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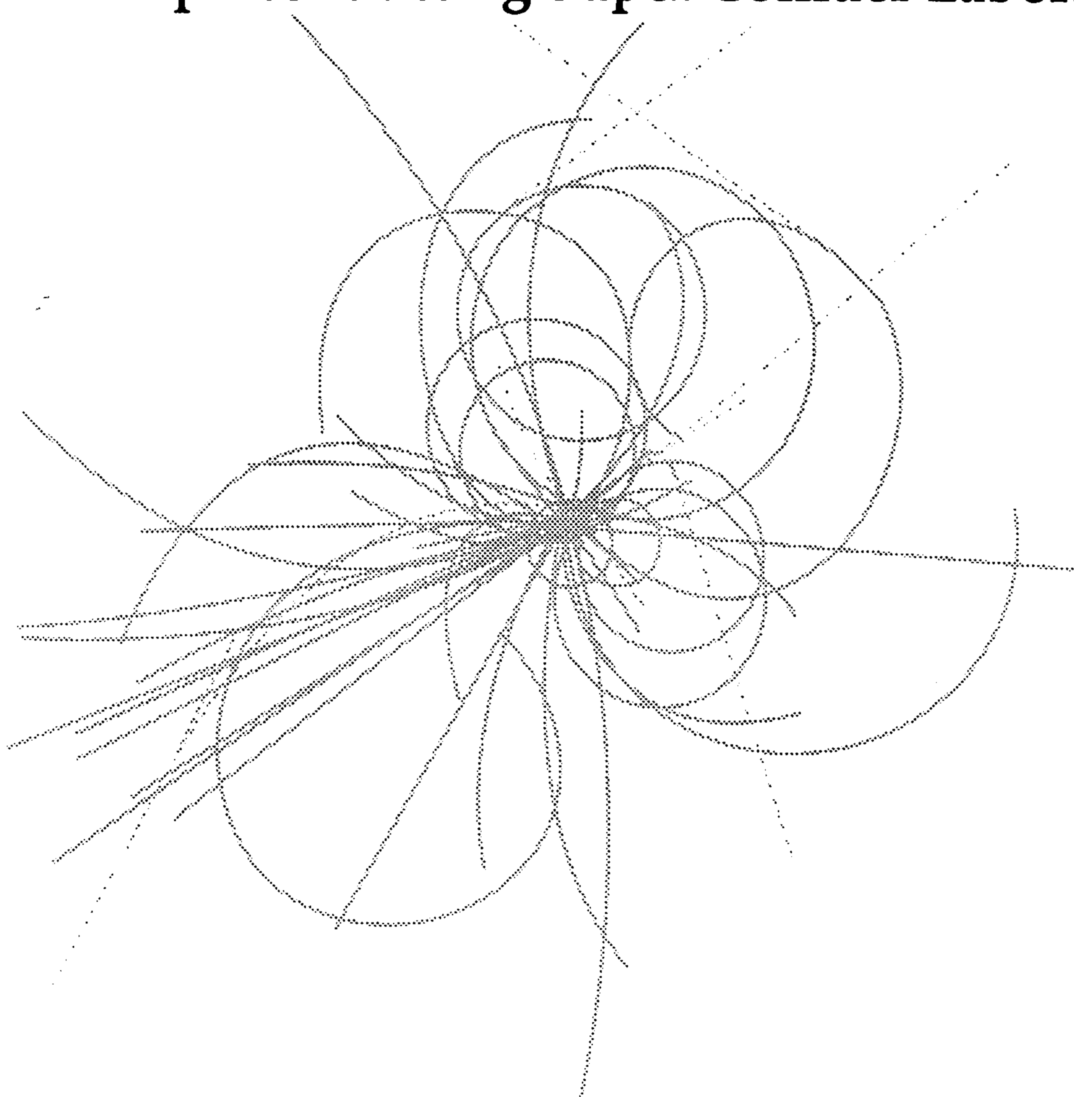

A Method to Render Second Order Beam Optics Programs Symplectic

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A METHOD TO RENDER SECOND ORDER BEAM OPTICS PROGRAMS SYMPLECTIC*

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Violations of Liouville's theorem have been observed in various tracking studies employing programs based on a second order matrix ray trace [1]. This is because the truncated Taylor's series transformation used by the codes in question is not canonical. We describe a simple method, based on a generating function, that brings the ray trace to canonical form.

