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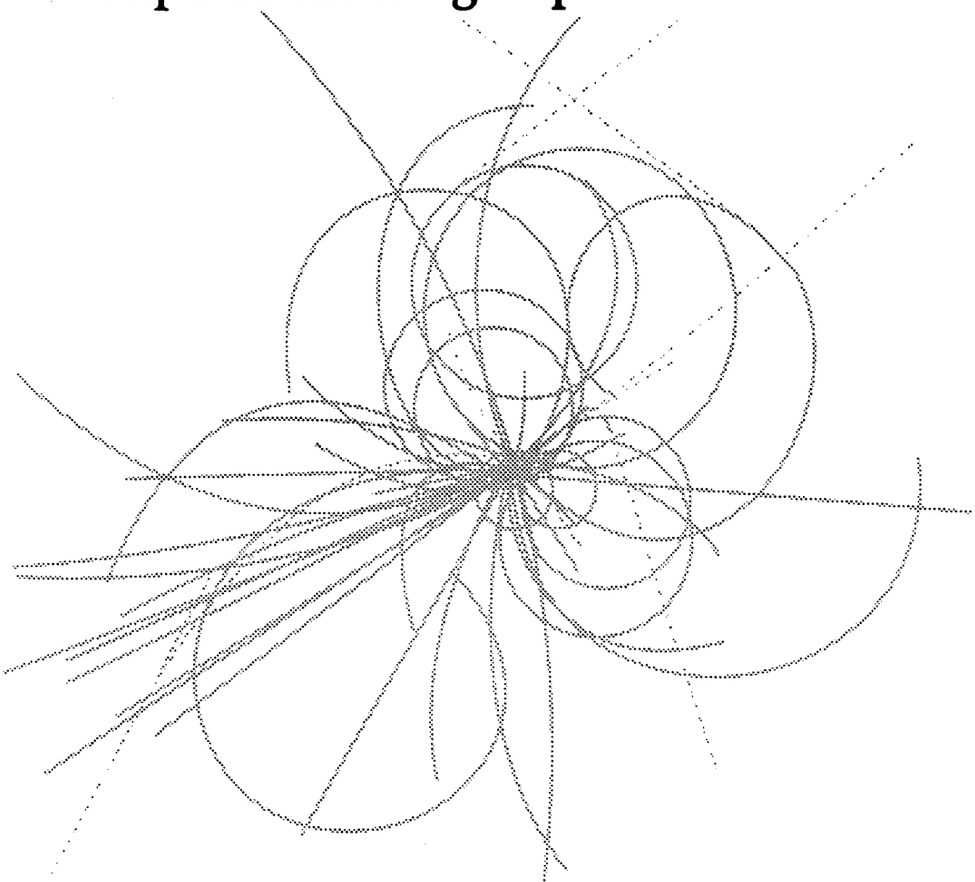

Random Sextupoles in the SSC Lattice

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April 1985

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RANDOM SEXTUPOLES IN THE SSC LATTICE

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Distortion has been calculated as a function of rms of random sextupoles in regular cells for a typical SSC - kind lattice.

The lattice was submitted under TAC14060, to the LBL data base. In the lattice with superperiodicity 6 there are two straight sections and an arc per period. Each straight section contains a low-beta triplet, a matching section, a trombone and a dispersion killer. There are 93.5 cells in the arc.

The main parameters of the lattice are as follows:

tune per ring $Q_x = 109.30297$, $Q_z = 109.29668$;

natural chromaticity per period $Q_x' = -43.437$, $Q_z' = 34.852$;

total length of the period 30639.42m;

$(\beta_x)max = 3956.41m$; $(\beta_z)max = 3957.37m$; $(\eta)max = 3.978m$

$\beta_x^* = \beta_z^* = 1m$; $\eta_x^* = \eta_x^* = 0$

in the cells $\beta = 535m$; $\eta = 4m$;

Phase advance per cell 60°

half of the cell length 145m

Two sextupole families in regular cells were used to cancel linear chromaticity. They were placed in 90 cells according to the A. Chao scheme (two in one half cell, another half cell is empty).

Random sextupoles were modelled as kicks in each regular dipole. These were 9 independent kicks per dipole (one kick per 12.5m). The correlation length of random sextupoles is unknown and, the chosen length may be too pessimistic. The magnitude of the random kicks was generated according to the Gaussian distribution with a given rms value σ. The following algorithms was used to generate random variable x:

$$x = \sqrt{12/n} \quad \Sigma(\xi_i - 0.5) \quad * \sigma$$

where n = 12, and ξi are random numbers uniformly distributed on the interval (0,1). The distribution of 10^4 random values x is shown on fig. 1. A comparison with that algorithm with parameter n = 48 and with the algorithm:

$$x = \sqrt{-20 * \mathrm{ALOG}(\xi_1)} \ \mathrm{Cos}\ (2\pi \cdot \xi_2)$$

did not show which one is better, so the simplest one was used. The random numbers ξ_i were generated by the standard algorithm:

$$\xi_{i+1} = y_{i+1}/(d-1);$$

$$y_{i+1} = \mathrm{mod}\ (7.10^5 * y_i;\ d); \quad d=2^{31}$$

The initial seed y_1 determines the set of random numbers. The dependence of distortion on the initial seed is shown in fig. 2. Each time the last value of y_i was used as the initial seed for the next run. Because there are big fluctuations, the seed which gives the typical distortion was chosen for the following calculations:

$$y_1 = 0.182081\ \mathrm{D{+}08}.$$

The 'distortion' D(s) was calculated, which is defined as:

(1) $$D(s) = (Q_x \cdot \Delta J_x + Q_y \cdot \Delta J_y)/(Q_x \cdot J_x + Q_y \cdot J_y)$$

where the action variables J_x, J_y are related to initial emittances ε_x, ε_y and coordinates as usual:

$$\varepsilon_x = J_x/p\ ; \quad \varepsilon_y = J_y/p$$

$$x = \eta(\Delta p/p) + \sqrt{2\ J_x\beta_x(s)/p}\ \ \mathrm{Sin}\ (\psi_x)$$

$$y = \sqrt{2J_y\beta_y(s)/p}\ \ \mathrm{Sin}\ (\psi_y)$$

It is easy to show, (see appendix) that the variation of the emittance $\Delta J_x(s)$, $\Delta J_y(s)$ with s due to sextupoles is given by the formula:

(2) $$\Delta J_{x,y} = \frac{\partial}{\partial X_{x,y}} A(X,j,s) + o(b^2);$$

$$A(X,j,s) = \sum_m A_m(X,s)\ (\Delta p/p)^{\ell_1}\ (\varepsilon x/2)^{\ell_2}\ (\varepsilon y/2)^{\ell_3}$$

where $$\ell_1 = n-m_1-m_2-m_3-m_4$$

$$\ell_2 = 0.5\ \ (m_1 + m_2)$$

$$\ell_3 = 0.5\ \ (m_3 + m_4)$$

(3) $$A_m(X,s) = \frac{Ps}{2}\ \ \exp\ \{i\Delta mx(X_x - Q_x\ \ s/R) + i\Delta m_y\ (X_y - Q_y\ \ s/R)\}$$

$$\{\int ds'\, G_{n'm}(s') + (\exp\ (2\pi i/N\ (\Delta m_x\, Q_x + \Delta m_y\, Q_y)) - 1)^{-1} \int_0^{2\pi R/N} ds' G_{nm}(s')\}$$

and $$Gnm(s) = \frac{1}{\rho(s)} g_{nm}(s) \exp \frac{is}{R} (\Delta m_x\, Q_x + \Delta m_y\, Q_y)$$

$$(4) \qquad g_{nm}(s) = b_{n-1} \frac{(n-1)!\ i^{(m_2 - m_1)}\ ((-)^{m_3} + (-)^{m_4})}{(n-m_1-m_2-m_3-m_4)!\, m_1!\, m_2!\, m_3!\, m_4!}$$

$$\beta x^{(m_1 + m_2)/2}\ \beta_y^{(m_3 + m_4)/2}\ \eta^{n-m_1-m_2-m_3-m_4} \exp i(\Delta m_x \theta_x + \Delta m_y \theta_y);$$

$$\Delta m_x = m_1 - m_2\ ; \quad \Delta m_y = m_3 - m_4$$

$$\theta_{x,y} = \mu_{x,y}(s) - Q_{x,y}\, s/R; \quad \psi_{x,y} = \phi_{x,y} + \theta_{x,y}$$

$\mu_{x',y}$ is the phase advance, $\mu_{x',y}(2\pi R) = 2\pi Q_{x',y}$;

$Q_{x',y}$ includes tune shift due to sextupoles;

N=6 is super periodicity; $m = (m_1, m_2, m_3, m_4)$ and for sextupoles $n=3$.

In the first approximation in b_n the 'new' action $j_{x,y}$ is constant, and 'new' phases are

$$X_{x,y} = (s).\ Q_{x,y}/R + X_{x,y}(o)$$

The 'old' phase

$$\phi_{x',y} = X_{x',y} - \frac{\partial A(X,j,s)}{\partial j}$$

so that the mapping (J,ϕ) to (j,X) is simplectic with the same accuracy $o(b^2)$.

The distortion D(s) was calculated according to the formulas (1-4). Beta functions β_x, β_y, η function and phase advances $\mu_{x,y}$ were taken from the output of the program MAD. MAD calculates these functions along the lattice with zero strength of all sextupoles but for a given value of $\Delta p/p$. The program DISTOR generates the set of random kicks, sets the strength of systematic sextupoles in the lattice to cancel linear chromaticity, and after that calculates the tune shift and the variations of the distortion (1) along the lattice. The example of these variations are shown on fig. 3 due to regular sextupoles in the lattice only and on fig. 4 the same with random and regular sextupoles. On fig. 5, 6 the result of calculations are given as a function of the rms value of random sextupoles $\sigma(b2)$ for σ = 1,5,10,20 in units $10^{-4}\, 1/cm^2$. The bars corresponds to the maximum and minimum value ot D(s) along the lattice.

To control the calculations the tune shift was calculated for each set of random kicks. Given $\Delta p/p$ and emittances, the distortion still depends on initial phases $X_{x,y}(o)$. Nevertheless, the range of variations is not very sensitive to the initial phases, because the distortion oscillates many times per period. This statement is illustrated by the following table:

Phase Dependence of the Distortion

$\Delta p/p = 0$; $\varepsilon_x = \varepsilon_y = 0.18.10^{-6}$; $b_2 = 10$ units seed $= 1.82081 \cdot 10^7$; $X_y(o) = 0$

$X_x(o)/2\pi$	MAX distortion	Minimum Distortion
0	0.294	-0.326
0.25	0.241	-0.291
0.5	0.326	-0.294
0.75	0.291	-0.241

Fig.5 corresponds $\Delta p/p = 0$ and fig. 6 corresponds to $\Delta p/p = 5 \cdot 10^{-4}$, and emittances $\varepsilon_x = \varepsilon_y = 0.18.10^{-6}$ and $\varepsilon_x = \varepsilon_y = 0.5.10^{-8}$m were used.
The maximum value $\Delta p/p = 5 \cdot 10^{-4}$ was chosen because it gives the tune shift $\Delta Q = 0.08$, if the systematic decapoles in dipoles of order of $b_4 = 10^{-4}$ 1/cm^4, with $\beta \simeq 535$m and $\eta = 4$m. The tune shift with $\Delta p/p = 10^{-3}$ would be 8 times larger, what seems to be too big, and b_4 hardly will be less than the used value.

CONCLUSION

1. The results are very sensitive to the set of random kicks. It means that the properly done shuffling can improve the quality of the system significantly.

2. There is a strong dependence of the distortion modulation on the magnitude of the emittances. The dependence on $\Delta p/p$ is moderate. That corresponds to perturbation of order $\eta^2(\Delta p/p)^2$ in the second case and of order of $\varepsilon\beta$ in the first case.

3. Let us assume, that at injection (1TeV) $X_c = \eta \cdot \Delta p/p \simeq 0.2$cm and rms beam size $1\sigma = 0.073$cm. It corresponds to $\Delta p/p = 5 \cdot 10^{-4}$ and emittances $\varepsilon = 0.18.10^{-6}$m corresponds to beams size of 13σ. In this case the 10 percent modulation occurs for rms random sextupoles $\langle b_2 \rangle \simeq 2 \cdot 10^{-4}$ 1/cm^2 and 20 percent modulation with

$$\langle b_2 \rangle \simeq 5 \cdot 10^{-4} \text{ 1/cm}^2.$$

4. In the storage mode (20TeV) emittance $\varepsilon = 0.5 \cdot 10^{-8}$m corresponds to rms beam size 10σ, $\sigma = 0.016$cm.
The 10 percent modulation corresponds to

$$\langle b_2 \rangle \simeq 6.0 \cdot 10^{-4} \text{ 1/cm}^2 \text{ for } \Delta p/p = 5 \cdot 10^{-4}$$

and

$$\langle b_2 \rangle \simeq 20 \cdot 10^{-4} \text{ 1/cm}^2 \text{ for } \Delta p/p = 0.5.10^{-8}$$

5. In all cases the strength of sextupole magnets, which is necessary to set linear chromaticity to zero, was rather small, $|b_2 l| \leq 10^2$ 1/m. This value has to be compared with $|b_2| \simeq 666$ 1/m^2 in LEP.

