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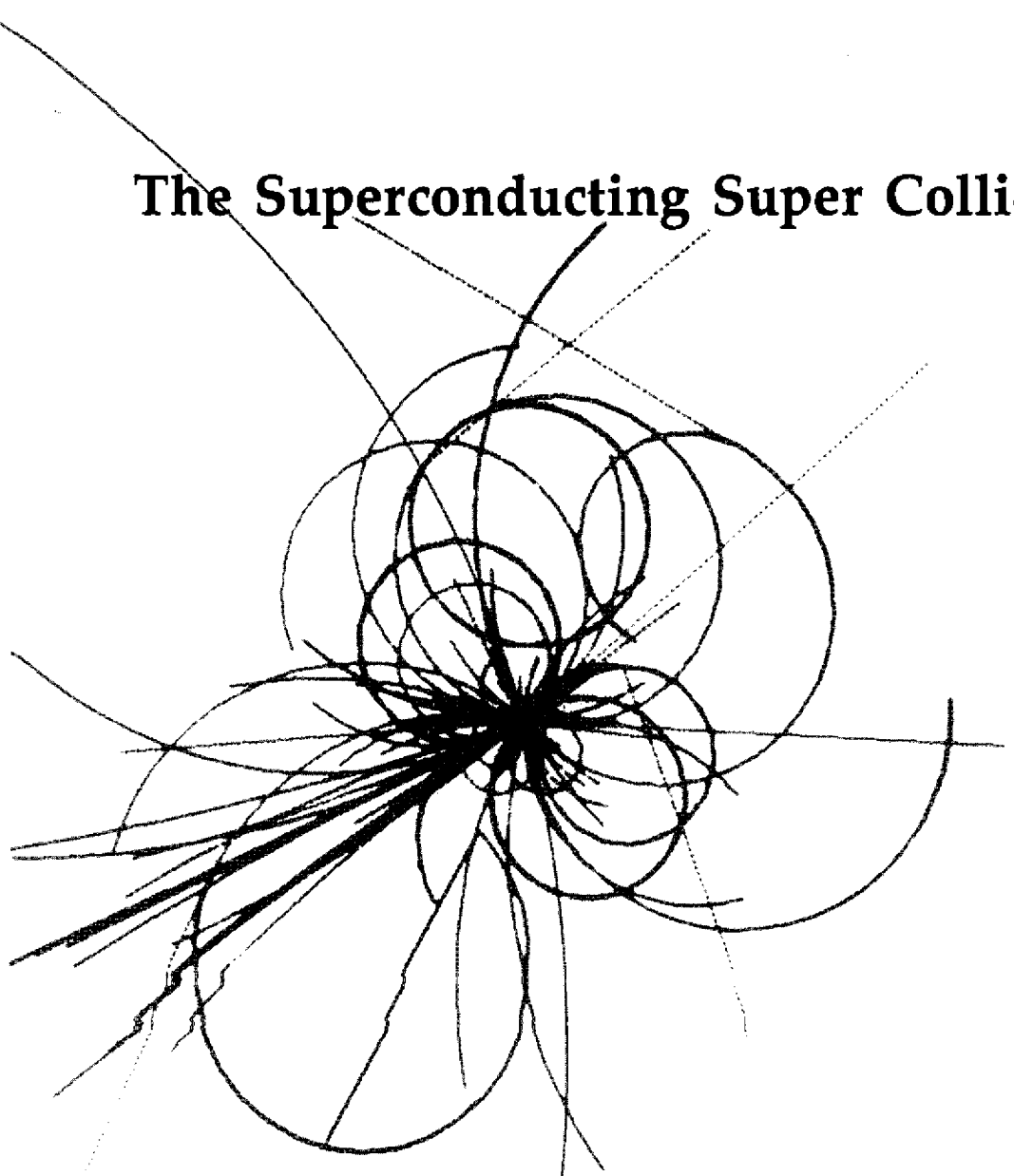

Differential Maps, Difference Maps, Interpolated Maps, and Long Term Prediction

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ABSTRACT

Mapping techniques may be thought to be attractive for the long term prediction of particle motion in accelerators, especially because a simple map can approximately represent an arbitrarily complicated lattice. The intention of this paper is to develop prejudices as to the validity of such methods by applying them to a simple, exactly solveable, example. It is shown that a numerical interpolation map, such as can be generated in the accelerator tracking program TEAPOT, predicts the evolution more accurately than an analytically derived differential map of the same order. Even so, in the presence of "appreciable" nonlinearity, it is shown to be impractical to achieve "accurate" prediction beyond some hundreds of cycles of oscillation. This suggests that the value of nonlinear maps is restricted to the the parameterization of only the "leading" deviation from linearity.

1. Introduction.

Linear maps have been basic to accelerator theory almost from the beginning of the subject. Accelerator lattice descriptions are, by their very nature, complicated and special, and a linear map (i.e. transfer matrix) represents an elegant distillation of the essence of this complexity. It has been natural to generalize this description to include nonlinear effects which become important at large particle amplitudes. There are at least three attitudes which can accompany such a generalization to nonlinear maps:

(*i*) Just as the Twiss parameters, α, β, and γ constitute a compact representation of the lattice, "canonical" parameterizations of nonlinear maps can compactly codify the "leading" or "near-linear" behaviour.

(*ii*) The nonlinear map into which a complicated lattice has been distilled can be studied to find, qualitatively, the sort of large amplitude behaviour that is possible. The best example of this is the Hénon map[1] , the study of which has given much useful insight. This approach can be numerical, looking phenomenologically for common features, or mathematical, finding rigorous features of maps. Such results apply reliably only to the map, not to the complicated system it is supposed to represent. In this attitude, maps serve as analogies to the actual system of interest, and confidence that the particular system will exhibit the same behaviour as the map amounts to being conjecture when (as it usually is) the map is used to obtain nonrigorous results.

(*iii*) Another attitude is to regard mapping as a mathematical technique (for "solving" the equations of motion) which is to be held to the same standard as any other mathematical method; that standard being that any prediction about the evolution of the system represented by the map (within estimateable errors) be absolutely reliable.

The present paper is relevant only to the third of these attitudes. If it is my conclusion (and it is) that high order differential mapping is not a good method

for long term prediction, that is not intended to convey any lack of respect for the use of nonlinear mappings with either of the other attitudes.

Mapping techniques are very seductive in the context of predicting the motion of particles in accelerators. Even a crude representation of a realistic lattice has thousands of independent parameters; (their lack of correlation may, for example, be due to errors and imperfections.) Using statistical methods and language, simple and meaningful statements can be made about such a system but that is not of present interest. We are interested in the time evolution in one particular system, out of the ensemble of possible systems, with this plethora of independent parameters somehow suppressed.

