\lambda_2:$. The linear eigenfunctions of $:\lambda_2:$ are given by the equations

$$V_i^{\pm} = X_{i,o} \mp ip_{i,o} \tag{2.3a}$$

$$:\lambda_2:V_i^{\pm} = \pm i2\pi\nu_i V_i^{\pm} \tag{2.3b}$$

In general, W can be expanded in powers $v_i^{\pm}$. For the sake of the argument, let us assume that the system has three degrees of freedom (rf-cavity turned on). Then W becomes:

$$W(\xi;s) = \sum_{\phi} A_{\phi}(s)|\phi> \tag{2.4a}$$

$$\phi = \left\{(a-b)\nu_1 + (c-d)\nu_2 + (e-f)\nu_3\right\}2\pi$$

$$|\phi> = V_1^{+a}\ V_1^{+b}\ V_2^{+c}\ V_2^{+d}\ V_3^{+e}\ V_3^{+f} \tag{2.4b}$$

If we compute $f(\xi_o)$ in the $|\phi>$ basis the result is given by:

$$f(\xi_o) = \sum_{\phi} f_{\phi}|\phi> \tag{2.5a}$$

$$f\phi = -\int_0^1 \exp(-is\phi)\ A_{\phi}(s)\ ds \tag{2.5b}$$

We are now ready for the calculation of the map A_ε, which brings M into normal form.

III. The Map A_ε

Suppose we apply a similarity transformation A_ε on M.

$$A_\varepsilon = \exp(:\varepsilon g:) \tag{3.1a}$$

$$M^+ = A_\varepsilon \, M \, A_{-\varepsilon} \tag{3.1b}$$

To order ε, we can combine the exponents in (3.1b):

$$\begin{aligned} M^+ &= \exp(:\varepsilon g:) \exp(:\varepsilon f:) \exp(:-\lambda_2:) \exp(:-\varepsilon g:) \\ &= \exp(:-\lambda_2:) \exp(\varepsilon:(-\exp(:-\lambda_2:)+I)g+f:) + O(\varepsilon^2) \\ &= \exp(:-\lambda_2:) \exp(\varepsilon:\Lambda:) \end{aligned} \tag{3.2}$$

We can decompose Λ and g in the $|\phi\rangle$ basis. The equation for Λ becomes:

$$\{1 - \exp(-i\phi)\} \, g_\phi + f_\phi = \Lambda_\phi \tag{3.3}$$

If ϕ equals zero for all tunes or equivalently if a=b, c=d and e=f, then Λ_ϕ cannot be set to zero. These are the tune shifts. The other terms can be set to zero if so desired. Michelotti, for example, keeps some of the terms contributing to dangerous resonances. In order to compare with his approach, we rewrite (3.3) in terms of W using (2.5b):

$$\{1 - \exp(-i\phi)\} \, g_\phi - \int_0^1 \exp(-is\phi) \, A_\phi(s) \, ds = \Lambda\phi \tag{3.4}$$

In the next section, we use the concept of a time-dependant canonical transformation in the normal form algorithm.

IV. The Michelotti Method

Consider the map $M(\xi_0;s)$ which brings ξ_0 to $\xi(s)$. It obeys the equation:

$$\frac{d}{ds} M(\xi_0;s) = M(\xi_0;s):-H(\xi_0;s): \tag{4.1}$$

Now, we make a canonical transformation to a set ξ^*:

$$\xi^*(s) = \exp(:\epsilon G(\xi(s);s):)\xi(s) \tag{4.2}$$

One can show that to order ϵ, the Hamiltonian $H^*(\xi^*;s)$ governing the motion of ξ^* is given by:

$$H^* = \lambda_2 + \epsilon W + \epsilon:\lambda_2:G - \epsilon\frac{\partial G}{\partial s} + O(\epsilon^2) \tag{4.3}$$