6. The results shown on fig. 5,6 could be compared with the ''diffusion extreme'' of the distortion

$$D \sim D_1 \cdot \sqrt{N}$$

where D_1 is a distortion due to one random kick

$$D_1 \simeq 2\pi\Delta Q \simeq b_2 l/R \cdot \eta\beta \cdot (\Delta p/p)$$

and N is the number of kicks per period

$$\sqrt{N} \sim 41$$

It gives for $\langle b_2 \rangle = 10 \cdot 10^{-4}$ 1/cm^2 the estimation

$$D \simeq 0.19 \quad \text{or 19 percent}$$

Of course, random sextupoles do not give a real diffusion, but only some periodic distortion. Nevertheless, the amplitude of the distortion remains of order of the ''diffusion extreme'' even after the average part of the distortion is cancelled out with sextupole magnets.

This paper has been done according to A. Chao request.

The author is thankful to Sho Ohnuma, A. Chao and C. Leeman for the useful discussion.

APPENDIX

Here is a brief derivation of the f (2),(3). As usual, the Hamiltonian

$$\text{(A.1)}\quad H(x, p_x, y, p_y, s) = P_x^{\,2} + P_y^{\,2}/2 + P_s/2\left[k^2x^2 + Kb_1(x^2-y^2)\right] - \Delta p \cdot K \cdot x + 1/2\, P_s K \cdot \sum_{n=3} b_{n-1}/n\left[(x+iy)^n + c.c\right]$$

can be rewritten in the phase-action variables (J,ϕ) as follows:

$$\text{(A.2)}\quad H(J, \phi, s) = Q_x\, J_x/R + Q_y\, J_y/R + H_{int}\,,$$

$$\text{(A.3)}\quad H_{int} = -\frac{PsK}{2}\sum_{n=3}\sum_{m} g_{n,m}(s)\cdot(\Delta p/p)^{\ell_1}$$

$$(Jx/2p)^{\ell_2}\,(Jy/2p)^{\ell_3}\ \exp\,(i\Delta m_x\,\phi_x + i\Delta m_y\,\phi_y)$$

where m (m_1, m_2, m_3, m_4), and Δm_x, Δm_y, are defined in f.(4); ℓ_1, ℓ_2, ℓ_3 are the same as in f. (2); $K = 1/\rho$ is the curvature of a trajectory.

The canonical variables (J_x, ϕ_x), (J_y, ϕ_y) are related with coordinates and momentums as usual (see formulas after f. (1)) The periodical function g(s), given by f. (4), depends on η and beta-functions, $\beta = \omega^2$ for ideal lattice.

$$\text{(A.4)}\quad \eta'' + (p_s/p)\,(Kb_1 + K^2)\eta = K$$

$$\omega_x'' + (p_s/p)\,(Kb_1 + K^2)\,\omega_x = \omega_x^{-3}$$

$$\omega_y'' - (p_s/p)\,Kb_1\ \omega_y = \omega y^{-3}$$

The nonresonance term in H_{int}

$$\text{(A.5)}\quad \overline{H}_{int} = -\,Ps/2\sum_{n=3}\sum_{m_1}\sum_{m_3}\left[Kg_{nm}\right](\Delta p/p)^{n-2m_1-2m_3}\ (Jx/2p)^{m1}\ (Jy/2p)^{m3}$$

gives the well-known tune shifts due to multipoles

$$\text{(A.6)}\quad \Delta Q_{x,y} = R\ \partial\overline{H}_{int}/\partial J_{x,y}$$

For example, the sextupole tune shift (n=3) is

$$\Delta Q_{x,y} = ps/p\ (\Delta p/p)\left[R/\rho\ b_2\ \eta\beta_x\right]$$

the octupole tune shift is

$$\Delta Q_x = -\,3/2\ (ps/p)\left[R/\rho\ b_3\ \left(+\beta_x\eta^2\ \sigma^2 - \beta_x\beta_y\epsilon_y + \beta_x^{\,2}\ \epsilon_x\right)\right]$$

The decapole tune shift is

$$\Delta Qx = -\ 2(ps/p)\left[R/\rho\ b_4\ \left(\beta_x(\eta\sigma)^3 - 3\beta_x\beta_y\eta\sigma\varepsilon_y + 3/2\ \beta_x\eta\delta\varepsilon_x\right)\right]$$

and so on.

From now on we assume, that the tune shift (A.6) is included in the tune O_x, O_y, so that perturbation (A.3) does not contain nonresonace terms. It has to be done to avoid secular terms.

As far as perturbation (A.3) is negligible, the actions J_x, J_y, are constant and define the emittance $\varepsilon_{x,y} = J_{x,y}/p = \sigma_{x,y}^2/\beta_{x,y}$.

To take perturbation into account, the second conical transformation (J,ϕ) to (j,X) can be done with the generating function

$$F(\phi,j,s) = \phi_x j_y + \phi_y j_y + A(j,\phi,s)$$

where

$$\text{(A.7)}\quad A(j,\phi,s) = \sum_{\ell} A\ \ (\phi,s)\ (\Delta p/p)^{\ell_1}\ (jx/2p)^{\ell_2}\ (jy/2p)^{\ell_3}$$

The old variables (J,ϕ) related to the new variables (j,X) as follows:

$$\text{(A.8)}\quad J_{x,y} = j_{x,y} + \partial/\partial\phi_{x,y}\ A(j.\phi,s)$$

$$X_{x,y} = \phi_{x,y} + \partial/\partial j_{x,y}\ A(j,\phi,s)$$

To eliminate terms of order of b_n in the new Hamiltonian

$$H(j,x,s) = H(J,\phi,s) + \partial A(j,\phi,s)/\partial s$$

it is enough to demand

$$\partial A_\ell(\phi,s)/\partial s + Q_x/R\ \ \partial A_\ell/\partial\phi_x + Q_y/R\ \partial A_\ell/\partial\phi_y =$$

$$= PsK/2 \sum_m \left(g_{nm}\ \exp.\ i(\Delta m_x\phi_x + \Delta m_y\ \phi_y) - Ps/2\ \left[Kg_{nm}\right]\delta_{m1m2}\ \delta_{m3m4}\right)$$