The proponent of mapping techniques must be willing to apply methods of any required degree of complexity to describe by a single map the motion of a particle for one complete turn around the accelerator. From then on, since the particle passes through the same sequence of elements on every turn, that map can be iterated to predict the subsequent motion. For motion restricted to small amplitudes in a single transverse plane a 2×2 map suffices, and that is certainly a simplification compared to the thousands of initial parameters. The two other coordinates can be incorporated by using 4×4 or 6×6 linear maps. It is natural to extend such mappings into the nonlinear region and our purpose is to see how well that works and to compare different methods.

These issues are of interest (to the author anyway) only in the context of accelerator physics, but this paper is almost entirely devoted to one simple, and only feebly related problem of mechanics. Before attempting to explain and justify the long-winded title I will pose that problem:

> **Predict for all time the angular position of a lossless gravity pendulum whose initial state of motion is known.**

The exact solution to this problem, well-known and not difficult, will be given shortly but that is only incidental to my purpose. For most problems of nonlinear mechanics it is necessary to revert to approximate solution techniques and the

present purpose is to investigate the class of approximate solution going by the name "mapping". The exact solution will serve as a standard in terms of which the precision of these other solutions can be judged.

There are certain universal features of nonlinear oscillations (such as harmonic generation) but no proof of universality will be attempted here. The most that will be attempted is a kind of argument by analogy in which the reader is encouraged to infer the effectiveness of maps for complicated systems from their demonstrated effectiveness, or lack thereof, on a simple pendulum. The analogy is far from perfect. There is not even an unambiguous scale comparison by which one can say that the "degree of nonlinearity" of a particular transverse particle amplitude is the same as for a particular pendulum amplitude. We now manufacture such a scale comparison. Though this procedure is not particularly persuasive, it will turn out that the qualitative conclusions are not very sensitive to it. In any case numerical values will be given and can be interpreted as the reader wishes.

Particles circulating in a storage ring with amplitudes greater than the so-called "dynamic aperture" are quickly lost. Let us call that "unit amplitude." Successful functioning of the storage ring depends on the long term stability of particles with amplitudes small, but not tiny, compared to this. Let us say that the amplitudes of interest are in the range 1/20 to 1/2. For the pendulum a "characteristic" amplitude is the amplitude at which the qualitative character of the motion changes from libration (i.e. back-and-forth) to rotation and we call that "unit amplitude". This choice of scale seems to be in reasonable analogy with the above accelerator choice at least as regards longitudinal motion, which is not unlike pendulum motion. For transverse motion the analogy is more feeble but I make it nonetheless and declare, for the pendulum, that half-swing values in the range from 10° to 90° are analogous to the above described particle amplitudes of interest.

The maps to be studied are differential maps, difference maps, and interpo-

lated maps. These will be defined only casually at this point as they will be illustrated by example below. Differential maps are Taylor series expansions of the exact map; the coefficients are assumed to be obtainable exactly by analytic formulas. Difference maps are the same, but the coefficients are obtained by numerical differentiation. Interpolated maps are obtained by interpolating exact maps.

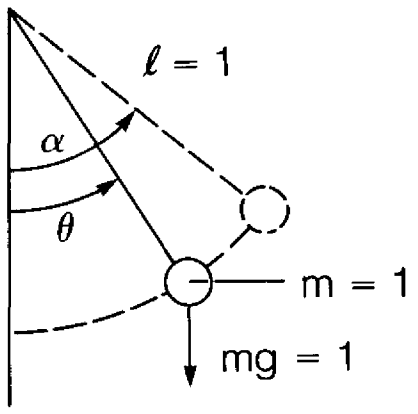
Sophisticated methods have been applied to the problem of finding differential maps for complicated lattices and elegant results have been obtained. Examples are MARYLIE,[2] due to Dragt and others, and more powerful methods due to Forest.[3] Also the differential algebra methods of Berz[4] have been used to obtain high order maps which have been shown to be correct by comparison with exact solutions evaluated to the same order.[5]

With the accelerator tracking program TEAPOT[6] (which uses exact analytical propagation formulas, and does not regularly use transfer matrices or any other kind of map for particle tracking) difference maps can be obtained numerically for complicated lattices. Computer word length limitations would eventually lead to reduced accuracy as the order of differentiation is increased but for low orders difference and differential maps are essentially equivalent. There are many possibilities for interpolation but we will restrict attention to the simplest possibility; by a simple change of parameter a difference map can be converted into an interpolated map.

In what follows the performance of these three types of maps for long term prediction of pendulum motion will be compared. My approach is purely elementary and numerical. It is possible, even probable, that the same conclusions could be obtained more briefly and more persuasively by purely analytical mathematical methods.

2. Mathematical Formulation and Exact Solution.

The system to be analysed is shown in the diagram. The maximum swing angle is α and the instantaneous angle is θ. The gravity constant g and the pendulum length ℓ and mass are set to 1 for the time being.



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The kinetic energy is given by

$$T = \dot{\theta}^2/2 \tag{2.1}$$

and the potential energy is given by

$$V = -\cos\theta + 1 = 2\sin^2\theta/2 \tag{2.2}$$

To make contact as quickly as possible with standard notation, the total energy E is expressed in terms of a parameter k so that

$$E \equiv 2k^2 = T + V = 2\sin^2\alpha/2 \tag{2.3}$$

and the equation of motion is

$$\dot{\theta}^2/2 = 2k^2 - 2\sin^2\theta/2 \tag{2.4}$$

By the change of variable

$$\sin\theta/2 = k\sin\phi \tag{2.5}$$

the equation of motion is transformed to

$$\dot{\phi}^2 = 1 - k^2\sin^2\phi \tag{2.6}$$

The "reduction to quadratures" solution of this equation is

$$u = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{2.7}$$

where the symbol u is being used for time, again to conform to standard notation of elliptic functions.[7] The functions which will appear are $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$. The solution of (2.7) is

$$\operatorname{sn} u = \sin\phi \tag{2.8}$$

or, in terms of the original variable

$$\theta = 2\sin^{-1}(k\operatorname{sn} u) \tag{2.9}$$

aside: the pendulum length and gravity constant can be restored by the relation

$$u = \sqrt{g/\ell}\, t \tag{2.10}$$

In other words u is time measured in units such that the small amplitude period is 2π.

