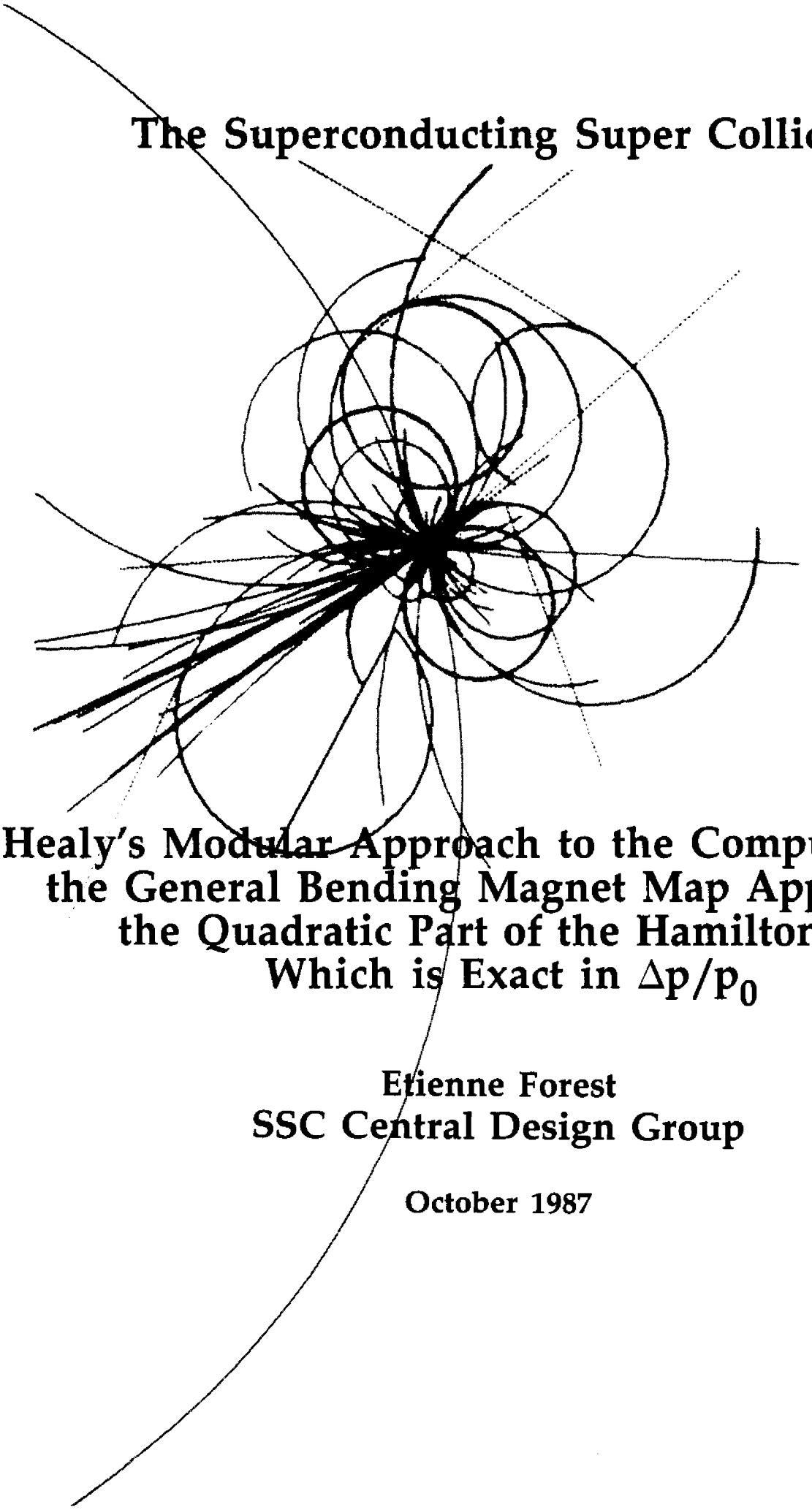
SSC-141

The Superconducting Super Collider



Healy's Modular Approach to the Computation of the General Bending Magnet Map Applied to the Quadratic Part of the Hamiltonian Which is Exact in $\Delta p/p_0$

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HEALY'S MODULAR APPROACH TO THE COMPUTATION OF THE GENERAL BENDING MAGNET MAP APPLIED TO THE QUADRATIC PART OF THE HAMILTONIAN WHICH IS EXACT IN $\Delta P/P_0$

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† Operated by Universities Research Association for the U.S. Department of Energy

0. INTRODUCTION

In the context of large machines the approximate Hamiltonian K, which is obtained by expanding to second order in the transverse variables while keeping the δ-dependence ($\delta = \frac{p-p_0}{p_0}$) exact, is often a legitimate representation of circular machines. One nebulous element has always been the general bending magnet. With the exception of the sector bend, many knowledgeable accelerator physicists have accepted the postulate that the parallel face bends can be obtained as

$$P = QSQ \quad ,$$

where Q are quadrupoles, and S is the sector bend matrix. This result is correct for δ-independent linear maps. However, in this paper we will show how one proceeds to a correct solution using the Hamiltonian K and in particular, we will show that the trick of quadrupole edge breaks down in a proper δ-independent treatment.

We will derive a slightly extended Maxwellian fringe field effect from what Dragt used in his paper on the chromaticity of small rings. Also, we will provide a δ-dependent matrix for the body of a parallel face bend. The approach we will follow relies heavily on the work of Liam Healy, who solved the general bend problem in the context of the code MARYLIE.