Here ℓ_1, ℓ_2, ℓ_3 are as given by f. (2). The periodical solution

$$A_\ell(\phi,s + 2\pi R/N) = A\ (\phi,s)$$

takes the form

$$A_\ell(\phi,s) = Ps/2 \qquad \exp.\ (i\Delta m_x\phi_x + i\Delta m_y\phi_y)$$

$$\Delta m_{x,y}$$

$$\text{(A.10)}\quad \{F(s,s) + \left[\exp.\ 2\pi i/N(\Delta m_x Q_x + \Delta m_y Q_y) - 1\right]^{-1} F(s,\ 2\pi R/N)\}$$

where

$$F(s,s_0) = \int_0^{s_0} ds^1/\rho \; g_{nm}(s^1) \exp. -i\left(\Delta m_x Q_x + \Delta m_y Q_y\right)(s-s')/R$$

Neglecting terms of the second order $O(b^2)$, the new Hamiltonian takes the form $H = H(j_x, j_y)$ so that j_x, j_y) are integrals of motion, and the new phases are given by relation

$$X_{x,y}(s) = X_{x,y}(o) + s/R \; Q_{x,y}$$

With the same accuracy, the ralation (A.8) gives the old variables as functions of the new variables

$$\text{(A.11)} \quad \begin{aligned} J_{x,y} &= j_{x,y} + \partial/\partial X_{x,y} \; A(j,X,s) \\ \phi_{x,y} &= X_{x,y} - \partial/\partial j_{x,y} \; A(j,X,s) \end{aligned}$$

It is easy to see, that variations (ΔJ, $\Delta\phi$) at the points s_2, s_1 are related by a matrix $M(s_2,s_1)$

$$\begin{pmatrix} \Delta J \\ \Delta\phi \end{pmatrix}_{s2} = M(s_2,s_1) \begin{pmatrix} \Delta J \\ \Delta\phi \end{pmatrix}_{s1}$$

which satisfies the condition of simplecticity

$$M^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So, the transformation (A.11) is simplectic.

The formulas (A.11,7,10) were used to determine the distortion along the lattice (compare with f.(2)).

$$\Delta J_{x,y} = J_{x,y}(s) - j_{x,y}$$

This formula gives the simple estimation for the relative distortion due to the n-th multipole

$$\text{(A.12)} \quad (\Delta J/J) \sim \sum 1/Q \;\; b_{n-1} \;\; o \;\; \Delta x^{n-2}$$

Here $b_{n-1}\, \ell_o$ is the kick due to a single localized multipole with the length ℓ_o, the sum is taken over all kicks, and the size of the beam of order

$$\Delta x \simeq \max(\eta \; \Delta p/p; \; \sqrt{\beta \cdot \varepsilon})$$

This estimation can be obtained directly from the initial Hamiltonian (A.1). The multipole b_{n-1} perturbes the motion so that

$$(dp_x/ds)_n = - PsK/2 \;\; b_{n-1}\left[(x + iy)^{n-1} + cc\right]$$

In the thin lense approximation with $b_{n-1} \neq 0$ on the length ℓ_o, it gives

$$(\Delta p_x)_n \sim \quad ps/p \quad \ell_o b_{n-1} \quad \Delta x^{n-1}$$

the sum over all kicks.

With the transverse momemtum P_x of order of

$$P_x \sim Ps \quad \Delta x \quad Q/R$$

it gives the same extruation as in (A.12). If the motion is stable, the coherent length of order of the betatron wave length, so that the distortion due to the n-th multipole

$$\text{(A.13)} \quad (\Delta p_x/p_x)n \simeq \frac{(\ell \cdot b_{n-1}) \quad \Delta x^{n-2} \quad N_c}{Q^2}$$

where N_c is the number of cells per ring and (ℓb_{n-1}) is the sum of kicks per cell.

If a particle goes through K periods, the distortion at $s_k = (2\pi R/N) \cdot K$ is determined by the distortion on the one period

$$(\Delta J_x/J_x)_K = \sum_{n,m} \frac{\text{Sin } (\pi K/N \; \partial Q)}{\text{Sin } (\pi/N \; \partial Q)} \left(\frac{\Delta J_x}{Jx}\right) \qquad s = 2\pi R/N$$

where $\partial Q = (m_1 - m_2)Q_x + (m_3 - m_4)Qy$

The estimation of the distortion due to random multipoles is similar, only $\sqrt{N_c}$ has to be used instead of N_c in f.(A.13).

Generally speaking, the tune shift (A.6) $2\pi\Delta Qx$ is the same order of magnitude, as the distortion (A.12). The only difference is that coherent length in the sum over all kicks is bigger, than for distortion, because with $m_1 = m_2$, $m_3 = m_4$ there are oscillating terms exp. i $\{(m_1-m_2)\ \mu_x(s) + (m_3-m_4)\ \mu_y(s)\}$. Exception is the sextupole distortion, because the linear chromaticity normally is set to zero. This conclusion is confirmed by numerical simulations.

1. Fig.1
   The distribution of 10^4 random numbers.

2. Fig. 2a, 2b
   The dependence of distortion on the initial seed for two different initial conditions.

3. Fig. 3
   The example of distortion variation along the lattice. Regular sextupoles only.

   $\varepsilon_x = \varepsilon_y = 0.5 * 10^{-8}$m; $\Delta p/p = 0$

4. Fig. 4
   The example of distortion variation along the lattice. Regular and random sextupoles.

   $\langle b2 \rangle = 5 \cdot 10^{-4}$ 1/cm^{-2}; $\varepsilon_x = \varepsilon_y = 0.5 \cdot 10^{-8}$;