Standard mapping techniques require a phase space description and that suggests following a Hamiltonian approach. Before doing that we make the following *ad hoc* choice of phase space variables with a view to making the mapping

as simple as possible.

$$\begin{aligned} x &= \sqrt{V/E} = k^{-1}\sin\theta/2 \\ p &= \sqrt{T/E} \end{aligned} \tag{2.11}$$

Combining the energy conservation equation

$$V + T = E \quad \text{or} \quad x^2 + p^2 = 1 \tag{2.12}$$

with the identity

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1 \tag{2.13}$$

it follows that the system point (x, p) in phase space is restricted to the unit circle. Taking the initial configuration to be vertical the cartesian coordinates are

$$\begin{aligned} x &= \mathrm{sn}\, u \\ p &= \mathrm{cn}\, u \end{aligned} \tag{2.14}$$

To obtain the relation of these coordinates to canonical coordinates the Lagrangean can be written in terms of x and $\dot{x}$,

$$L = \frac{2k^2\dot{x}^2}{1 - k^2x^2} - 2k^2x^2 \tag{2.15}$$

The momentum conjugate to x is

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{4k^2\dot{x}}{1 - k^2x^2} \tag{2.16}$$

and the Hamiltonian is

$$H = 2k^2x^2 + p_x^2(1 - k^2x^2)/(8k^2) \tag{2.17}$$

Finally then, the relation of our variable p to the canonical variable p_x is

$$p = p_x\sqrt{1 - k^2x^2}/(4k^2) \tag{2.18}$$

3. The Exact Transfer Map.

When the time evolves from u to $u+v$, substitution into (2.14) shows that the phase space point moves as

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \operatorname{sn} u \\ \operatorname{cn} u \end{pmatrix} \Longrightarrow \begin{pmatrix} x_v \\ p_v \end{pmatrix} = \begin{pmatrix} \operatorname{sn}(u+v) \\ \operatorname{cn}(u+v) \end{pmatrix} \tag{3.1}$$

Using "addition formulas"[8] valid for elliptic functions this evolution can be expressed by the exact nonlinear mapping

$$\begin{pmatrix} x_v \\ p_v \end{pmatrix} = \begin{pmatrix} \operatorname{cn} u \operatorname{dn} v & \operatorname{sn} v(1-k^2x^2)^{1/2} \\ -\operatorname{sn} u \operatorname{dn} v(1-k^2x^2)^{1/2} & \operatorname{cn} v \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \frac{1}{1-k^2x^2\operatorname{sn}^2 v} \tag{3.2}$$

To predict the long term motion of the pendulum this map could be iterated but of course it is simpler just to use (2.14). If we ambitiously attempt to track accurately over, say, 10^{10} cycles we seem to be faced with the task of evaluating expresssions such as $\operatorname{sn} 10^{10}$. Actually it is easier than that since whole cycles (or for that matter whole half-cycles) can be subtracted off assuming that the period is known with sufficient accuracy. Fortunately that is easy.[9] As a result the exact motion can be predicted with an accuracy millions of times better than can be obtained with any of the approximate maps that will be discussed.

4. The Differential Map.

The linear transfer map can be obtained directly from (3.2) by setting $x=0$. Furthermore the x-dependent factors appearing there can be expanded easily. For example, setting $a=k^2x^2$ and $b=k^2x^2\operatorname{sn}^2 v$ we have

$$(1-a)^{1/2}(1-b)^{-1} = 1 + (b-a/2) + (b^2 - ba/2 - a^2/8) + \ldots \tag{4.1}$$

The last term exhibited is quartic in x but the formula is easily extended. For lattice calculations it is impractical[10] to go much beyond term of order x^8 and that is how far calculations in this paper are carried.

Even apart from the fact that we have a system simple compared to an accelerator lattice, this map exhibits especially simple features. For one thing it is independent of p. Note though that this is not basic. By (2.13) it would be possible to express x in terms of p. We will return to this below when discussing symplecticity. Another simple feature is that only even powers of x appear. To be analogous in this regard an accelerator lattice would contain no sextupoles or other even multipoles. We make the observation, which we then proceed to ignore, that this feature makes the pendulum not quite analogous to the most common nonlinear betatron motion. The leading qualitative manifestation of this lack of analogy is probably that the pendulum frequency is perturbed in the lowest nonvanishing order of perturbation; the tune of a lattice containing only sextupoles is unperturbed in lowest order.

Though less easy than has been true for our simple example, corresponding differential maps of quite high order (say 8 or 10) may soon be obtainable analytically for accelerators using methods mentioned above. Up to at least order 5 it has already been done[6] . These can be iterated to predict the evolution of the motion. To repeat what was said in the introduction my purpose is to develop a prejudice as to the accuracy of this procedure, especially as compared to the other maps to be discussed.

To make contact with standard transfer map terminology I will temporarily, just for this paragraph, generalize the notation. A general matrix element has the form $f(x^1, x^2, ...)$ where $x \equiv x^1, p \equiv x^2$ and if there were more degrees of freedom they would be x^3, x^4 etc. This matrix element has the form (summation convention implied)

$$f = R + T_i x^i + U_{ij} x^i x^j + V_{ijk} x^i x^j x^k + ... \tag{4.2}$$

Note that f itself should have one "input" and one "output" index; they have been suppressed. The notation R for linear order and T for quadratic order, introduced in the program TRANSPORT,[11] is well established.

In practice differential maps are necessarily truncated and written in this form. If only terms up to and including U_{ij} are kept the map will be in error to terms in V_{ijk}. This error is quickly made manifest by non-symplectic behavior. For the pendulum description this corresponds to the wandering of the phase point off the unit circle. An oft-employed practice is to force symplecticity artificially; we could do it by normalizing the phase space point to unity after each iteration. Such procedures are not unique and they don't make the description right, they only make it not manifestly wrong. It seems to me that enforcing symplecticity artificially is unsound. Usually, in numerical work, the existence of a rigorous relation such as this is greatly valued as a check on the numerical accuracy of the results. Since we have the exact solution available, we do not need to avail ourselves of this possibility, but I choose not to distinguish between radial and azimuthal phase space deviations, and do not enforce symplecticity.

For our problem the expansion (4.2) can be replaced by a simpler formula.

$$f(x) = R + Ux^2 + Wx^4 + Yx^6 + ... \tag{4.3}$$

As mentioned previously, the T-matrix elements vanish.

5. Interpolated Maps.

Any one of the four matrix elements of (3.2) can be written as a function $f(x)$. In preparation for calculating what will be called an interpolated map this function can be evaluated at points on a regular grid. Since the functions do not depend on p we need only use points on the x-axis, and for simplicity we use equally-spaced values, $0, \pm x_{typ}, \pm 2x_{typ}, ...$ Here I am using the somewhat clumsy notation of TEAPOT.[7] The value x_{typ} is a free parameter to be specified later. If there were p-dependence there would be another parameter, p_{typ}.

