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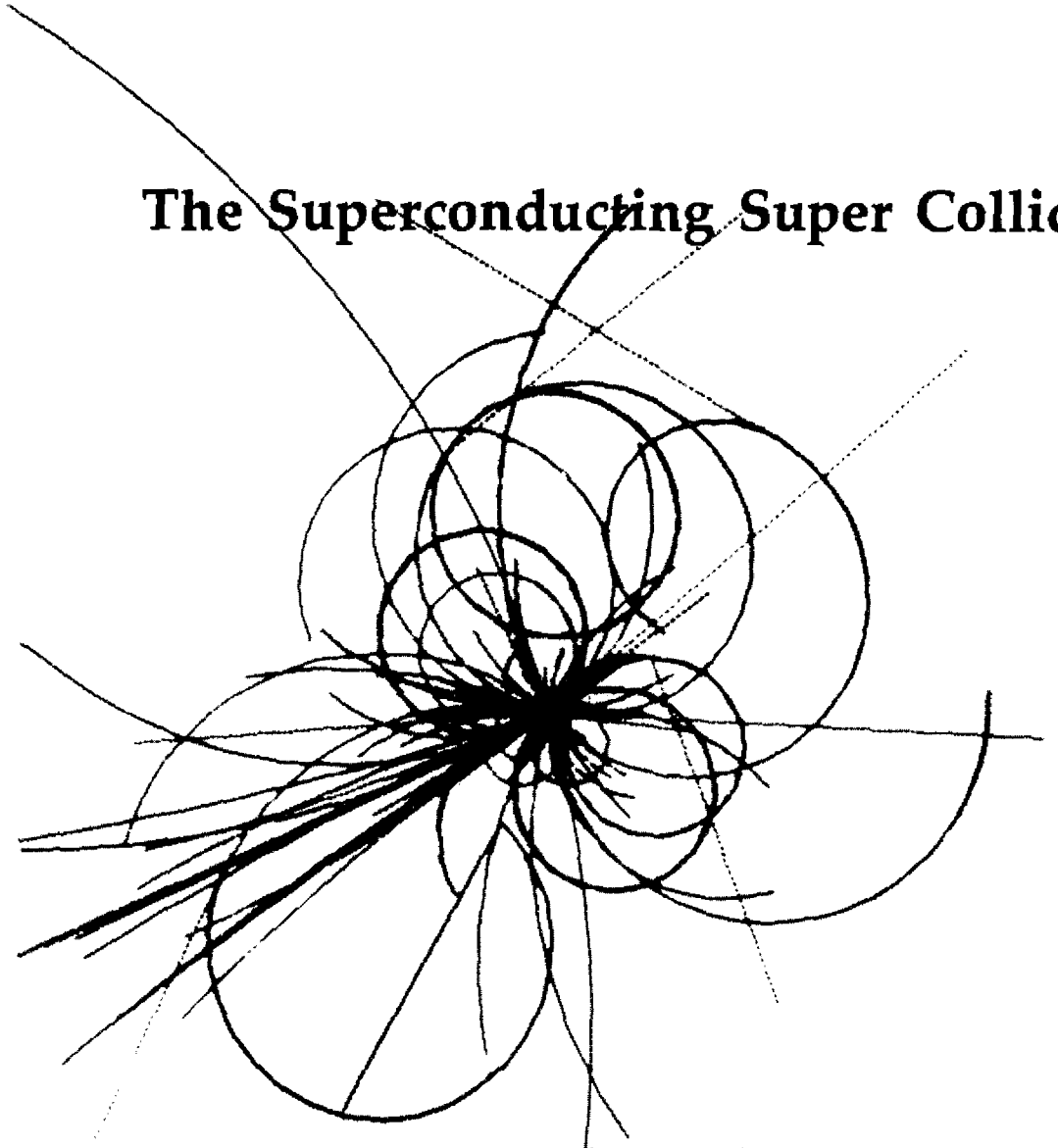
# The Superconducting Super Collider



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# Generalization of TEAPOT to Nonplanar Lattices

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## ABSTRACT

Formulas needed to convert the accelerator analysis program, TEAPOT, from planar to full three-dimensional geometry, are derived. This generalization is required for analyzing folded, "terrain-following", or any other nonplanar lattice. As in the original version the tracking is exact, and hence certainly symplectic. This report will form part of the TEAPOT manual.

# 1. Introduction.

This report describes a revision of the accelerator analysis and tracking program TEAPOT[1] , to permit the analysis of nonplanar lattices. The essential features of TEAPOT, of being exact and hence symplectic to all orders, and of handling arbitrary lattice errors, are preserved. The original version correctly treated vertical bend errors but required the ideal orbit to lie in a single plane. For many purposes such a restriction is acceptable, or even desirable, if the relatively simpler formulas result in faster program execution. What is required, and what this report describes, is an upgrade which correctly handles the few vertical bends typically present —say for beam separation— while preserving computation speed at the vast majority of lattice points at which out-of-plane bends do not occur. In effect, the lattice is broken into planar sectors within which the previous formalism holds; the new formalism describes the transitions between planar sectors. The example lattice best illustrating the economy of this strategy is the so-called "folded" SSC lattice, with the two half-rings lying in different planes; only a few elements near the fold require a full three dimensional treatment. A fully general vector treatment is given; it is shown to reduce to the previous TEAPOT planar formalism within plane sectors.

In verbal description of three dimensional geometry, it is difficult to avoid ambiguity. Some of the phrases which will appear are defined here. It is assumed that, even with vertical bends present, the entire lattice lies close to a single plane; the term horizontal will always refer to that plane, and vertical will mean normal to that plane. (Normally, vertical is also the direction defined by a gravity-plumbbob, but that is not required by the formalism, and a somewhat pathological example will be given for which it is definitely false.) On the other hand, the terms "up" and "down" bends will refer to deviations from the local plane. In that sense they are both "out-of-plane" bends. For normal polarities (positive charge, magnetic field up) the lattice will follow a roughly circular path; clockwise when viewed from above. The same thing can be expressed

by the statement that particles bend to the right in most bend elements. The term "generally outward" will mean outward in this global picture, and the term "generally forward" specifies that tangential direction in which the arc length parameter increases. Terms like these occur in the context of resolving sign ambiguities, and the precise directions do not matter. Even within a single plane, there is the possibility of "reverse bends", which cause the orbit to curve outward instead of inward. The "principal normal" points in the direction opposite to the "acceleration direction." As such, in most cases, it points in the generally outward direction; but for reverse horizontal bends, it points in the generally inward direction, and for a non-horizontal bend, it is not horizontal.