   $\Delta p/p = 5 \cdot 10^{-4}$

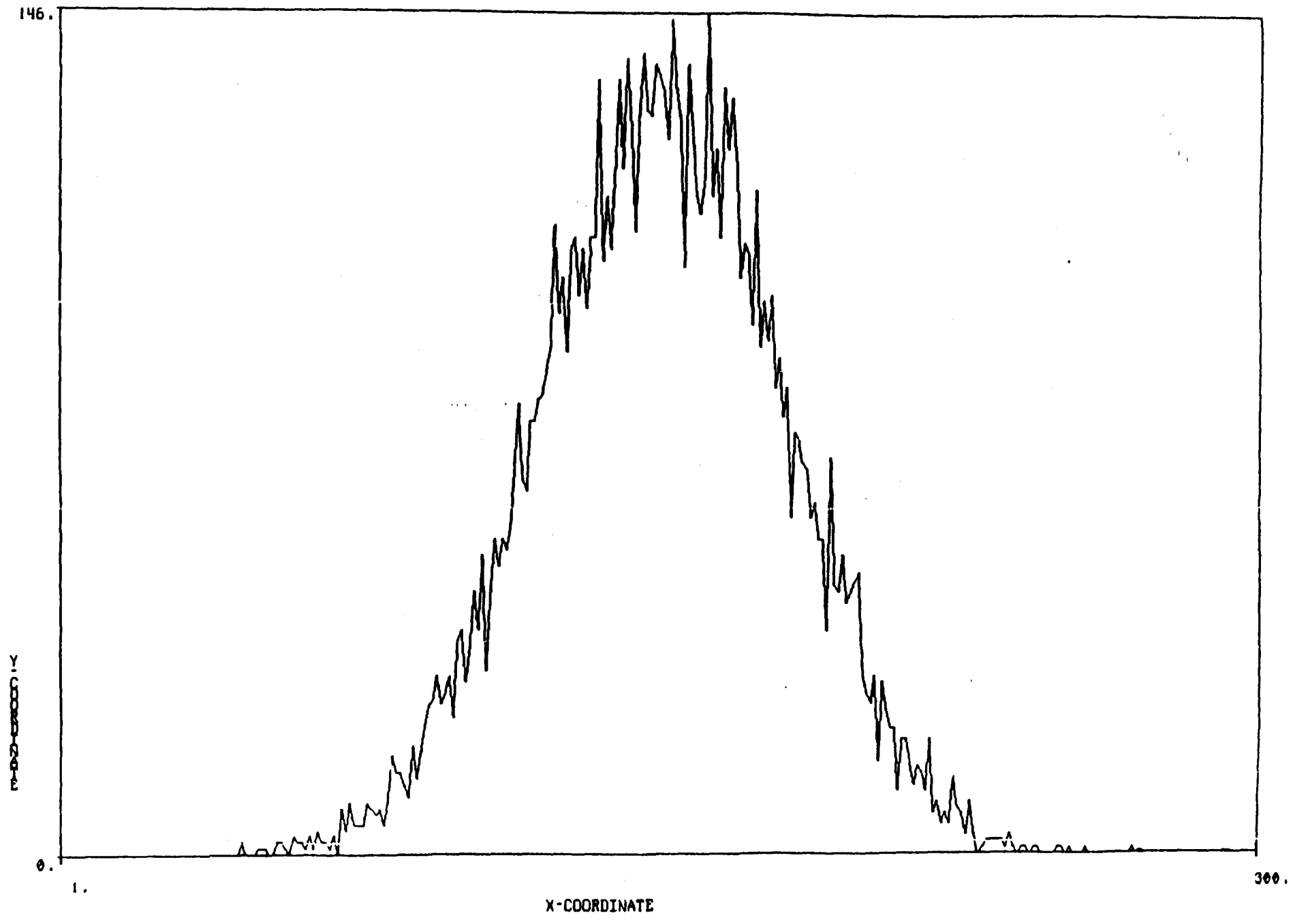
5. Fig. 5
   The distortion vrs $\langle b_2 \rangle$; $\Delta p/p = 0$