For writing difference formulas we introduce the notation $f_n = f(nx_{typ})$. We express x in units of x_{typ} by introducing the variable $r = x/x_{typ}$. The function

$f(x(r))$ can be expressed approximately in terms of the tabulated values f_n by a Lagrange interpolation formula. Since our functions are all even in r, only values f_n with positive n are required. The following sequence of formulas, valid to successively higher powers of r^2, are almost simple enough to be checked mentally.

$$
\begin{aligned}
f^{(0)} =& f_0 \\
f^{(2)} =& -(r^2-1)f_0 + r^2 f_1 \\
f^{(4)} =& \frac{(r^2-1)(r^2-4)}{4} f_0 - \frac{r^2(r^2-4)}{3} f_1 + \frac{r^2(r^2-1)}{12} f_2 \\
f^{(6)} =& -\frac{(r^2-1)(r^2-4)(r^2-9)}{36} f_0 + \frac{r^2(r^2-4)(r^2-9)}{24} f_1 \\
& - \frac{r^2(r^2-1)(r^2-9)}{60} f_2 + \frac{r^2(r^2-1)(r^2-4)}{360} f_3 \\
& \cdots
\end{aligned}
\tag{5.1}
$$

By calculating each of the matrix elements of (3.2) using these formulas, interpolated maps of successively higher order can be obtained.

6. Difference Maps.

By comparison with (4.3) approximate difference formulas can be written for the Taylor expansion coefficients of the differential map. For example, from $f^{(4)}$ we obtain

$$
\begin{aligned}
R \approx R^{(4)}(x_{typ}) =& f_0 \\
U \approx U^{(4)}(x_{typ}) =& \frac{-15f_0 + 16f_1 - f_2}{12x_{typ}^2} \\
W \approx W^{(4)}(x_{typ}) =& \frac{3f_0 - 4f_1 + f_2}{12x_{typ}^4}
\end{aligned}
\tag{6.1}
$$

To obtain a rigorous connection between these newly introduced, x_{typ}-dependent expressions with the corresponding coefficients of the differential map it is clearly necessary to take the limit $x_{typ} \to 0$. That cannot be done numerically, but, depending on the computer word length of the computer being used, a value of

x_{typ} can be used that is sufficiently small that the leading coefficients can be accurately evaluated. Used in (4.3) they express what I have called a difference map. It can be seen, as the terms have been defined, that an interpolated map and a difference map differ only in the value of x_{typ} used.

7. Numerical Results.

The various maps have been iterated for various swing amplitudes; the results are plotted in the following figures. In every case the deviation, approximate minus exact, $\sqrt{(x_{ap} - x_{ex})^2 + (p_{ap} - p_{ex})^2}$, is calculated. Since the exact phase point remains on the unit circle, this is both the absolute value and the fractional absolute value of the vector phase space displacement of the approximate solution from the exact solution; we take it as the latter, evaluating it as a percent error. Rather than plotting this error as a function of iteration number one can note the iteration number at which the error first exceeds some value, such as 10%.

Data of that sort is plotted in Fig. 1. For this data, and all other plots in this paper, the map period v in (3.1), has been taken to be $1.6 \times \pi$; that corresponds to an evolution time interval equal to eight tenths of the small amplitude period. As a result, the number of iterations and the number of periods have comparable magnitudes. Defining "accurate" iterations to be those for which the error is less than 10%, Fig. 1 is a plot of number of accurate iterations versus the approximation order for various pendulum swing amplitudes. Clearly long term accuracy degrades as the amplitude increases. For one amplitude, 30 degrees, the 1% error data is shown; it shows no striking qualitative difference from the 10% data. What is striking is that the number of accurate iterations varies linearly with the approximation order, at least over the range studied. The data of Fig. 1 can be described semi-quantitatively by the following crude and simple formula:

$$\text{number of accurate iterations} \approx 10 \frac{\text{approximation order}}{\text{amplitude}^2} \tag{7.1}$$

where "amplitude" is the maximum pendulum angle in radians and "order" is

one for linear transfer matrices and increases by one for each power of x or p.

The remaining figures are intended to compare the abilities of different maps to predict the motion over long times. The figures all have the same format; the solid curves in the upper and lower plots are what is being compared; for convenience the upper data is replotted in the lower graph, joined by broken curves. In every case the solid curves in the lower graph come from a differential map. Fig. 2 compares seventh and ninth order differential maps. An example comparison is indicated. For 30 degree amplitude, if 2% accuracy is required, then the number of accurate iterations with a seventh order map is $\log_{10}^{-1} 2.29 = 195$ turns. Going to ninth order yields $\log_{10}^{-1} 2.45 = 282$ accurate iterations.

The purpose of Fig. 3 is to compare an interpolated and a differential map of the same (ninth) order. For interpolated maps the parameter x_{typ} must be specified. In this paper it is always chosen so that, in the particular order, the largest grid point is $x = 1.0$; that is, $x_{typ} = 2/(\text{order} - 1)$. Somewhat smaller values in the range 0.9 to 1.0 gave greater precision, but they have not been used, as the improvement was not great and the optimum depends on pendulum amplitude. Depending on how the comparison is made the interpolated map can be said to be either somewhat more accurate, or far more accurate, than the differential map. Again working on the 30 degree case, and demanding 10% accuracy, the differential map yields 335 accurate iterations. After this many iterations the interpolated map is still yielding much better than 1% accuracy; it could be said to be one hundred times "better". On the other hand, after 457 turns, the interpolated map inaccuracy exceeds 10%; this is not that great an improvement over the 335 turns of accurate differential map tracking.

The remaining figures compare differential and difference maps. For the difference maps x_{typ} was chosen according to the formula $x_{typ} = 0.2/(\text{order} - 1)$. For seventh order maps the agreement is very good as shown in Fig. 4. The overplotting in the lower graph of difference map results is just barely visible. The absence of a 10 degree differrence curve is due to the occurrence of a computer

overflow occurring during its evaluation. Going to ninth order yields Fig. 5, which shows that the difference map has broken down and is inferior to the seventh order map; that is presumeably due to roundoff error in the numerical differentiation employed in evaluating the difference map. To confirm this, Fig. 6 was generated using extended precision in the computer. The precision was "real*16", meaning that a floating point number is represented by a 128 bit binary number in the computer; such numbers are accurate to about 33 decimal digits. For the previous calculation, 64 bit numbers, accurate to about 16 decimal digits, were used. With the extended precision the difference and differential maps are indistinguishable graphically. Numerically the greatest deviation in the logarithm of the number of accurate iterations is 2.5276 instead of 2.5302.