Unit vectors which are most natural for particle tracking using a differential geometric description of the orbit do not seem to be appropriate either for the betatron oscillation formalism, or for relating lattice elements to the outside world. For that reason two sets of unit vectors will be specified at every lattice site.

A set with superscript $I$, for internal, is a right-handed, orthogonal triad of unit vectors, $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^I, \hat{\mathbf{s}}^I)$, defined loosely as follows. $\hat{\mathbf{s}}^I$ is tangential to the orbit, $\hat{\mathbf{x}}^I$ points in the principal normal direction, and $\hat{\mathbf{y}}^I = \hat{\mathbf{s}}^I \times \hat{\mathbf{x}}^I$. For smooth curves these directions are unambiguous, but for the kinky straight-line orbits of TEAPOT that is not true, which is why the word "loosely" was added to the previous sentence. Exact definitions will be given below. (These axes are used also in MAD[2] , according to the MAD manual. This has not been useful, however, as the elements are broken up differently and the use of these axes is internal to the program.) This set has the desirable feature that $\hat{\mathbf{y}}^I$ is always normal to the bend plane in bend elements; as a result it has the orientation assumed for the standard multipole expansion describing the bend element. It has the undesirable effect that, in straight sections, its orientation is inherited from the previous bend; that makes its orientation possibly different for clockwise and counter-clockwise ordering of the elements. Also, in these coordinates there is an undesirable spurious local coupling between $x$ and $y$ motions at the positions

of out-of-plane bends, owing to the exchange of $x$ and $y$ axes.

Another triad of orthogonal unit vectors, with superscript $E$, for external, and differing from the previous set only by a rotation around $\hat{\mathbf{s}}^I$, is defined as follows. $\hat{\mathbf{s}}^E = \hat{\mathbf{s}}^I$, and $\hat{\mathbf{x}}^E$ is that horizontal vector normal to $\hat{\mathbf{s}}^E$. Finally $\hat{\mathbf{y}}^E = \hat{\mathbf{s}}^E \times \hat{\mathbf{x}}^E$. This will also be spelled out more explicitly below. This frame is especially appropriate for betatron description, as it avoids the above-mentioned spurious local coupling. Furthermore, since $\hat{\mathbf{x}}^E$ is always horizontal, the orientation of a multipole element is directly usable externally, for example in positioning the element with the use of a gravity-level.

For planar lattices the two triads of unit vectors are identical. For lattices with two sectors in different but parallel planes, they are identical in the planar regions. For folded lattices, they differ (slightly if the fold is slight) in at least one or the other half-lattice.

## 2. Global Coordinates.

As in the original TEAPOT, the global coordinates $(X, Y, Z)$, of the accelerator design program MAD[2] are employed. The position vector, $\mathbf{R} = (X, Y, Z)^T$, points from the origin to the point $(X, Y, Z)$. It is assumed that the lattice consists of $N$ thin multipoles whose nominal centers, $P_i$, are located at $\mathbf{R}_i; i = 1, 2, ..., N$. (The symbol $P_i$ will be used both for the point on the central trajectory and the multipole plane, normal to the central trajectory at that point.) These $N$ points completely define the geometry; there are no other orientation parameters to be externally imposed; bends in TEAPOT will be such as to guarantee that the central orbit does, in fact, pass through those points. It is, however, necessary to be careful that the orientation specification of other multipole elements is interpreted by TEAPOT as was intended by the lattice design program —probably MAD— which generated the points $P_i$.

Global unit vectors are given by $\hat{\mathbf{X}} = (1, 0, 0)^T$, $\hat{\mathbf{Y}} = (0, 1, 0)^T$, and $\hat{\mathbf{Z}} = (0, 0, 1)^T$, where the $(X, Z)$ plane is the global horizontal plane, and $Y$ is the,

positive-upward axis normal to that plane. It is assumed that there is a null multipole element at the origin and that the closed orbit starts out directed along $\hat{\mathbf{Z}}$ there. The following special assignments assure these conditions; to facilitate re-cycling at the origin the global coordinate arrays are extended redundantly, at the top and bottom, by defining the origin both as $\mathbf{R}_0$ and $\mathbf{R}_{N+1}$.

$$\begin{aligned}
\mathbf{R}_0 &: X_0 = Y_0 = Z_0 = 0.0 \\
\mathbf{R}_1 &: X_1 = Y_1 = 0.0; \qquad Z_1 = L_{0+} \\
\mathbf{R}_N &: X_N = Y_N = 0.0; \qquad Z_N = -L_{N+} \\
\mathbf{R}_{N+1} &: X_{N+1} = Y_{N+1} = Z_{N+1} = 0.0
\end{aligned} \tag{2.1}$$

The geometry of three successive points, $P_{i-1}$, $P_i$, and $P_{i+1}$ is indicated in Fig. 1.