If we denote by ϵL the new ϵ-contribution to H^*, we get:

$$L = :\lambda_2:G - \frac{\partial G}{\partial s} + W \quad . \tag{4.4}$$

This equation is solved by Michelotti with the help of a Green's function. This is not essential to the problem. What is important are the boundary conditions imposed on the Green's function. Let us assume that we choose G such that:

$$G(\xi_0;0) = G(\xi_0;1) \tag{4.5}$$

With this condition, what is M^+ for one turn in terms of M?

$$M(\xi_0^*)\ \xi_0^* = \xi_1^* \tag{4.6a}$$

$$\exp(:\epsilon G(\xi_0;0):)M^+(\xi_0)\xi_0 = \exp(:\epsilon G(\xi_1;1):)\xi_1$$

$$= M(\xi_0)\exp(:\epsilon G(\xi_0;1):)\xi_0$$

But, according to (4.5), we must have:

$$= M(\xi_0)\exp(:\epsilon G(\xi_0;0):)\xi_0 \tag{4.6b}$$

We deduce from (4.6b) that the expression for M^+ is:

$$M^+(\xi_0) = \exp(:-\epsilon G(\xi_0;0):)\ M(\xi_0)\exp(:\epsilon G(\xi_0;0): \tag{4.7}$$

Comparing (4.7) and (3.1b) implies:

$$A_\epsilon = \exp(:-\epsilon G(\xi_0;0):) \tag{4.8a}$$

or,

$$G(\xi_0;0) = -g(\xi_0) \quad . \tag{4.8b}$$

In the next section, we prove the equality (4.8).

V. Derivation of the Michelotti Map

We first decompose all the relevant quantities in the $|\phi\rangle$ basis:

$$W = \sum_\phi A_\phi(s)\ |\phi\rangle \tag{5.1a}$$

$$G = \sum_\phi G_\phi\ |\phi\rangle \tag{5.1b}$$

$$L = \sum_\phi L_\phi\ |\phi\rangle \quad . \tag{5.1c}$$

We substitute (5.1) into (4.4):

$$L_\phi = i\phi\ G_\phi - \frac{\partial G_\phi}{\partial s} + A_\phi \tag{5.2}$$

There are two possibilities. Either A_ϕ is left untouched ($G_\phi = 0$) or it

is removed ($L_\phi = 0$). Of course, for the sake of comparison, we want to remove it. In that case, (5.2) becomes:

$$i\phi\, G_\phi - \frac{\partial G_\phi}{\partial s} = -A_\phi \quad . \tag{5.3}$$

We first solve the homogeneous equation:

$$i\phi\, G_\phi - \frac{\partial G_\phi}{\partial s} = 0 \tag{5.4a}$$

$$G_\phi = \Gamma \exp(i\phi s) \tag{5.4b}$$

Using the techniques of variation of parameter, we now solve (5.3) for a particular solution:

$$-A_\phi(s) = -\frac{d\tilde{\Gamma}}{ds} \exp(i\phi s) \tag{5.5a}$$

$$\tilde{\Gamma}(s) = \int_0^s \exp(-i\phi s')\, A_\phi(s')ds' \tag{5.5b}$$

The general solution is:

$$G_\phi(s) = \exp(i\phi s) \int_0^s \exp(-i\phi s')\, A_\phi(s')ds' + \Gamma \exp(i\phi s) \quad . \tag{5.6}$$

We now apply the boundary condition, which is:

$$G_\phi(0) = G_\phi(1) \tag{5.7a}$$

$$\Gamma = \Gamma \exp(i\phi) + \exp(i\phi) \int_0^1 \exp(-i\phi s)\, A_\phi(s)ds \tag{5.7b}$$

Solving for $\Gamma(=G_\phi(0))$, we obtain:

$$G_\phi(0) = \Gamma = \frac{\exp(i\phi)}{1 - \exp(i\phi)} \int_0^1 \exp(-i\phi s)\, A_\phi(s)ds,$$

$$= - \frac{1}{1 - \exp(-i\phi)} \int_0^1 \exp(-i\phi s)\, A_\phi(s)ds \quad . \tag{5.8}$$

If one sets A_ϕ to zero in equation (3.4), the resulting g_ϕ is given by:

$$g_\phi = \frac{1}{1 - \exp(-i\phi)} \int_0^1 \exp(-i\phi s)\, A_\phi(s)ds \quad . \tag{5.9}$$

Hence, as expected, g_ϕ is just $-G_\phi$. Q.E.D.