8. Conclusions.

The main conclusions have been stated in the abstract. Quantitative results have been given for the accuracy achievable in predicting the long term motion of a pendulum by various procedures. For this particular simple system, it is shown that, when the nonlinearity is "appreciable", iteration of approximate maps yields accurate prediction for only some hundreds of periods of oscillation; of course, this depends on approximation order and swing amplitude, as has been spelled out. The number of accurate iterations increases roughly proportional to the approximation order. This leads eventually to a diminishing return, since the factor by which the number of map coefficients increases, as the order is increased by one, is roughly equal to the number of degrees of freedom.

This suggests that map-iteration is not promising for long term prediction, though that has been shown only for the pendulum system. If the pendulum is fully typical of nonlinear motion, then the same can be said for all nonlinear systems, and that is my subjective opinion. It is however possible, for another system such as an accelerator containing nonlinear elements, that maps could be useful for long term prediction. To be persuasive in that case, it would be

desireable to show ways in which the pendulum system is atypical and gives misleading results, and what characterizes that class of nonlinear system that are conducive to map-prediction.

It has also been shown that greater accuracy can be achieved using interpolated maps than using differential maps. This is not at all surprising since the differential map amounts to **extrapolation** from the origin using Taylor series; this should be expected to be less accurate than **interpolation** based on a grid of exact values. By using interpolation formulas of sufficient sophistication, one supposes that maps almost as accurate as the grid point maps can be obtained, but that has not been investigated here.

Low order differential maps can also be used for a near-linear description which generalizes the Twiss-parameter lattice representation. It has been shown that such maps can be obtained numerically as difference maps which are essentially identical to the corresponding differential maps.

Though long term prediction is impractical using maps, it is practical using analytical formulas. This has been exhibited for the pendulum and it is also possible in accelerator tracking; the program TEAPOT performs exact tracking in an approximate lattice. (Here "exact" should more correctly be expressed as "to machine precision".) This procedure corresponds to the long-established scientific methodology in which idealized models based on plausible physical arguments are subjected to rigorous mathematical analysis. As always in physics, if the model is wrong or too simple its predictions will not agree with observations on the system the model is supposed to represent.

I am pleased to thank my collaborators Lindsay Schachinger and Vern Paxson and, especially, Tjet Sun, for his assistance in performing the calculations and generating the graphs.

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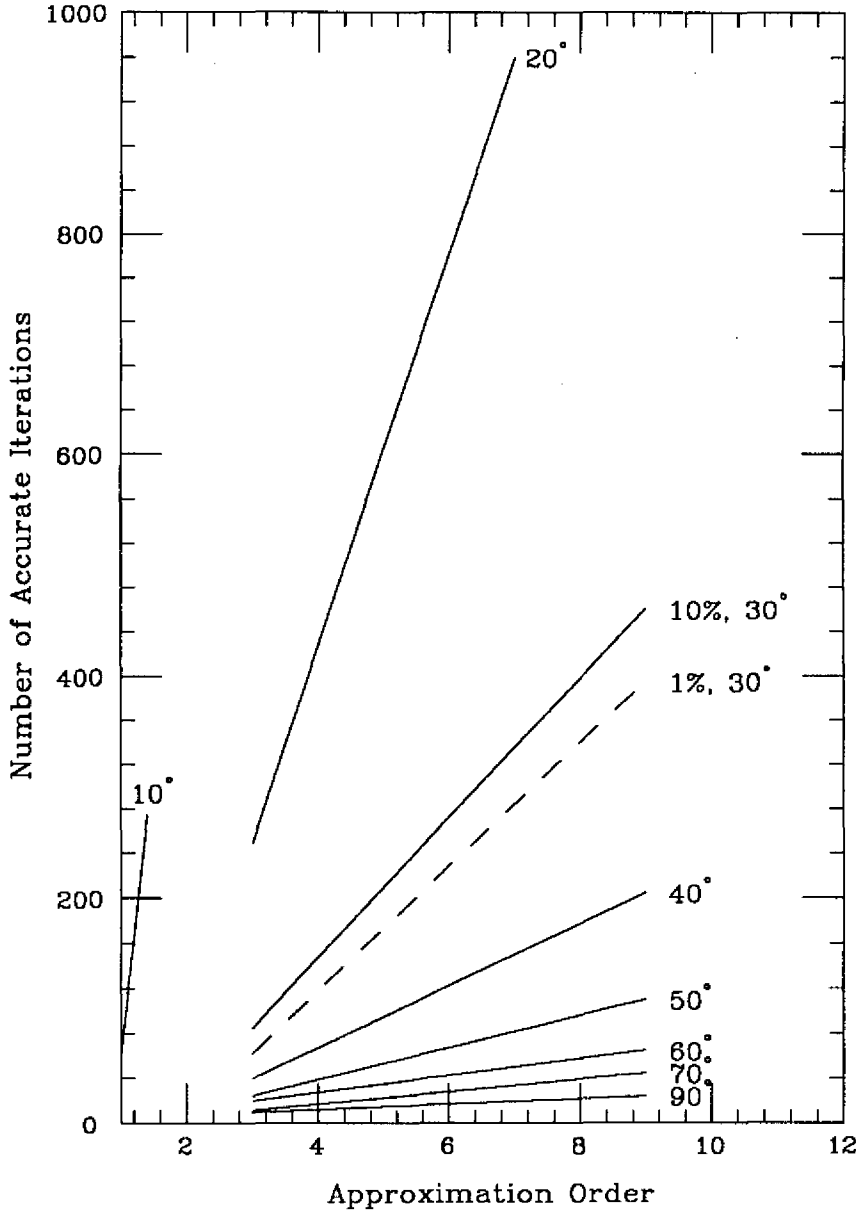
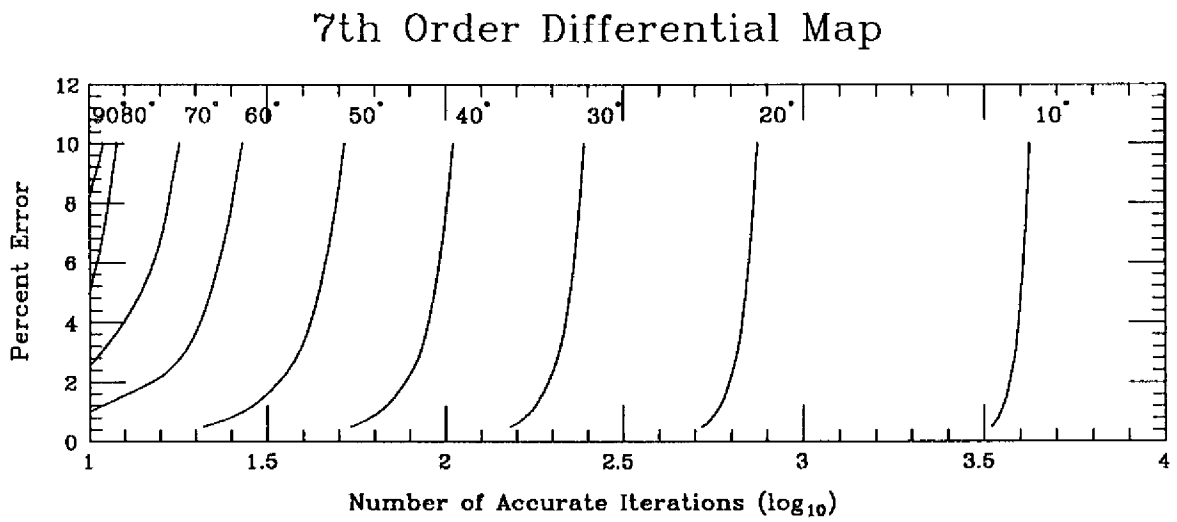