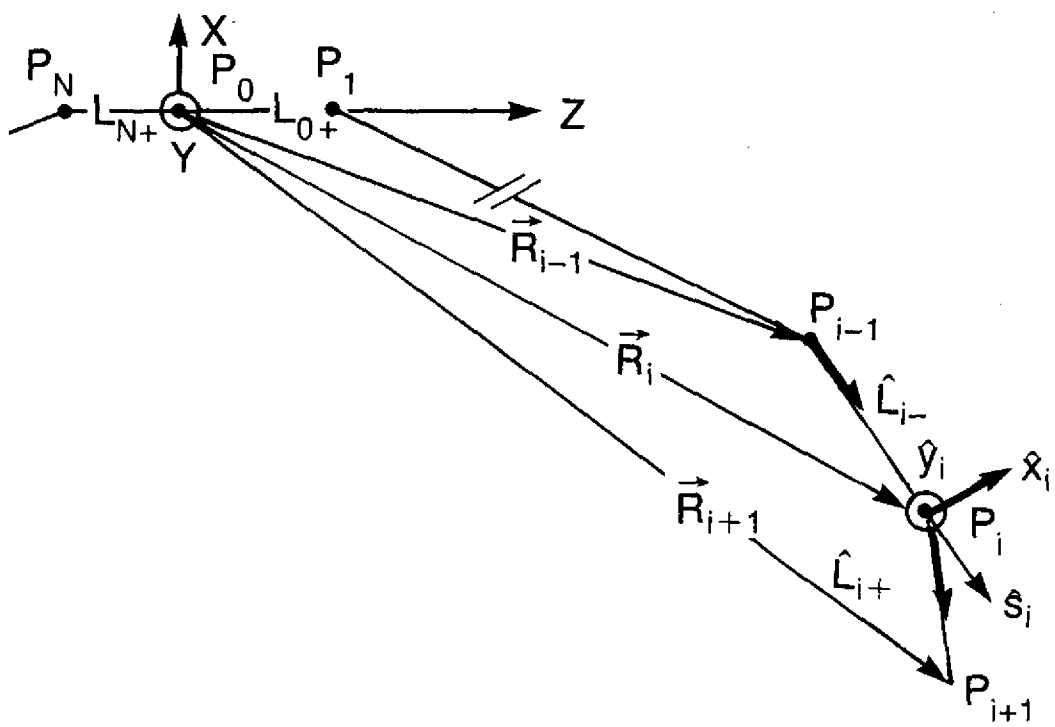


Figure 1.

## 3. General Element Types.

In the original TEAPOT, all lattice elements are formally equivalent, except drifts, which do not count. Drifts still do not count but now there are three general element types: non-bend elements, in-plane bend elements, and out-of-plane bend elements. These distinctions are internal to TEAPOT, and are transparent to the user generating the standard format input file, except for the following restrictions.

(*i*) Miniscule bends ($< 10^{-10}$ radians) are assumed to be exactly zero and are treated as non-bends. The user should fear anomalous results when allowing a bend angle to become very small, for example in testing the code. To achieve this end the bend can be replaced by a drift.

(*ii*) Zero-length bends are not allowed.

These restrictions are needed so that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^I, \hat{\mathbf{s}}^I)$ can, without exception, be calculated from $(\mathbf{R}_{i-1}, \mathbf{R}_i, \mathbf{R}_{i+1})$ in the case that element $i$ is a bend. Non-bend elements can have arbitrary length and strength since they do not change the coordinate system orientation.

(*iii*) To assure the unambiguous discrimination between in-plane and out-of-plane bends, bends in planes tilted by only a miniscule amount with respect to the previous bend plane are disallowed.

As previously, only sector bends are allowed; other entrance and exit pole-face angles are to be modeled with thin multipoles. Bend elements of length comparable to transverse dimensions must be treated carefully; see manual.

## 4. Determination of Local Unit Vectors.

The triads of local unit vectors $(\hat{\mathbf{x}}_i^I, \hat{\mathbf{y}}_i^I, \hat{\mathbf{s}}_i^I)$, introduced above, are calculated sequentially, starting at element $i = 0$). At every stage there is a tentative set, $(\hat{\mathbf{x}}^T, \hat{\mathbf{y}}^T, \hat{\mathbf{s}}^T)$ defined, to be adopted in case the element is a non-bend element. Initially these are taken to be

$$\hat{\mathbf{x}}^T = \hat{\mathbf{X}}; \quad \hat{\mathbf{y}}^T = \hat{\mathbf{Y}}; \quad \hat{\mathbf{s}}^T = \hat{\mathbf{Z}} \tag{4.1}$$

and they are updated as appropriate.

The condition for element $i$ to be a **non-bend element** is that

$$\begin{aligned} \text{either} : |\mathbf{R}_{i+1} - \mathbf{R}_i| &< 10^{-10}\text{m} \\ \text{or} : |(\mathbf{R}_{i+1} - \mathbf{R}_i) \times (\mathbf{R}_i - \mathbf{R}_{i-1})| &< 10^{-10}\text{m}^2 \end{aligned} \tag{4.2}$$

If element $i$ is a non-bend element then the tentative values are accepted,

$$\hat{\mathbf{x}}_i^I = \hat{\mathbf{x}}^T; \quad \hat{\mathbf{y}}_i^I = \hat{\mathbf{y}}^T; \quad \hat{\mathbf{s}}_i^I = \hat{\mathbf{s}}^T \tag{4.3}$$

and the same tentative values are retained. Also, input and output vectors are defined by

$$\hat{\mathbf{L}}_{i-} = \hat{\mathbf{L}}_{i+} = \hat{\mathbf{s}}^T. \tag{4.4}$$

For bend elements the following relations are used. (See Fig. 1.)

$$\hat{\mathbf{L}}_{i-} = \frac{\mathbf{R}_i - \mathbf{R}_{i-1}}{|\mathbf{R}_i - \mathbf{R}_{i-1}|} \tag{4.5}$$

$$\hat{\mathbf{L}}_{i+} = \frac{\mathbf{R}_{i+1} - \mathbf{R}_i}{|\mathbf{R}_{i+1} - \mathbf{R}_i|} \tag{4.6}$$

$$\hat{\mathbf{s}}_i^I = \frac{\mathbf{L}_{i-} + \mathbf{L}_{i+}}{|\mathbf{L}_{i-} + \mathbf{L}_{i+}|} \tag{4.7}$$

$$\hat{\mathbf{y}}_i^I = -\frac{\mathbf{L}_{i-} \times \mathbf{L}_{i+}}{|\mathbf{L}_{i-} \times \mathbf{L}_{i+}|} \tag{4.8}$$

$$\hat{\mathbf{x}}_i^I = \hat{\mathbf{y}}_i^I \times \hat{\mathbf{s}}_i^I \tag{4.9}$$

All these vectors are represented in the computer by $(X, Y, Z)$ components. Note that $\hat{\mathbf{s}}_i^I$ bisects the input and output directions, making it tangential to the orbit at the center of the element. It points generally forward. $\hat{\mathbf{x}}_i^I$ and $\hat{\mathbf{y}}_i^I$ are principal normal and binormal, respectively, with $\hat{\mathbf{y}}_i^I$ being normal to the bend plane. Except in regions of vertical or reverse bends, $\hat{\mathbf{x}}_i^I$ points generally outward, and $\hat{\mathbf{y}}_i^I$ points generally vertically upwards.