This method utilizes the second order matrix elements provided by existing beam optics programs, and may be implemented by the use of a few FORTRAN subroutines. As an example, the program DIMAT [2] has been modified to accept the symplectric ray trace; results obtained using both symplectic and nonsymplectic transformations are compared.

Choice of Coordinates

The coordinates x,x',y,y',ℓ,δ employed by most matrix programs are not an exactly canonical set (as they cannot be described by a Hamiltonian system which leads to the Lorentz force law without approximation). In order to insure the result of any calculation is obtained in a canonical fashion, we will use coordinates x,p_x,y,p_y,t,p_t, where x and y are transverse displacements, p_x and p_y their conjugate moment, t is the time of flight deviation of a nonideal particle relative to the synchronous particle, and p_t is -(energy deviation) of a nonideal particle relative to the synchronous particle.

These two sets of coordinates are related by the following (noncanonical) transformation.

$$\begin{aligned}
x &= x \\
p_x &= (1+\delta)x'/\sqrt{1+(x'^2+y'^2)} \\
y &= y \\
p_y &= (1+\delta)y'/\sqrt{1+(x^2+y'^2)} \\
t &= \ell/\beta \\
&\quad +((L_0+\ell)/\beta)[\beta\sqrt{(1+\delta)^2+(1-\beta^2)/\beta^2}-(1+\delta)]/(1+\delta) \\
p_t &= 1/\beta - \sqrt{(1+\delta)^2+(1-\beta^2)/\beta^2}
\end{aligned} \qquad (1)$$

Here, $\beta = v_0/c$ and L_0 is the design orbit length in the accelerator, element or beamline under consideration.

Second Order Matrix Transformation

Most existing matrix codes employ transformations of the noncanonical variables (x,x',y,y',ℓ,δ), which may be written as follows.

$$\bar{z}_i = \sum_{j=1}^{27} A_{ij} v_j \qquad i = 1, 2, 3, 4, 5, 6 \quad (2)$$

In this relation, $\bar{z} = (\bar{x},\bar{x}',\bar{y},\bar{y}',\bar{\ell},\bar{\delta})$ is the image of $z = (x,x',y,y',\ell,\delta)$ under the second order transformation, and v is the following 27-component vector.

$$v = (x,x',y,y',\ell,\delta,x^2,xx',xy,xy',x\ell,x\delta,x'^2,x'y,x'y', \ldots, \ell^2,\ell\delta,\delta^2)$$

Equation (2) is a nonsymplectic transformation of noncanonical variables. (The nonsymplecticity results from the truncation of a Taylor's series, the first two orders of which are represented by (2).) That (2) is a nonsymplectic transformation may be seen by computation of the Poisson bracket $[\bar{x}, \bar{x}']$; if $[x, x'] \equiv 1$, we find $[\bar{x}, \bar{x}'] = 1 +$ (second order terms), in violation of the symplectic condition. Thus, (2) is symplectic only to second order.