Figure 1. Dependence of the number of accurate iterations on the approximation amplitude, for various pendulum swing amplitudes.



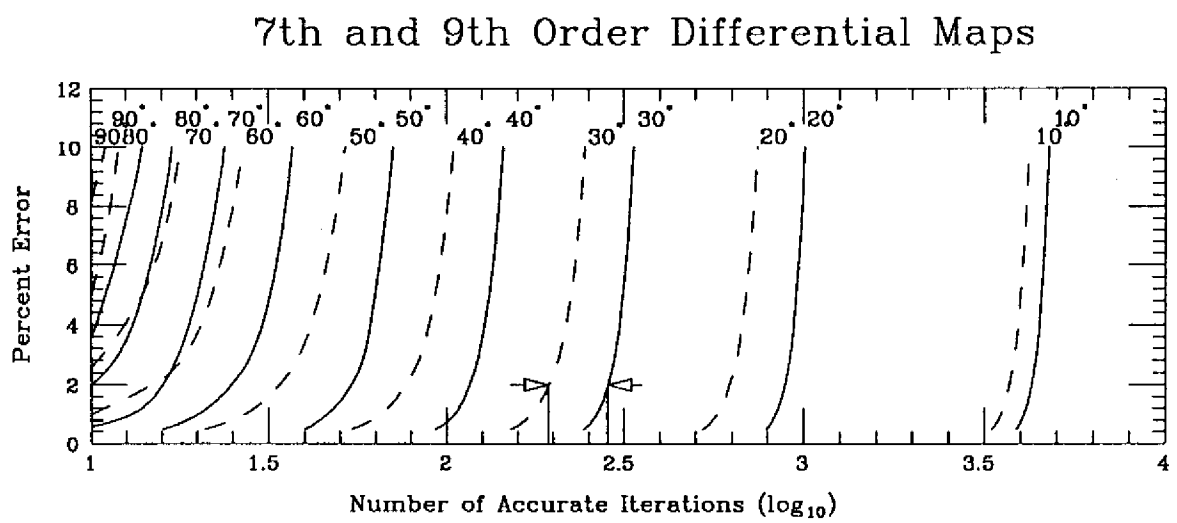


Figure 2. Comparison of results of iterating seventh and ninth order differential maps. As explained in the text, this and all following figures have a similar format; comparison is facilitated by the dashed-line over-plotting, in the lower graph, of the upper graph curves.