   $\varepsilon_x = \varepsilon_y = 0.5 \cdot 10^{-8}$

   $\varepsilon_x = \varepsilon_y = 0.18 \cdot 10^{-6}$

6. Fig. 6
   The same as Fig. 5, but $\Delta p/p = 5 \cdot 10^{-4}$

Fig. 1



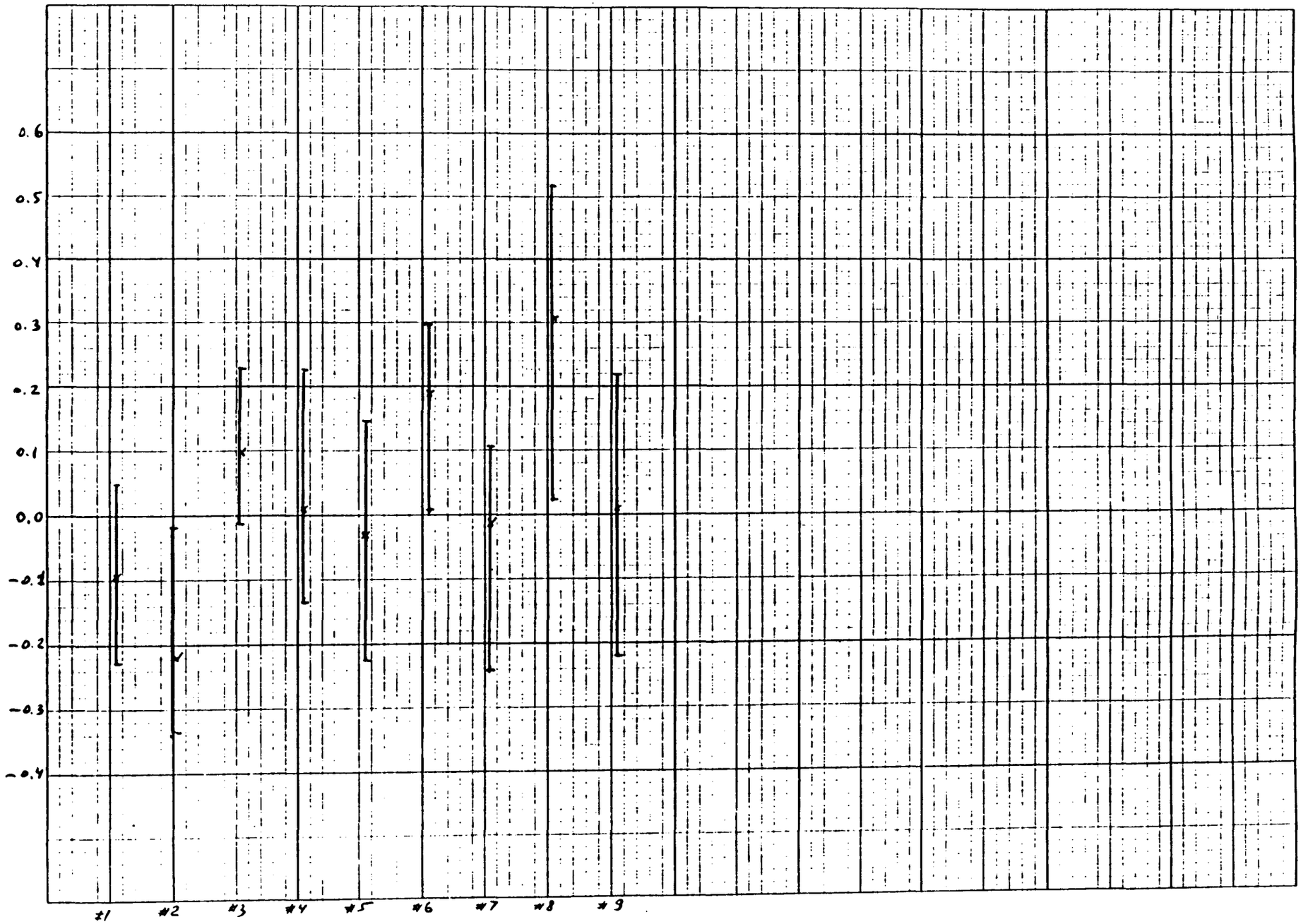


Fig. 2(a)

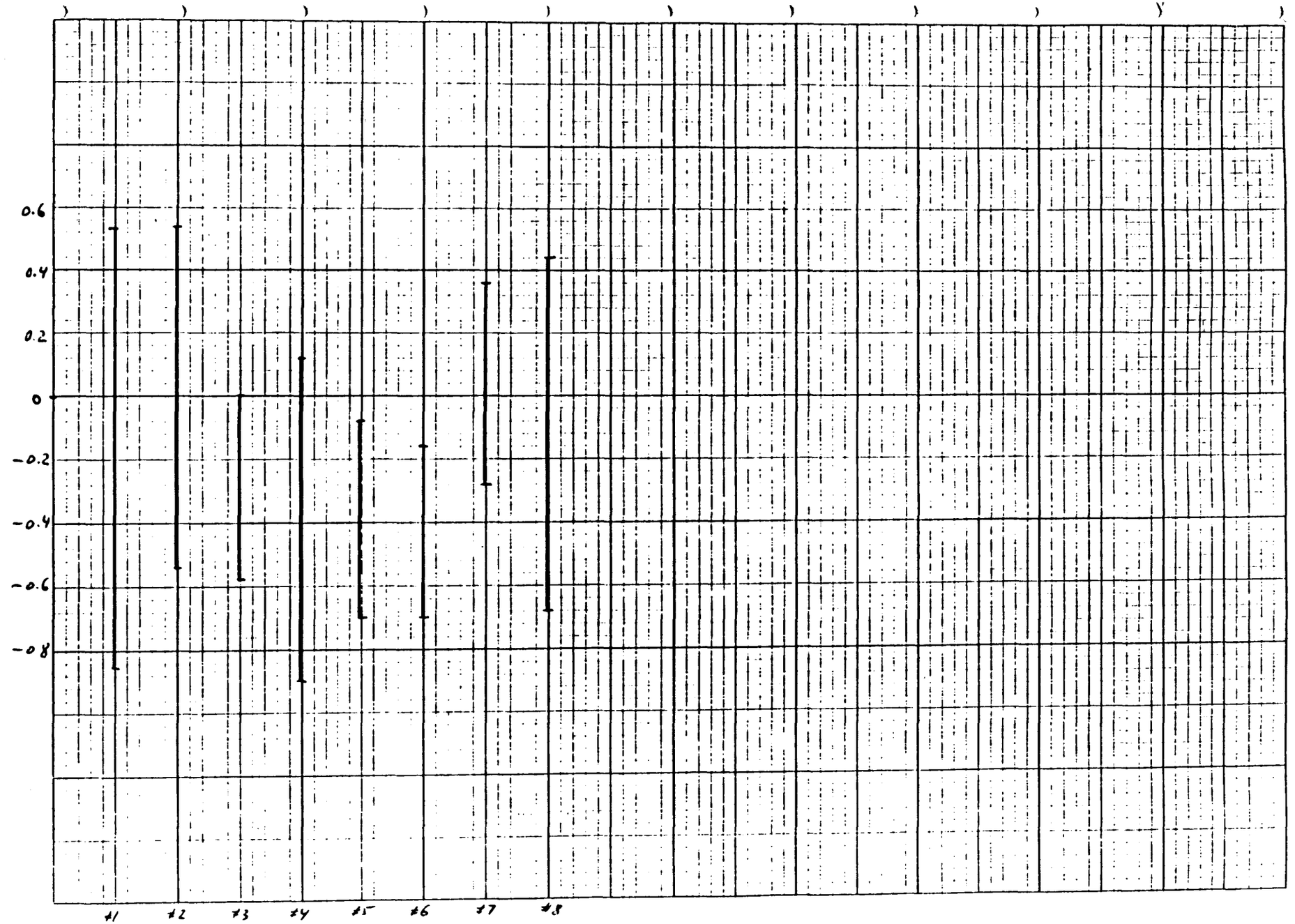


Fig. 2(b)