The calculations indicated by (4.5) through (4.9) are performed only for bend elements. They are also used to make the distinction between "in-plane" and "out-of-plane" bends.

$$\begin{aligned} \text{in-plane bend} : &\hat{\mathbf{y}}_i^I = \hat{\mathbf{y}}_i^T; \quad |\hat{\mathbf{y}}_i^I \times \hat{\mathbf{y}}_i^T| < 10^{-10} \\ \text{out-of-plane bend} : &\hat{\mathbf{y}}_i^I \neq \hat{\mathbf{y}}_i^T; \quad |\hat{\mathbf{y}}_i^I \times \hat{\mathbf{y}}_i^T| \geq 10^{-10} \end{aligned} \tag{4.10}$$

It is because these are calculated quantities, that it was necessary to introduce the small quantity to make this distinction. Unfortunately the "TILT" entry for bend elements in the standard input language cannot be used for this purpose, since non-zero TILT refers to non-horizontal bends; that is, it distinguishes elements tilted in our external, not internal, reference frame. During initial set up a flag is set, or not, depending on the condition (4.10). Subsequently, the decision whether to use planar or non-planar formulas is based on this flag setting. If there is an out-of-plane bend at $P_i$, the three-dimensional formalism is required to track particles from the $P_{i-1}$ frame to the $P_i$ frame.

On leaving the bend element, new tentative unit vectors are given by

$$\begin{aligned} \hat{\mathbf{y}}^T &= \hat{\mathbf{y}}_i^I, \\ \hat{\mathbf{s}}^T &= \hat{\mathbf{L}}_{i+}, \\ \hat{\mathbf{x}}^T &= \hat{\mathbf{y}}_i^T \times \hat{\mathbf{s}}_i^T. \end{aligned} \tag{4.11}$$

At every element the external triad can be calculated by

$$\begin{aligned}
\hat{\mathbf{s}}_i^E &= \hat{\mathbf{s}}_i^I, \\
\hat{\mathbf{x}}_i^E &= \frac{\hat{\mathbf{Y}} \times \hat{\mathbf{s}}_i^E}{|\hat{\mathbf{Y}} \times \hat{\mathbf{s}}_i^E|}, \\
\hat{\mathbf{y}}_i^E &= \hat{\mathbf{s}}_i^E \times \hat{\mathbf{x}}_i^E .
\end{aligned} \tag{4.12}$$

(The orbit is assumed to be nowhere vertical.) Note that $\hat{\mathbf{x}}_i^E$ points in the generally outward direction. The external and internal triads differ only by a rotation $\phi_i^{EI}$ around the $\hat{\mathbf{s}}_i^D$ axis. It is given by

$$\begin{aligned}
\sin\phi_i^{EI} &= \hat{\mathbf{y}}_i^E \cdot \hat{\mathbf{x}}_i^I, \\
\cos\phi_i^{EI} &= \hat{\mathbf{x}}_i^E \cdot \hat{\mathbf{x}}_i^I, \\
\phi_i^{EI} &= \arctan(\sin\phi_i^{EI} / \cos\phi_i^{EI}).
\end{aligned} \tag{4.13}$$

Suppressing the index $i$, the following abbreviations will be used below:

$$\begin{aligned}
S_n^{EI} &= \sin(n\phi^{EI}) \\
C_n^{EI} &= \cos(n\phi^{EI})
\end{aligned} \tag{4.14}$$

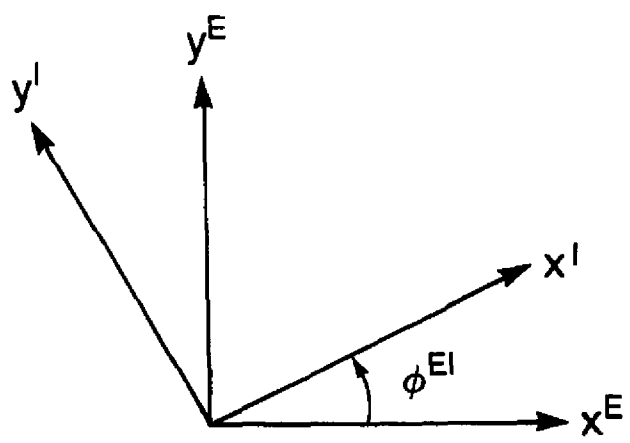


Figure 2.

During particle tracking the particle trajectory will pass a point in the multipole plane specified by

$$\begin{aligned}\hat{\mathbf{r}}_i &= \overset{I}{x}_i \overset{I}{\hat{\mathbf{x}}}_i + \overset{I}{y}_i \overset{I}{\hat{\mathbf{y}}}_i, \\ &= \overset{E}{x}_i \overset{E}{\hat{\mathbf{x}}}_i + \overset{E}{y}_i \overset{E}{\hat{\mathbf{y}}}_i .\end{aligned} \tag{4.15}$$

The components are related by

$$\begin{aligned}\overset{E}{x}_i &= \cos\overset{EI}{\phi}_i \overset{I}{x}_i - \sin\overset{EI}{\phi}_i \overset{I}{y}_i, \\ \overset{E}{y}_i &= \sin\overset{EI}{\phi}_i \overset{I}{x}_i + \cos\overset{EI}{\phi}_i \overset{I}{y}_i,\end{aligned} \tag{4.16}$$

and the inverse

$$\begin{aligned}\overset{I}{x}_i &= \cos\overset{EI}{\phi}_i \overset{E}{x}_i + \sin\overset{EI}{\phi}_i \overset{E}{y}_i, \\ \overset{I}{y}_i &= -\sin\overset{EI}{\phi}_i \overset{E}{x}_i + \cos\overset{EI}{\phi}_i \overset{E}{y}_i .\end{aligned} \tag{4.17}$$