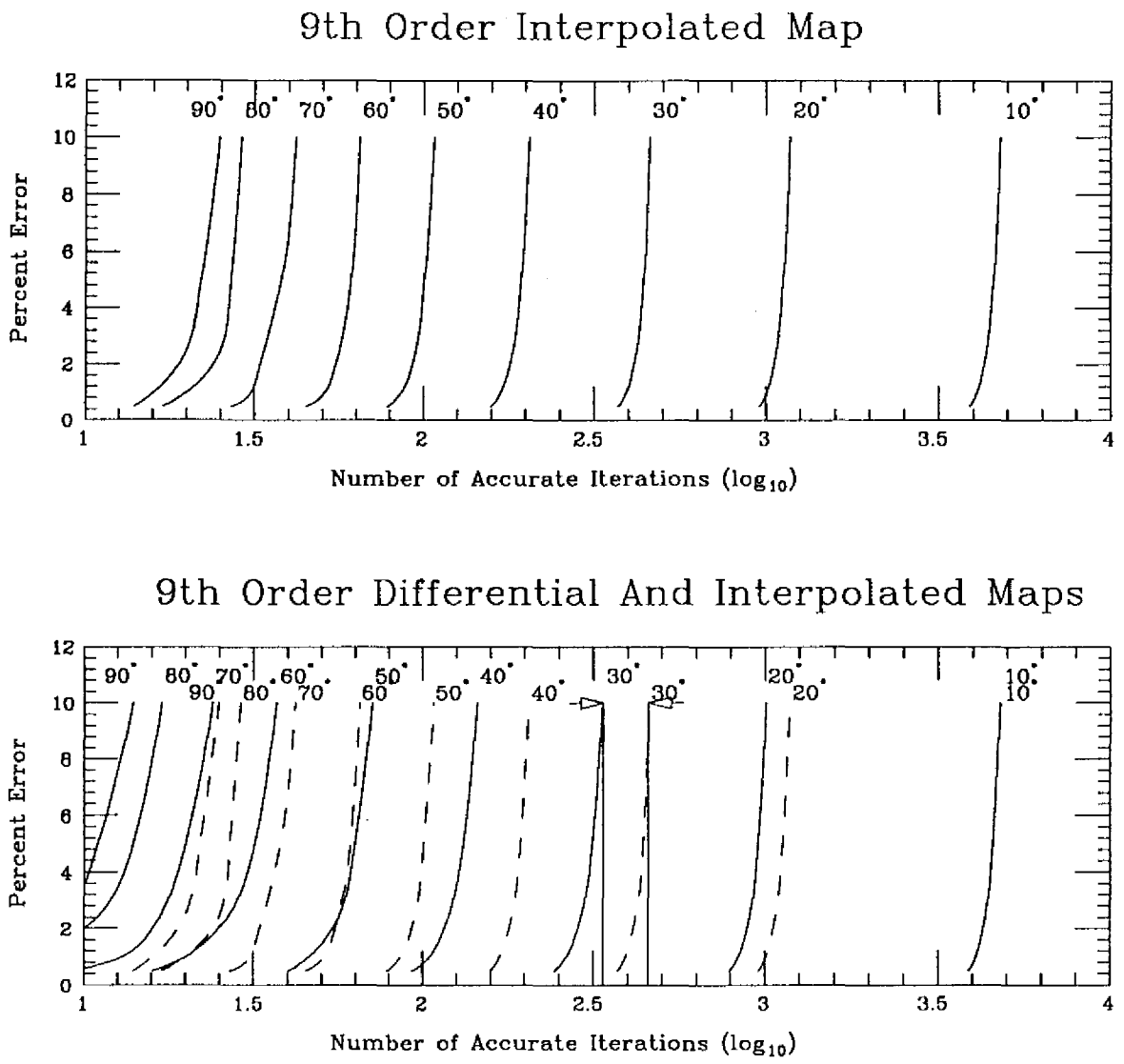


Figure 3. Comparison of results of iterating a differential map and an interpolated map, both of ninth order.

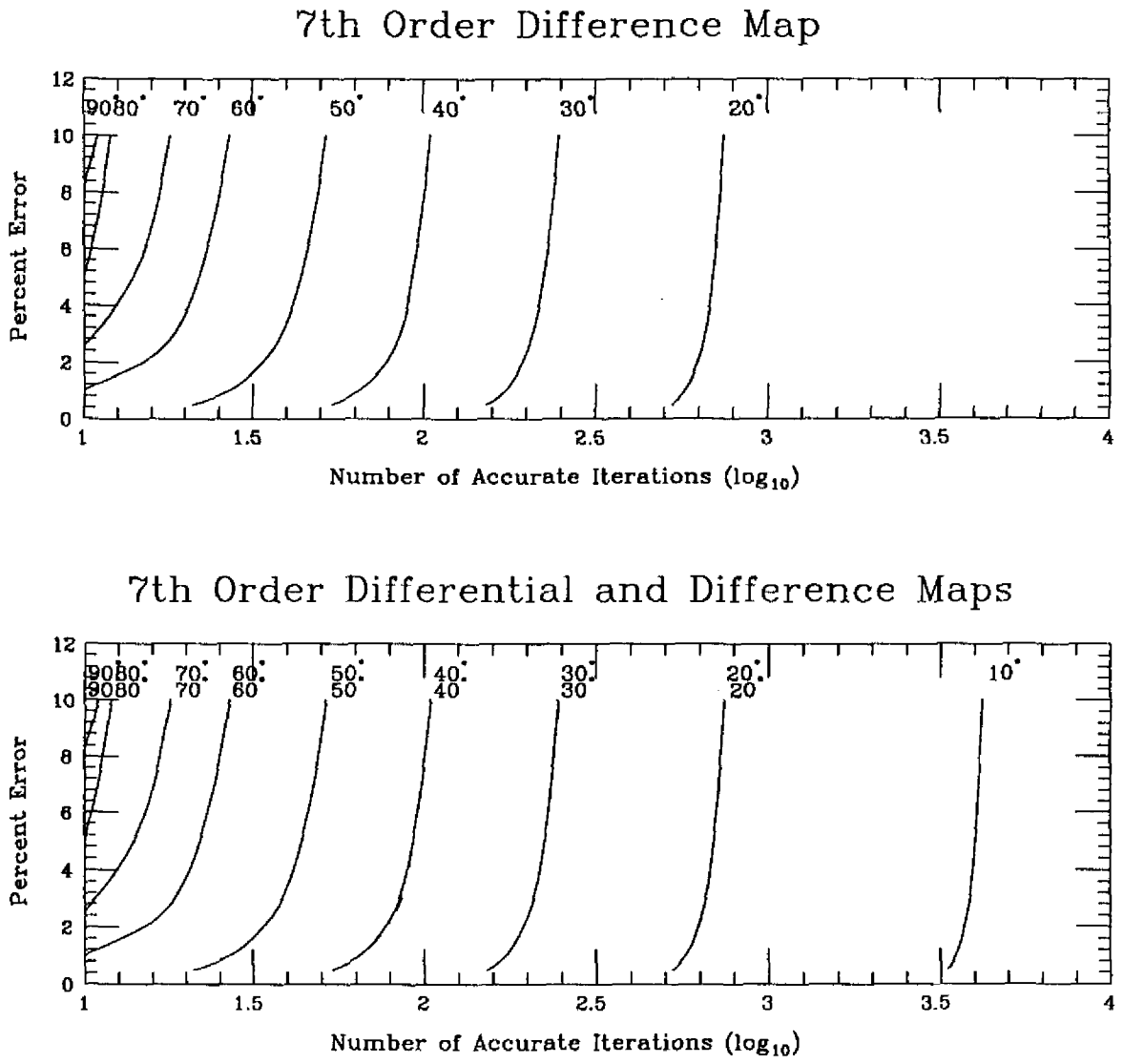


Figure 4. Comparison of results of iterating a differential map and a difference map, both of seventh order—"lower precision" computation.