We seek a canonical transformation (i.e., a symplectic transformation of canonical variables), which reproduces the results of (2) through second order (the advertised accuracy). As a first step, we rewrite (2) in terms of canonical variables using equations (1), to obtain a nonsymplectic transformation of canonical variables. In practice, as (2) is accurate to second order only, equations (1) may be inverted and expanded to second order in the canonical variables, and the results for $x, x',\ldots$ inserted in (2) to obtain a transformation of the following form.

$$\bar{\xi}_i = \sum_{j=1}^{27} \bar{A}_{ij} w_j \qquad i = 1, 2, 3, 4, 5, 6 \qquad (2')$$

Here, $\bar{\xi} = (\bar{x},\bar{p}_x,\bar{y},\bar{p}_y,\bar{t},\bar{p}_t)$ is the image of $\xi = (x,p_x,y,p_y,t,p_t)$ under the nonlinear transformation (2') and w is the following 27-component vector.

$$w = (x,p_x,y,p_t,t,p_t,x^2,xp_x,xy,xp_y,xt,xp_t,p_x^2,p_xy, \ldots, t^2,tp_t,p_t^2)$$

Equation (2'), like equation (2), is a nonsymplectic transformation, but is in terms of canonical variables and will reproduce all results of equation (2) through terms of second order (the advertised accuracy of either (2) or (2')).

Use of a Generating Function

It is possible, through the use of a generating function, to obtain a symplectic transformation of canonical variables (i.e., a canonical transformation), which reproduces (2') through second order while remaining symplectic to all orders. We decompose (2') into the following pair of transformations.

$$\bar{\xi}_i = \sum_{j=1}^{6} \bar{A}_{ij} \tilde{\xi}_j \qquad i = 1, 2, 3, 4, 5, 6 \quad (3a)$$

$$\tilde{\xi}_j = \sum_{k=1}^{27} B_{jk} w_k \qquad j = 1, 2, 3, 4, 5, 6 \quad (3b)$$

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The notation here is as in equation (2'). Equation (3a) is linear; by construction the 6x6 ("linear") portion of $\bar{A}$ is a symplectic matrix. The matrix B is defined as follows.

$$B_{jk} = \sum_{\ell=1}^{6} (\bar{A}^{-1})_{j\ell}\bar{A}_{\ell k}$$

Here, $(\bar{A}^{-1})$ is the inverse of the 6x6 (linear) part of the $\bar{A}$ and $\bar{A}_{\ell k}$ is the full 6x27 matrix. By construction, $B_{jk} = \delta_{jk}$ for j,k=1, 2,..., 6; that is, the transformation (3b) differs from the identity only in second order.

We now seek a generating function F such that the following equations *reproduce* (3b) though terms of second order.

$$\tilde{q}_i = \partial F(q,\tilde{p})/\partial\tilde{p}_i \qquad (4a)$$

$$i = 1, 2, 3$$

$$p_i = \partial F(q,\tilde{p})/\partial q_i \qquad (4b)$$

In relation, $q=(x,y,t)$ and $p=(p_x,p_y,p_t)$ with similar definitions for $\tilde{q}$ and $\tilde{p}$. As we wish to reproduce (3b) only to second order, and as (3b) is close to the identity, we may take

$$F(q,\tilde{p}) = \sum_{i=1}^{3} q_i\tilde{p}_i + F_3(q,\tilde{p}) \quad .$$

Here, F_3 is a homogeneous polynomial of order 3 in the components of q and $\tilde{p}$. Then, equations (4) read as follows.

$$\tilde{q}_i = q_i + \partial F_3(q,\tilde{p})/\partial\tilde{p}_i \qquad (5a)$$

$$i = 1, 2, 3$$

$$\tilde{p}_i = \tilde{p}_i + \partial F_3(q,\tilde{p})/\partial p_i \qquad (5b)$$

Solving (5) to second order by iteration for $\tilde{q}(q,p)$ and $\tilde{p}(q,p)$ and comparing the result to equation (3b) specifies the derivatives of F_3 in terms of the B_{ij}.

$$\partial F_3(q,p)/\partial p_i = \sum_{j=7}^{27} B_{2i-1,j}w_j$$

$$i = 1, 2, 3$$

$$\partial F_3(q,p)/\partial q_i = -\sum_{j=7}^{27} B_{2i}w_j$$

Here, w is as in equations (2') and (3b). Use of these derivatives in equations (5) give the following "symplectified" transformation.