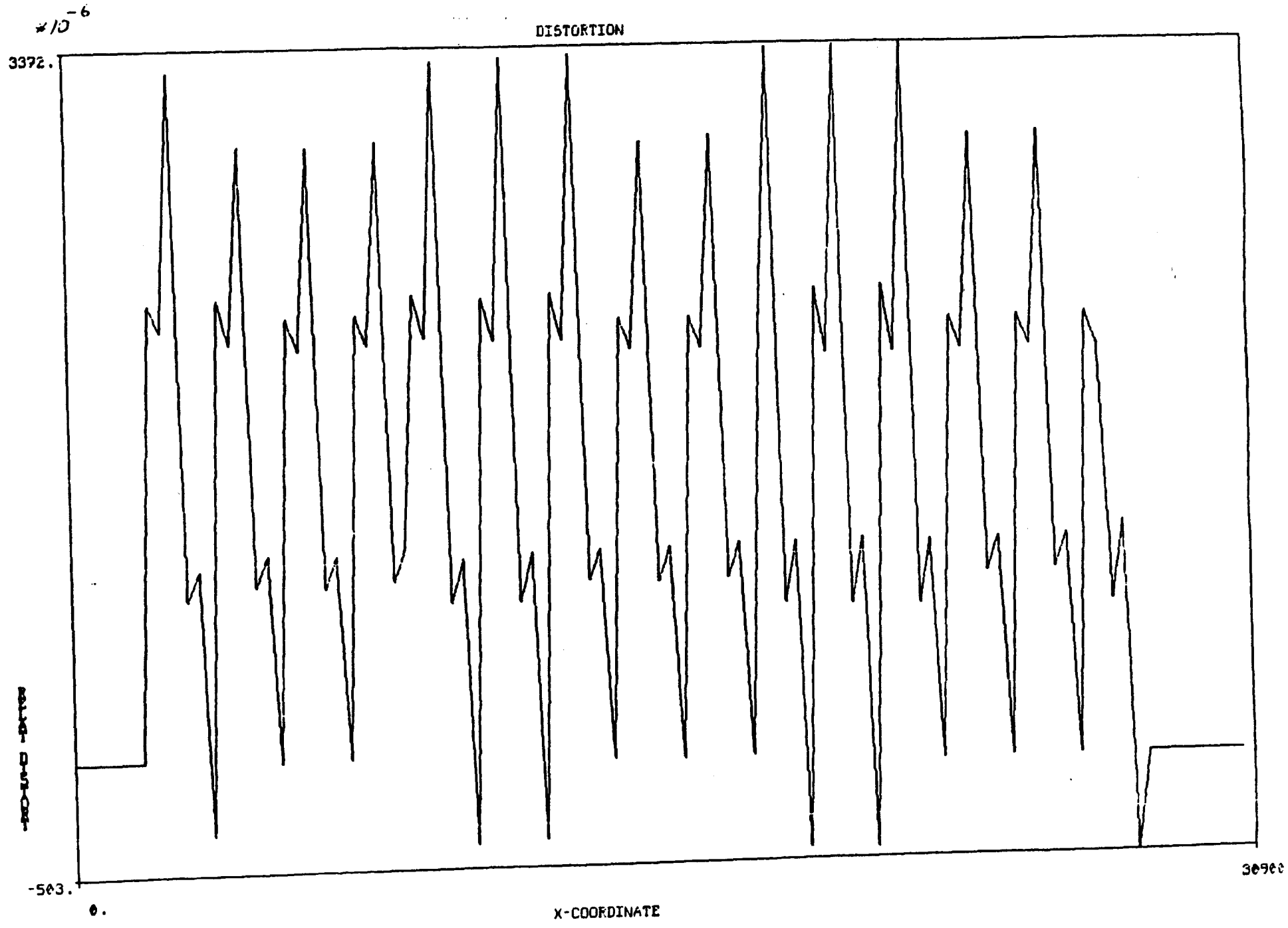


Fig. 3

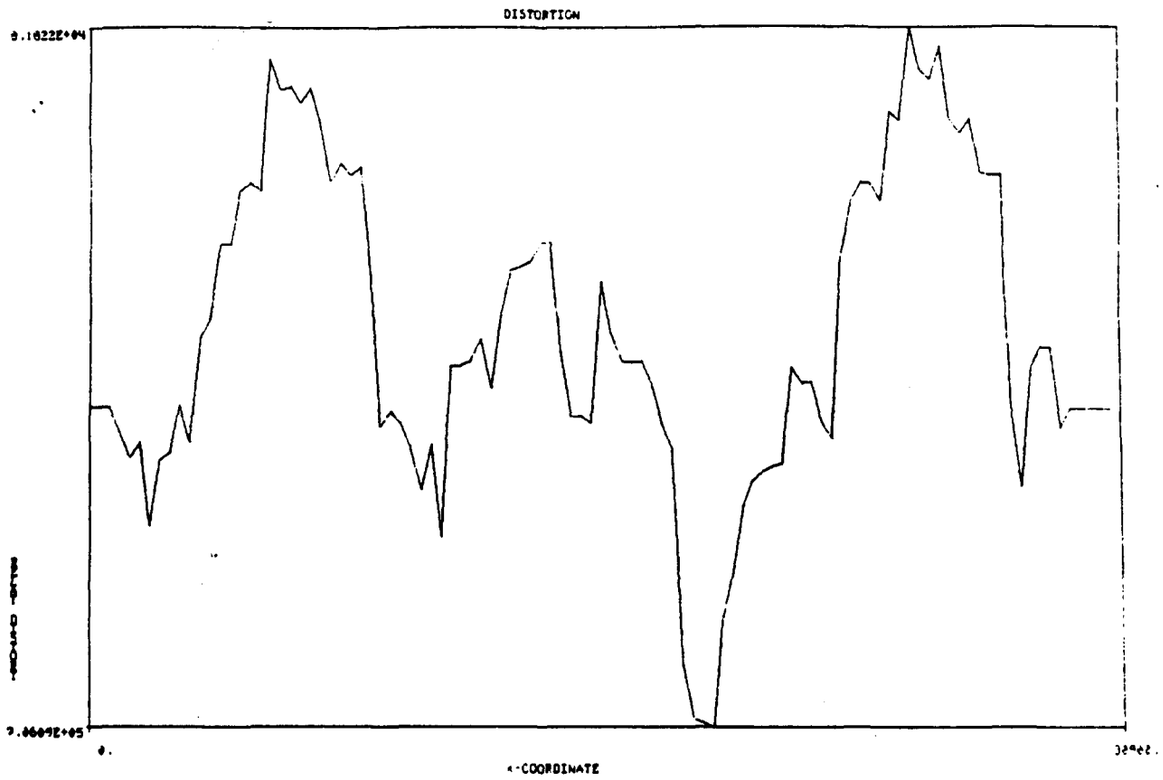


Fig. 4