## 5. Coordinates of $P_{i+1}$ in the $P_i$ Frame.

For particle tracking it is necessary to know the coordinates $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{S}_i)$ of the point $P_{i+1}$ in the $(\overset{I}{\hat{\mathbf{x}}}_i, \overset{I}{\hat{\mathbf{y}}}_i, \overset{I}{\hat{\mathbf{s}}}_i)$ local coordinate system at $P_i$. The formulas, written as if $P_i$ is a bend element, will simplify if $P_i$ is a non-bend element. As in TEAPOT (2.6) the angular deflection $\Delta\Theta_i$ at $P_i$ satisfies

$$\Delta\Theta_i = -2\sin^{-1}\frac{\widetilde{b}_{0i}}{2}, \tag{5.1}$$

where the multipole elements $\widetilde{b}_n$ are related to the "field integrals" $b_n L$ by

$$\widetilde{b}_n = \frac{LB_0 b_n}{p_0/e}, \tag{5.2}$$

and similarly for $\widetilde{a}_n$. We assume the axis orientation is such that $\widetilde{a}_0 = 0$; this is always possible even for bend elements, and just amounts to aligning the

multipole axis with the bend plane. From Fig. 3 we have

$$\sin(\Delta\Theta_i/2) = \hat{\mathbf{L}}_{i+} \cdot \hat{\mathbf{x}}_i, \tag{5.3}$$

and hence

$$\widetilde{b}_{0i} = -2\hat{\mathbf{L}}_{i+} \cdot \hat{\mathbf{x}}_i. \tag{5.4}$$

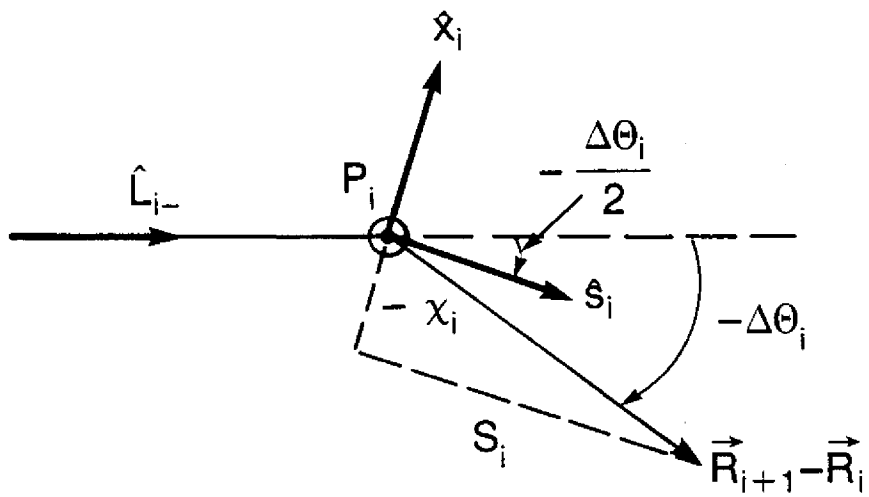


Figure 3.

The formulas given so far relate the geometry to the bend strengths. They are useful to check consistency.

The needed coordinates of $P_{i+1}$ are

$$\begin{aligned} \mathcal{X}_i &= (\mathbf{R}_{i+1} - \mathbf{R}_i) \cdot \hat{\mathbf{x}}_i, \\ \mathcal{S}_i &= (\mathbf{R}_{i+1} - \mathbf{R}_i) \cdot \hat{\mathbf{s}}_i, \\ \mathcal{Y}_i &= 0. \end{aligned} \tag{5.5}$$

The vanishing of $\mathcal{Y}_i$ follows from the fact that $\hat{\mathbf{y}}_i$ is normal to the plane specified by the points $P_{i-1}, P_i, P_{i+1}$. These formulas are valid for zero length elements as well.

## 6. Transformation From the $P_i$ to the $P_{i+1}$ Frame.

We wish to relate the frame $P_i$ components $(\chi_i, \xi_i, \sigma_i)$ of a vector $\mathbf{w}_i$ to the components $(\chi_{i+1}, \xi_{i+1}, \sigma_{i+1})$ of the same vector in the $P_{i+1}$ frame.

$$\begin{aligned} \mathbf{w}_i &= \chi_i \hat{\mathbf{x}}_i + \xi_i \hat{\mathbf{y}}_i + \sigma_i \hat{\mathbf{s}}_i \\ &= \chi_{i+1} \hat{\mathbf{x}}_{i+1} + \xi_{i+1} \hat{\mathbf{y}}_{i+1} + \sigma_{i+1} \hat{\mathbf{s}}_{i+1} \end{aligned} \tag{6.1}$$

The relationship can be expressed as the product of three rotations.

$$\begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} \xrightarrow[\frac{\Delta\Theta_i}{2}]{} \begin{pmatrix} \chi' \\ \xi' \\ \sigma' \end{pmatrix} \xrightarrow[\Delta\Phi]{} \begin{pmatrix} \chi'' \\ \xi'' \\ \sigma'' \end{pmatrix} \xrightarrow[\frac{\Delta\Theta_{i+1}}{2}]{} \begin{pmatrix} \chi_{i+1} \\ \xi_{i+1} \\ \sigma_{i+1} \end{pmatrix}. \tag{6.2}$$

Using (5.3) we introduce a briefer notation

$$\begin{aligned} s_1 &= \sin \Delta\Theta_i/2 = \hat{\mathbf{L}}_{i+} \cdot \hat{\mathbf{x}}_i, \\ c_1 &= \cos \Delta\Theta_i/2 = \hat{\mathbf{L}}_{i+} \cdot \hat{\mathbf{s}}_i, \\ s_3 &= \sin \Delta\Theta_{i+1}/2 = \hat{\mathbf{L}}_{i+1+} \cdot \hat{\mathbf{x}}_{i+1}, \\ c_3 &= \cos \Delta\Theta_{i+1}/2 = \hat{\mathbf{L}}_{i+1+} \cdot \hat{\mathbf{s}}_{i+1}. \end{aligned} \tag{6.3}$$