$$\tilde{q}_i = q_i + \sum_{j=7}^{27} B_{2i-1,j}u_j \qquad (6a)$$

$$i = 1, 2, 3$$

$$p_i = \tilde{p}_i - \sum_{j=7}^{27} B_{2i,j}u_j \qquad (6b)$$

Here, $u=(x,\tilde{p}_x,y,\tilde{p}_y,t,\tilde{p}_t,x^2,x\tilde{p}_x,\ldots,t^2,t\tilde{p}_t,\tilde{p}_t^2)$.

Equations (6) reproduce equations (3b) through second order, and are symplectic to all orders. They are therefore a canonical transformation which may be used to replace the original noncanonical transformation (2) in a second order matrix program.

Programming Considerations

To implement this method in a standard matrix program will require three types of routines. The first type must convert the variables x,x',... used within the program to canonical values $x,p_x,\ldots$ The second must convert the matrix elements A_{ij} used in the nonsymplectic ray trace (2) to the matrix elements A_{ij} and B_{ij} of equation (3), which will be used in the symplectic ray trace defined by equations (3a) and (6).

The final type of routine actually performs the ray trace. This may be done by solving the nonlinear system (6b) for $\tilde{p}(q,p)$ and using the result in (6a) to obtain $\tilde{q}(q,p)$. In practice, as the system (6b) differs from the identity only in second order, it is readily solvable by a number of techniques. For example, the program MARYLIE [3] solves such systems by use of a Newton's search procedure; in the following example, the program DIMAT employs an iteration technique [4] with acceleration factors for more rapid convergence.

The various routines under consideration will in general represent only minor programming effort and can in fact be written in a modular fashion, so as to be implementable in any standard second order matrix optics program.

Symplectification of DIMAT

Routines similar to those discussed above have been incorporated as FORTRAN subroutines in the program DIMAT [5]. We now give an example of their application.

Example - Combined Function FODO Lattice: A simple lattice of combined function, 2°, 4 m long bending magnets (separated by 1 m long drift spaces) was designed. The lattice consisted of a matched insertion, followed by three 60° FODO cells, followed by a second matched insertion, and ended by four more 60° FODO cells. The matched insertions were represented by a simple Twiss matrix and were included solely to adjust the tunes of the lattice. In the example shown below, the phase advances across the insertions were taken to be 79° horizontally and 48° vertically. The chromaticities were fit to zero by introducing sextupole components in the bend magnets.

Once all tunes and chromaticities were fit, a particle was launched in the midplane and tracked using a nonsymplectic ray trace. The result of the experiments are shown in Figures I; the growth of phase space (and associated violation of Liouville's theorem) is apparent. When the experiment was repeated using a symplectic ray trace of the type discussed above, no growth of phase space was observed (Figures II).

Conclusions

We have given evidence that a standard second order matrix ray trace can violate the symplectic condition. This problem may be effectively remedied by use of a simple generating function method. The procedure for doing so has been implemented in the program DIMAT. We conclude that it is possible to bring a second order matrix ray trace to canonical form.

Acknowledgments

The method used to evaluate the symplectic ray trace described herein is analogous to that employed in the program MARYLIE, which was written at the University of Maryland by a collaboration led by Dr. Alex Dragt and which included two of us (D.D. and E.F.). We would like to thank Dr. William Grieman

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Figure I a) Horizontal phase space for turns 1 to 50, nonsymplectic ray trace acting on particle launched in magnetic midplane.