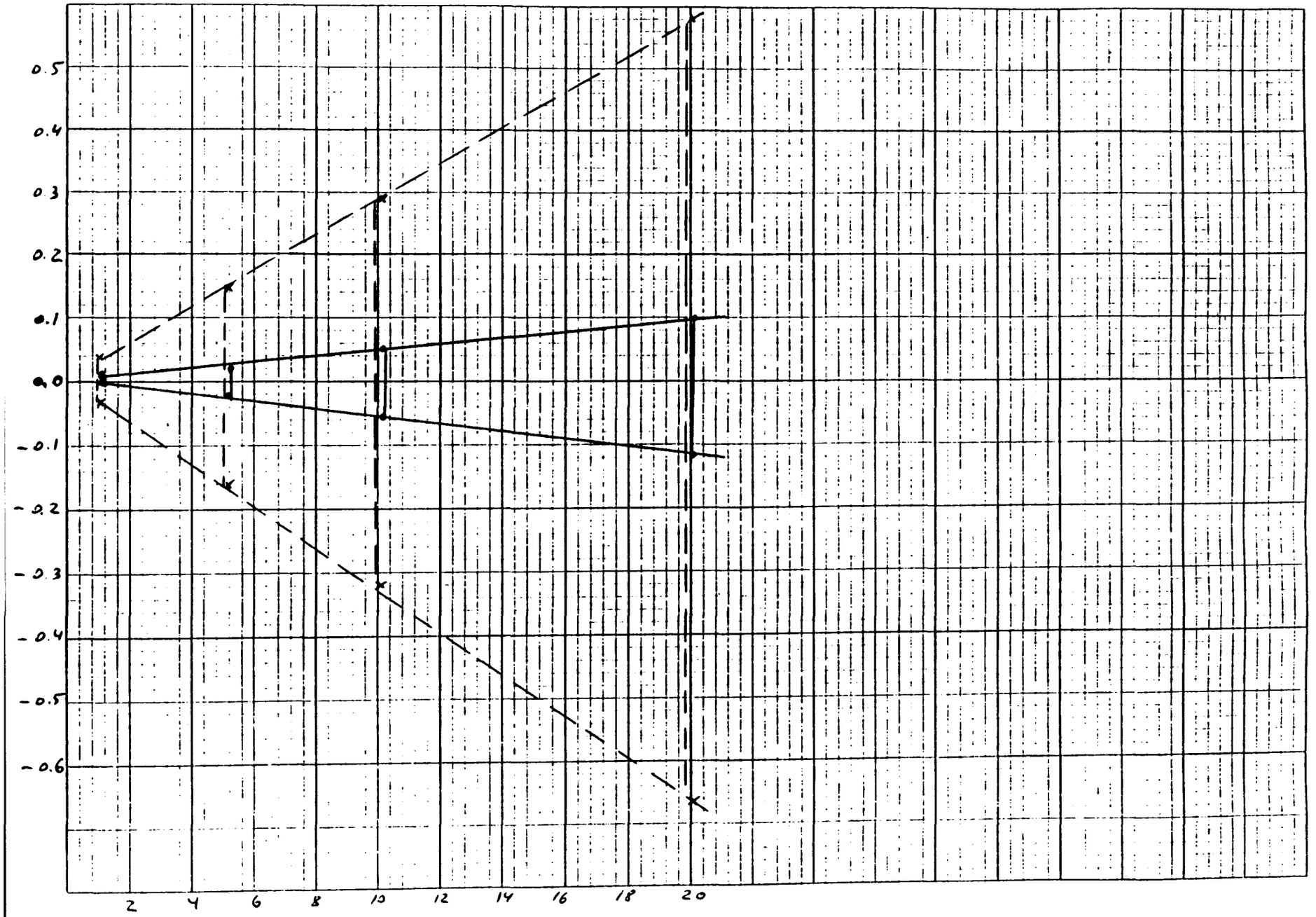


Fig. 5

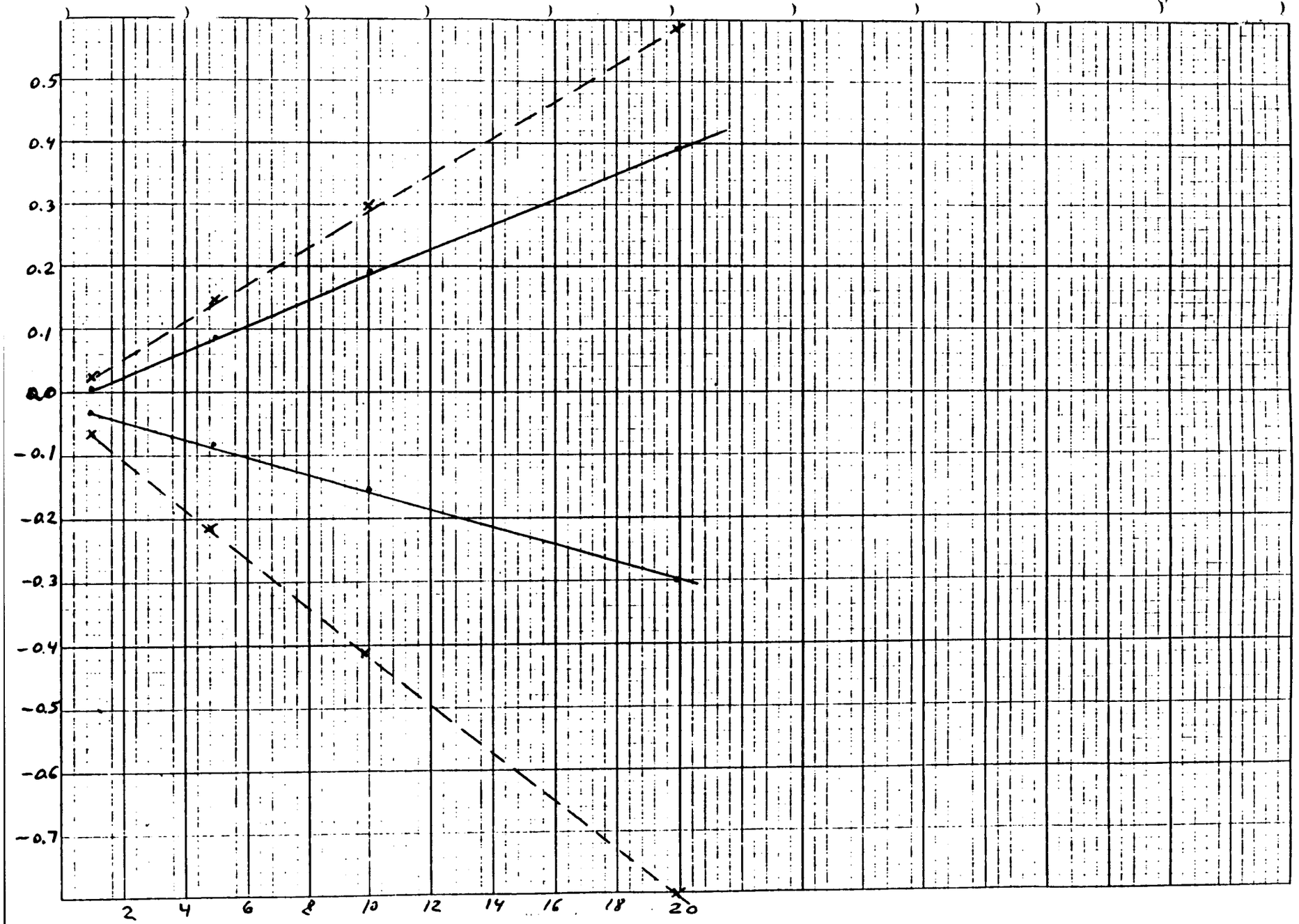


Fig. 6