The first transformation is a rotation around $\hat{\mathbf{y}}_i$ by an angle $-\Delta\Theta_i/2$ and the third is a rotation around $\hat{\mathbf{y}}_{i+1}$ by an angle $-\Delta\Theta_{i+1}/2$. The intermediate transformation is an azimuthal rotation around $\hat{\mathbf{s}}_i$. Note that $\hat{\mathbf{s}}_i$ lies in both the bend plane at $P_i$ and the bend plane at $P_{i+1}$. The unit vector $\hat{\mathbf{y}}_i$ is normal to the former plane and hence normal to $\hat{\mathbf{s}}_i$. $\hat{\mathbf{y}}_{i+1}$ is normal to the latter plane and hence also normal to $\hat{\mathbf{s}}_i$. As a result there is a rotation around $\hat{\mathbf{s}}_i$ making $\hat{\mathbf{y}}_i$ parallel to $\hat{\mathbf{y}}_{i+1}$. Calling the rotation angle $\Phi$, with a positive angle corresponding to a right hand screw motion,

$$\begin{aligned} s_2 &= \sin \Phi = (\hat{\mathbf{y}}_i \times \hat{\mathbf{y}}_{i+1}) \cdot \hat{\mathbf{L}}_{i+}, \\ c_2 &= \cos \Phi = \hat{\mathbf{y}}_i \cdot \hat{\mathbf{y}}_{i+1}. \end{aligned} \tag{6.4}$$

Writing these transformations explicitly we obtain for the three rotations

$$
\begin{aligned}
\begin{pmatrix} \chi_{i+1} \\ \xi_{i+1} \\ \sigma_{i+1} \end{pmatrix} &= \begin{pmatrix} c_3 & 0 & -s_3 \\ 0 & 1 & 0 \\ s_3 & 0 & c_3 \end{pmatrix} \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} \\
&= \begin{pmatrix} c_3c_2c_1 - s_3s_1 & c_3s_2 & -c_3c_2s_1 - s_3c_1 \\ -s_2c_1 & c_2 & s_2s_1 \\ s_3c_2c_1 + c_3s_1 & s_3s_2 & -s_3c_2s_1 + c_3c_1 \end{pmatrix} \begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} \\
&= \begin{pmatrix} m_{xx} & m_{xy} & m_{xs} \\ m_{yx} & m_{yy} & m_{ys} \\ m_{sx} & m_{sy} & m_{ss} \end{pmatrix} \begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} \\
&= \mathcal{M}_{i+} \begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} .
\end{aligned}
\tag{6.5}
$$

The matrix $\mathcal{M}_{i+}$, just introduced, can also be used to relate unit vectors in successive frames:

$$
\begin{pmatrix} \hat{\mathbf{x}}_i & \hat{\mathbf{y}}_i & \hat{\mathbf{s}}_i \end{pmatrix} \begin{pmatrix} \chi_i \\ \xi_i \\ \sigma_i \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}}_{i+1} & \hat{\mathbf{y}}_{i+1} & \hat{\mathbf{s}}_{i+1} \end{pmatrix} \mathcal{M}_{i+} \begin{pmatrix} \chi_{i+1} \\ \xi_{i+1} \\ \sigma_{i+1} \end{pmatrix} \tag{6.6}
$$

From this, using the relation $(\mathcal{M}_{i+}^T)^{-1} = \mathcal{M}_{i+}$, which follows because $\mathcal{M}_{i+}$ is a rotation matrix, we obtain

$$
\begin{pmatrix} \hat{\mathbf{x}}_{i+1} \\ \hat{\mathbf{y}}_{i+1} \\ \hat{\mathbf{s}}_{i+1} \end{pmatrix} = \mathcal{M}_{i+} \begin{pmatrix} \hat{\mathbf{x}}_i \\ \hat{\mathbf{y}}_i \\ \hat{\mathbf{s}}_i \end{pmatrix} \tag{6.7}
$$

The transformation of velocity components is given by

$$
\begin{pmatrix} v_{x(i+1)} \\ v_{y(i+1)} \\ v_{s(i+1)} \end{pmatrix} = \mathcal{M}_{i+} \begin{pmatrix} v_{xi} \\ v_{yi} \\ v_{si} \end{pmatrix} , \tag{6.8}
$$

and $|\mathbf{v}|$ is unchanged.

## 7. Deflection in the Multipole at $P_i$.

We can now generalize the steps which TEAPOT performs in propagating a particle which intersects the multipole plane $P_i$ at $(x, y)$ with velocity components $v_{x-}$,$v_{y-}$,$v_{s-}$ in the local $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{s}})$ frame. For brevity we have dropped the index $i$ from the subscripts. The velocity components of the particle leaving the multipole, $v_{x+}$,$v_{y+}$,$v_{s+}$, are evaluated as described in TEAPOT (4.1),

$$\begin{aligned} v_{x+} &= v_{x-} + F_x v/(1+\delta), \\ v_{y+} &= v_{y-} + F_y v/(1+\delta), \\ v_{s+} &= \sqrt{v^2 - v_{x+}^2 - v_{y+}^2}, \end{aligned} \tag{7.1}$$

where $v^2 = v_{x-}^2 + v_{y-}^2 + v_{s-}^2$ is a constant of the motion, unless synchrotron oscillations are being modeled. $F_x$ and $F_y$ are to be evaluated from the multipole coefficients $\overset{I}{a}_m, \overset{I}{b}_m$. Because the "internal" coordinates are in use, formulas can be copied directly from the original, planar formulas of TEAPOT (4.13). Naturally the multipole elements must be expressed in the internal system; that is indicated by the superscript $I$. The orientation of bend elements has been fixed previously to assure this orientation, but for non-bend elements it can only be assured by the correct generation of the input lattice file.

Unfortunately the three-dimensional aspects of the so-called "standard input format" are not completely unambiguous. In conformity to the MAD actual code performance (which disagrees with our reading of the MAD documentation) we declare the "standard" to be as follows: erect elements are erect in what we have called "external" coordinates. This can only be clarified by examples.