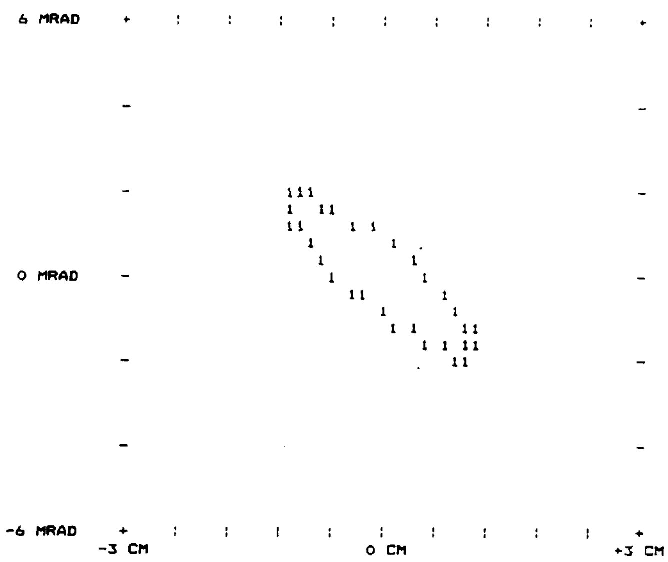

Figure I b) Horizontal phase space for turns 2050 to 2100, using nonsymplectic ray trace. Growth of phase space (in violation of Liouville's theorem) is apparent.

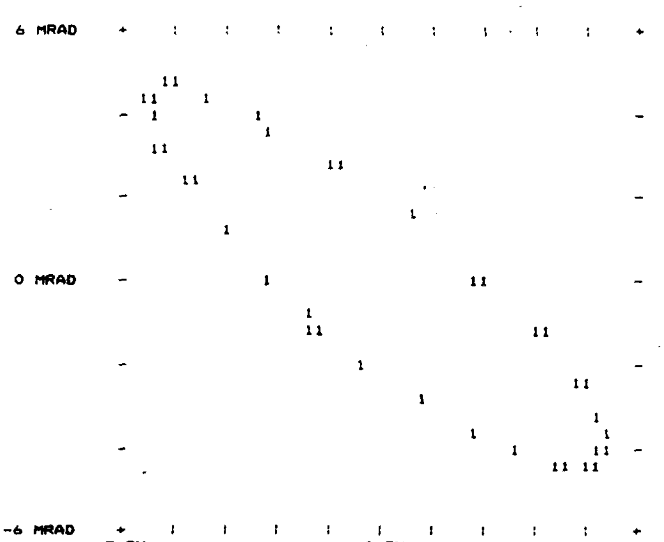

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Figure II a) Horizontal phase space for turns 1 to 50, symplectic ray trace acting on particle launched in magnetic midplane.

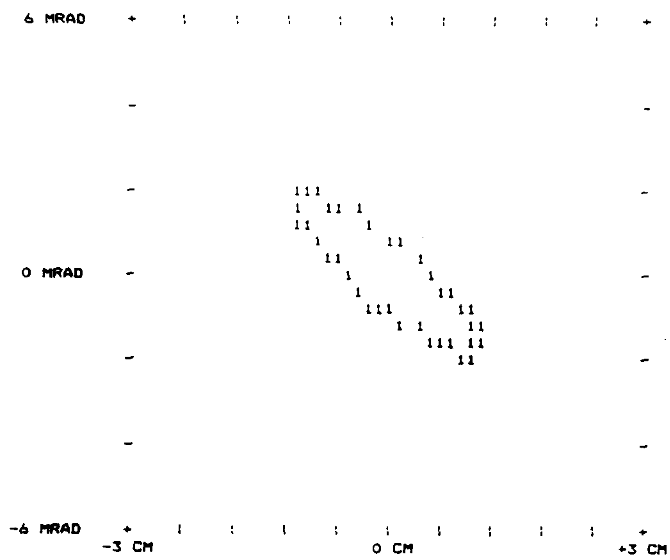

Figure II b) Horizontal phase space for turns 2050 to 2100, using symplectic ray trace. No growth over a 2000 turn interval is apparent; Liouville's theorem is satisfied.