VI. Conclusion

As one sees, there are no formal differences between the two methods. Our technique goes after the map (f_ϕ) and then computes the normal form (g_ϕ). Michelotti's technique goes after the map and the normal form at once.

The use of Green's function in Michelotti's paper is not central to the method. Perhaps his treatment would gain in simplicity if he used our interaction picture approach.

As far as higher orders in ϵ are concerned, the equivalence is obvious. Indeed, at each order in ϵ, the equation to be solved is of the form (5.2) for Michelotti and (3.4) for us.

Appendix

Here we show that we properly constructed the linear Floquet variables.

Let us write $M(\xi_0;s)$ in term of $\hat{M}(\xi_0;s)$. $\hat{M}$ is the original map, before turning into Floquet variables.

$$M(\xi_0;s) = F_2(\xi_0;0)\ \hat{M}(\xi_0;s)\ F_2^{-1}(\xi_0;s) \tag{A.1}$$

$$= F_2^0\ \ \hat{M}(s)\ F_2^{s-1}$$

Now, we take the s-derivative of (A1):

$$\dot{M}(\xi_0;s) = \dot{F}_2^0\ \ \hat{M}\ F_2^{s-1} + F_2^0\ \ \dot{\hat{M}}\ F_2^{s-1} + F_2^0\ \ \hat{M}\ \dot{F}_2^{s-1} \quad . \tag{A.2}$$

We must evaluate each term in (A2):

$$\dot{F}_2^{s-1} = \frac{d}{ds}\ (\hat{M}_2^{-1}(\xi_0;s)\ A_2^{-1}\ \exp(:-s\lambda_2:))$$

$$= F_2^{s-1}\ :-\lambda_2: + :K_2(\xi_0;s):\ F_2^{s-1} \tag{A.3}$$

Here we used the equation for $M_2^{-1}(\dot{M}_2^{-1} = :K_2(\xi_0;s):M_2^{-1})$.
Proceeding similarly for F_2^0:

$$\dot{F}_2^0 = \dot{A}_2(\xi_0) = 0 \quad . \tag{A.4}$$

Finally, the equation for $\dot{\hat{M}}$ is just:

$$\dot{\hat{M}} = \hat{M}:-K_2(\xi_0;s) - \varepsilon\ V(\xi_0;s): \quad . \tag{A.5}$$

Substituting in (A.2), we get:

$$\dot{M}(\xi_o;s) = F_2^o \ \hat{M}:-K_2(\xi_o;s) - \varepsilon \ V(\xi_o;s):F_2^{s-1}$$

$$+ F_2^o \ \hat{M}F_2^{s-1}:-\lambda_2:+F_2^o \ \hat{M}:K_2(\xi_o;s):F_2^{s-1}$$

$$= F_2^o \ \hat{M}F_2^{s-1} \ :-\lambda_2:+F_2^o \ \hat{M}F_2^{s-1} \ :-\varepsilon F_2^s \ V(\xi_o;s):$$

$$= M(\xi_o;s):-(\lambda_2+\varepsilon F_2^s \ V(\xi_o;s)): \tag{A.6}$$

Since $\dot{M} = M:-H:$, we deduce from (A.6):

$$H(\xi_o;s) = \lambda_2 + \varepsilon \ F_2^s \ V(\xi_o;s) \quad . \tag{A.7}$$

This is what we claimed at line (1.5).

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