A dipole specified to have TILT $= \pi/2$, (where, for this particular case of rotation by 90 degrees, entering TILT, with no value would be equivalent) bends vertically, or what is the same thing, the axis of rotation is horizontal. Both vertical and horizontal are used in the sense defined in the introduction; they refer to a single plane, the same for all lattice elements. As a pathological, but

correct, example, a machine with all dipoles having the attribute TILT would lie in a single plane, the $(Y, Z)$ plane. (To comply with the general spirit of what has gone before it would be more natural to choose the single external plane as being the $(Y, Z)$ plane, but then the words vertical and horizontal would be mixed up.) Continuing with the pathological example, observe that in "internal" coordinates, all the bends, except the first and last, are "in-plane"; they have $\overset{I}{a}_0 = 0.0$. (The first bend is special only because of the initial assignment (4.1) for the tentative unit vectors.)

Quadrupoles are even more confusing. For horizontal, planar, lattices a positive quadrupole strength implies horizontal focusing of positively charged particles; the value of $b_1$ is positive, and it is not necessary to distinguish between internal and external coordinates. Such a quadrupole could be said to have in-plane focusing. A vertically focusing quad can be modeled either by setting TILT $= \pi/2$, or by making the quadrupole strength negative. We now insist that these features apply to "external" coordinates, in the case that the accelerator does not lie in a single plane. Consider, for example, a large almost horizontal, ring consisting of nothing but FODO cells. Consider introducing a small vertical bend just after an F (horizontally focusing) element. We assume that the following D element will need to be more-or-less as it was before, and that its specification in the input file is changed, at most, by a little bit; it will have no TILT specified, and it will have negative strength. Note, however, that in internal coordinates, this element will have in-plane **focusing**; it will satisfy $\overset{I}{b}_1 = -\overset{E}{b}_1 =$ positive.

Since the input file is referred to external coordinates and particle tracking in TEAPOT uses internal coordinates, it is necessary to perform a transformation of the multipole elements as they are stored in the TEAPOT thin-element machine file. The constants for this transformation have been obtained in (4.14); the transformations are

$$\begin{aligned} \overset{I}{b}_n &= \overset{E}{b}_n \overset{EI}{C}_{n+1} - \overset{E}{a}_n \overset{EI}{S}_{n+1} \\ \overset{I}{a}_n &= \overset{E}{a}_n \overset{EI}{C}_{n+1} + \overset{E}{b}_n \overset{EI}{S}_{n+1} \end{aligned} \tag{7.2}$$

To improve the chances of getting the signs correct, and to make contact with conventional definitions for the multipole elements, this transformation will now be derived explicitly. Internal and external coordinates are related by

$$\phi^{E} = \phi^{I} + \phi^{EI} \tag{7.3}$$

The $z$ component (the only non-vanishing component) of the vector potential has the same value in both external and internal coordinates:

$$\begin{aligned} A_z &= -\sum_n \frac{(b_n^E + ia_n^E)(x^E + iy^E)^{n+1}}{n+1} \\ &= -\sum_n \frac{(b_n^E + ia_n^E) r^{n+1} \exp(i(n+1)\phi^E)}{n+1} \\ &= -\sum_n \frac{(b_n^E + ia_n^E)(C_{n+1} + iS_{n+1}) r^{n+1} \exp(i\phi^I)}{n+1} \\ &= -\sum_n \frac{(b_n^I + ia_n^I)(x^I + iy^I)^{n+1}}{n+1} \end{aligned} \tag{7.4}$$

Comparison of the coefficients yields (7.2). From the vector potential the magnetic field is obtained from $\mathbf{B} = \nabla \times \mathbf{A}$, which yields the conventional formula

$$B_y^E + iB_x^E = \sum_n (b_n^E + ia_n^E)(x^E + iy^E)^n. \tag{7.5}$$

From now on, only internal coordinates will be used and the superscript $I$ will be suppressed. The deflections are calculated recursively from the multipoles, as

in the original TEAPOT.

$$
\begin{aligned}
\widetilde{B}_y &= \widetilde{b}_M, \\
\widetilde{B}_x &= \widetilde{a}_M, \\
\widetilde{B}_y &= x\widetilde{B}_y - y\widetilde{B}_x + \widetilde{b}_{M-1}, \\
\widetilde{B}_x &= y\widetilde{B}_y + x\widetilde{B}_x + \widetilde{a}_{M-1}, \\
\widetilde{B}_y &= x\widetilde{B}_y - y\widetilde{B}_x + \widetilde{b}_{M-2}, \\
\widetilde{B}_x &= y\widetilde{B}_y + x\widetilde{B}_x + \widetilde{a}_{M-2}, \\
&\circ \\
&\circ \\
&\circ \\
-F_x = \widetilde{B}_y &= x\widetilde{B}_y - y\widetilde{B}_x + \widetilde{b}_0, \\
F_y = \widetilde{B}_x &= y\widetilde{B}_y + x\widetilde{B}_x + \widetilde{a}_0.
\end{aligned}
\tag{7.6}
$$

## 8. Finding the Point of Intersection With the Next Multipole Plane.

The parameter $s$ varies along the trajectory T, shown in Fig. 4; it satisfies the relation

$$s = v_s t, \tag{8.1}$$

where $t$ is time.

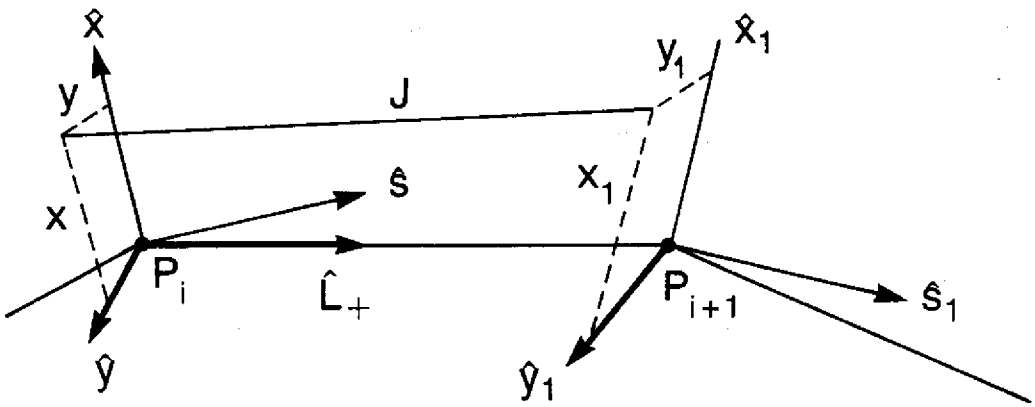


Figure 4.