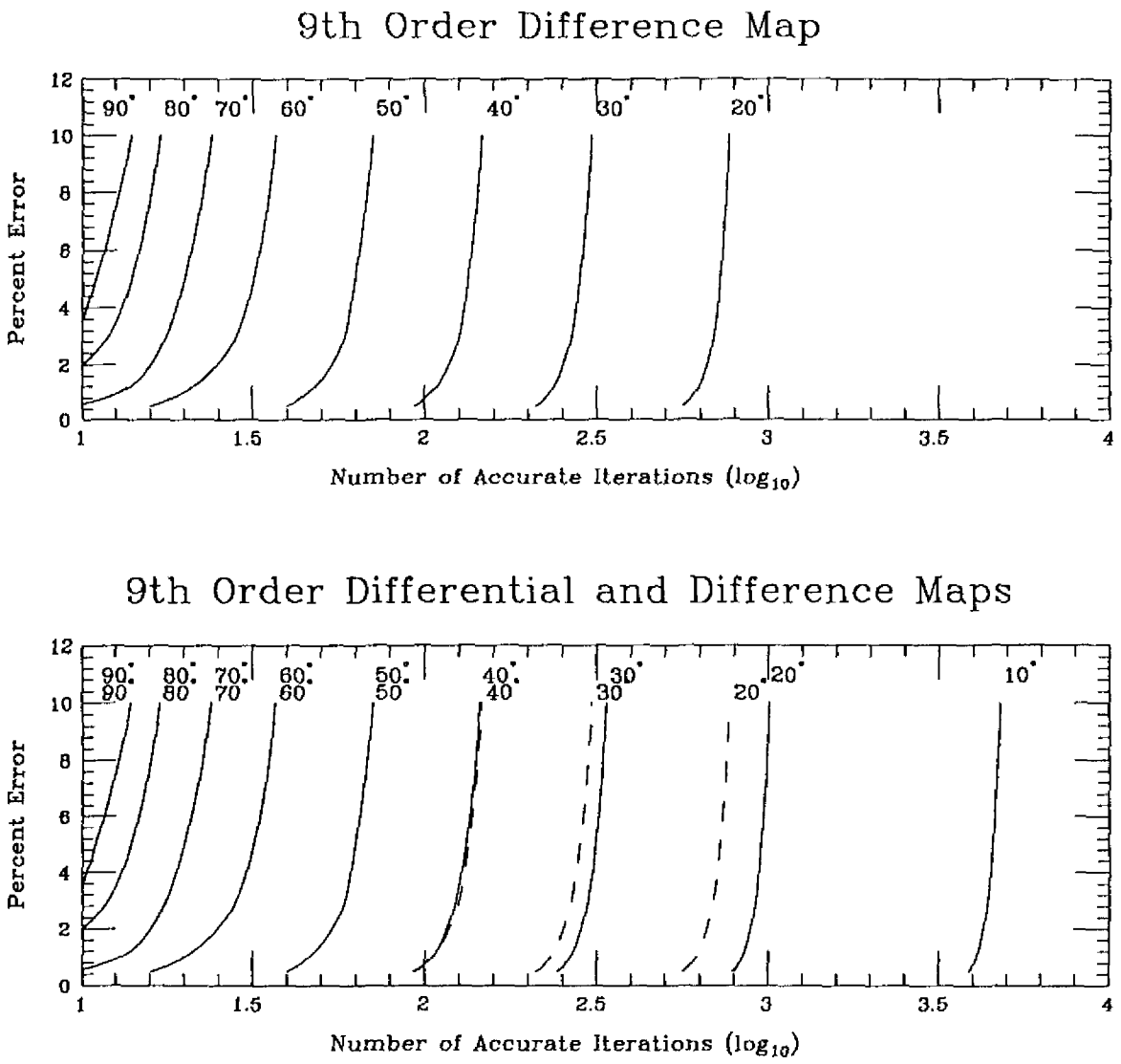
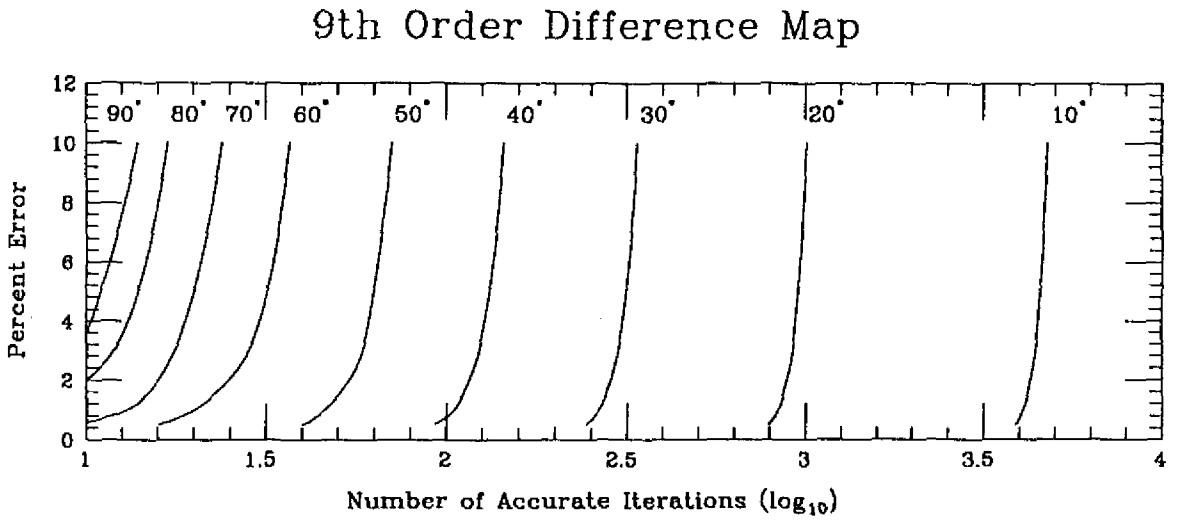


Figure 5. Comparison of results of iterating a differential map and a difference map, both of ninth order—"lower precision" computation.



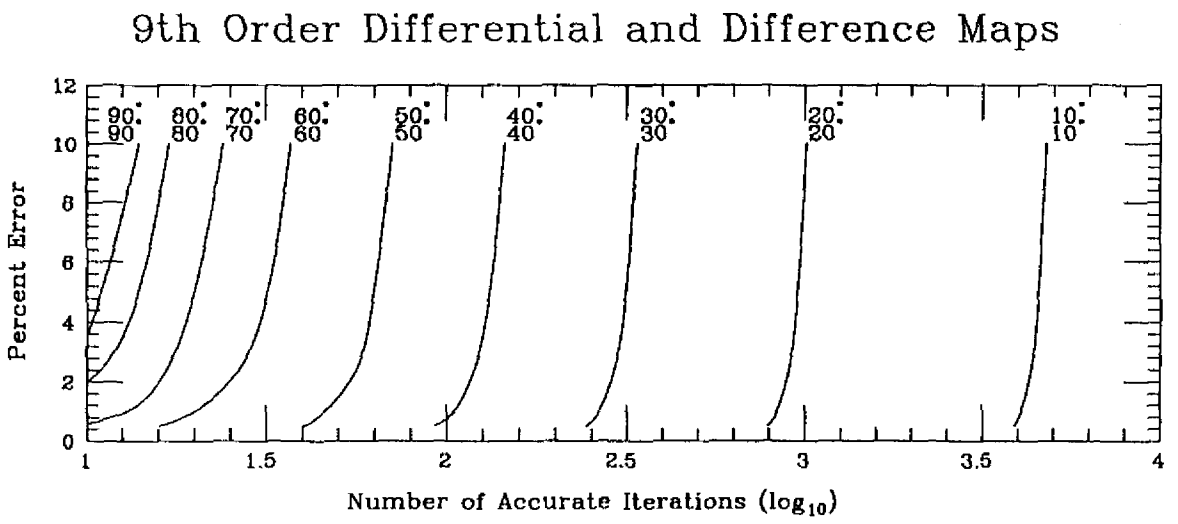


Figure 6. Comparison of results of iterating a differential map and a difference map, both of ninth order—"higher precision" computation.