Dropping the subscript $i$ from quantities in the present plane, and identifying quantities in the next multipole plane, $P_1$, with subscripts 1, the intersection of T with $P_1$ is determined by the condition

$$(x + \frac{v_x}{v_s}s)\hat{\mathbf{x}} + (y + \frac{v_y}{v_s}s)\hat{\mathbf{y}} + s\hat{\mathbf{s}} = \mathcal{X}\hat{\mathbf{x}} + \mathcal{S}\hat{\mathbf{s}} + x_1\hat{\mathbf{x}}_1 + y_1\hat{\mathbf{y}}_1 \tag{8.2}$$

Substituting from (6.7), and equating the coefficients of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{s}}$ to zero,

$$\begin{aligned} -x - \frac{v_x s}{v_s} + \mathcal{X} + x_1 m_{xx} + y_1 m_{yx} &= 0, \\ -y - \frac{v_y s}{v_s} + x_1 m_{xy} + y_1 m_{yy} &= 0, \\ -s + \mathcal{S} + x_1 m_{xs} + y_1 m_{ys} &= 0. \end{aligned} \tag{8.3}$$

Solving for $x_1, y_1$ and $s$ we get

$$\begin{pmatrix} x_1 \\ y_1 \\ s \end{pmatrix} = \mathcal{M}_{\mathbf{v}}^{-1} \begin{pmatrix} x - \mathcal{X} \\ y \\ -\mathcal{S} \end{pmatrix}, \tag{8.4}$$

where

$$\mathcal{M}_{\mathbf{v}} = \begin{pmatrix} m_{xx} & m_{yx} & -\frac{v_x}{v_s} \\ m_{xy} & m_{yy} & -\frac{v_y}{v_s} \\ m_{xs} & m_{ys} & -1 \end{pmatrix}. \tag{8.5}$$

Naturally the value of $s_1$ is zero. Since $\mathcal{M}_{\mathbf{v}}$ depends on $\mathbf{v}$, which is a function of the trajectory, and these calculations occur in the innermost tracking loop, it is sensible to invert the matrix explicitly.

$$\mathcal{M}_{\mathbf{v}}^{-1} = \frac{1}{|\mathcal{M}_{\mathbf{v}}|} \begin{pmatrix} m_{ys}\frac{v_y}{v_s} - m_{yy} & -m_{ys}\frac{v_x}{v_s} + m_{yx} & -m_{yx}\frac{v_y}{v_s} + m_{yy}\frac{v_x}{v_s} \\ -m_{xs}\frac{v_y}{v_s} + m_{xy} & m_{xs}\frac{v_x}{v_s} - m_{xx} & m_{xx}\frac{v_y}{v_s} - m_{xy}\frac{v_x}{v_s} \\ m_{xy}m_{ys} - m_{xs}m_{yy} & m_{xs}m_{yx} - m_{xx}m_{ys} & m_{xx}m_{yy} - m_{xy}m_{yx} \end{pmatrix} \tag{8.6}$$

with

$$|\mathcal{M}_{\mathbf{v}}| = -(m_{xy}m_{ys} - m_{xs}m_{yy})\frac{v_x}{v_s} + (m_{xx}m_{ys} - m_{xs}m_{yx})\frac{v_y}{v_s} - (m_{xx}m_{yy} - m_{xy}m_{yx}). \tag{8.7}$$

## 9. Reduction to the Planar Case.

In the special case of a planar lattice, the angle $\phi^{EI} = 0$. The rotation matrix $\mathcal{M}_{i+}$ reduces to

$$\mathcal{M}_{i+,flat} = \begin{pmatrix} \cos\frac{\Delta\Theta_i+\Delta\Theta_{i+1}}{2} & 0 & -\sin\frac{\Delta\Theta_i+\Delta\Theta_{i+1}}{2} \\ 0 & 1 & 0 \\ \sin\frac{\Delta\Theta_i+\Delta\Theta_{i+1}}{2} & 0 & \cos\frac{\Delta\Theta_i+\Delta\Theta_{i+1}}{2} \end{pmatrix}. \tag{9.1}$$

With this substitution for $\mathcal{M}_{i+,flat}$, equation (6.8) agrees with TEAPOT (4.21) where the unfortunate choice of the symbol $\Phi_{i+1}$ for the rotation angle between successive multipole planes should now be suppressed as the multipole planes no longer necessarily contain a purely vertical axis.

The transformation matrix for the particle positions in the case of a planar lattice reduces to

$$\mathcal{M}^{-1}_{\mathbf{v},flat} = \frac{1}{|\mathcal{M}_{\mathbf{v},flat}|} \begin{pmatrix} -1 & 0 & \frac{v_x}{v_s} \\ \frac{v_y}{v_s}\sin\Delta\Theta_s & -\cos\Delta\Theta_s - \frac{v_x}{v_s}\sin\Delta\Theta_s & \frac{v_y}{v_s}\cos\Delta\Theta_s \\ \sin\Delta\Theta_s & 0 & \cos\Delta\Theta_s \end{pmatrix}, \tag{9.2}$$

where $|\mathcal{M}_{\mathbf{v},flat}| = -\cos\Delta\Theta_s - v_x/v_s \sin\Delta\Theta_s$ and $\Delta\Theta_s = (\Delta\Theta_i + \Delta\Theta_{i+1})/2$. Substituting this expression into equation (8.4) regenerates the formulae in TEAPOT (4.20).

# REFERENCES


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