1. THE MODULAR DECOMPOSITION OF AN ARBITRARY MAGNET

As described by Healy, the solution of an arbitrary bend can be obtained by a modular decomposition. Suppose we have a bend of entrance angle ψ and exit angle ϕ. If this bend has a total bend angle of α, then it can be physically decomposed as a half parallel face of angle ψ, a sector bend of α-ϕ-ψ, and a half parallel face of angle ϕ. This is shown in Fig. 1.

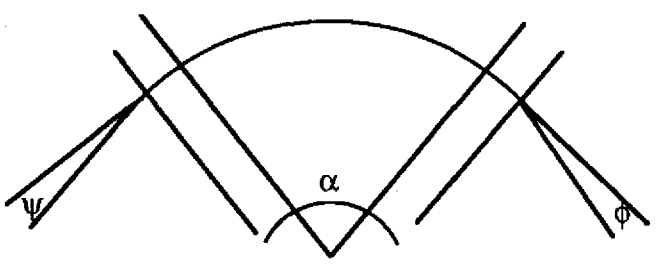

Fig. 1. General bend geometry.
(Borrowed from L. Healy.)

Hence, one needs to concentrate on the solution of the half bend of entrance angle ψ. In this paper, we will solve this problem within the framework of the approximate Hamiltonian K used in programs such as PATRIS or FASTRAC.

The half bend itself can be decomposed into three operations. They are represented in Fig. 2.

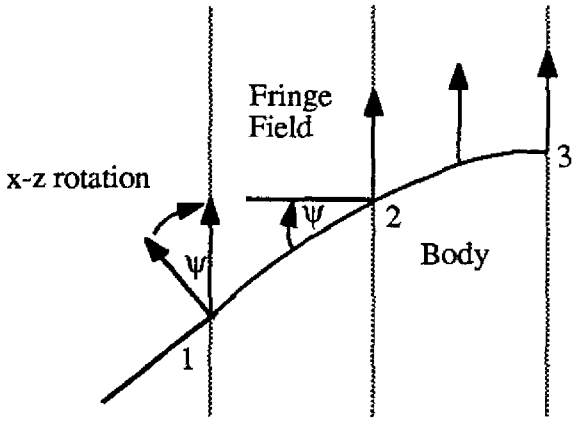

Figure 2

Firstly, the particle comes in a Frenet frame at location 1. There, the frame of reference is rotated by ψ (drift in polar coordinate). Secondly, the particle propagates into the fringing fields. This we will represent in some approximation defined in Sec. 3. Finally, the particle travels through the body of the magnet from location 2 to 3.

The purpose of the following sections is to derive these three operations within the approximate Hamiltonian of the FASTRAC style programs. We start in Sec. 2 with the x-z rotation.

2. The Approximate Hamiltonian and the x-z Rotation

In a drift, H is given by

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} - \frac{p_\tau}{\beta} \quad , \qquad (2.1a)$$

or, in polar coordinates,

$$\tilde{H} = -p_\phi = -x\left((1+\delta)^2 - p_x^2 - p_y^2\right)^{1/2} \quad . \tag{2.1b}$$

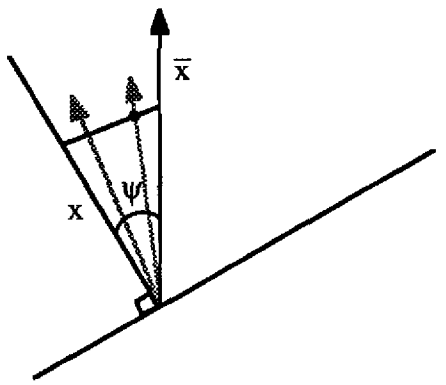

Figure 3

As seen in Fig. 3, the x-z rotation of the axis amounts to a drift in polar coordinates where x is just the radius r. By analyticity in x, we can extend the results to a negative x.

The rotation of angle ψ induced by $\tilde{H}$ can be computed exactly; we then expand in powers of the transverse variables

$$\bar{x} = \frac{x}{\cos\psi} \tag{2.2a}$$

$$\bar{p}_x = \cos\psi\, p_x + \sin\psi\, \delta \tag{2.2b}$$

$$\bar{y} = y \tag{2.2c}$$

$$\bar{p}_y = p_y \tag{2.2d}$$

$$\bar{\tau} = \tau + \left(\frac{1}{\beta} - p_\tau\right)\frac{x\tan\psi}{1+\delta} \tag{2.2e}$$

It is a simple exercise to check that Eq. (2.2) is symplectic. Equations (2.2) can thus be derived geometrically, since they correspond to a drift. We have also subtracted $\sin\psi$ from $\bar{p}_x$, so as to express it in deviation variables.

3. THE BODY OF THE HALF BEND

Using rectangular coordinates, we can propagate inside the bend using the exact Hamiltonian $-p_z$.

$$H = -p_z = -\left[(1+\delta)^2 - p_x^2 - p_y^2\right]^{1/2} - a_z \tag{3.1a}$$

$$a_z = -q\frac{B_0}{p_0}x \tag{3.1b}$$

Assuming that positive ρ bends inward, we can write

$$H = -\left[(1+\delta)^2 - p_x^2 - p_y^2\right]^{1/2} + \frac{x}{\rho} \quad . \tag{3.2}$$

To derive the transport matrix, we must first compute the design orbit (x_D, p_{xD}) at $\delta = p_y = 0$.

$$\frac{dx}{dz} = [x,H] = \frac{p_x}{\sqrt{1-p_x^2}} \quad , \tag{3.3a}$$

$$\frac{dp_x}{dz} = [p_x,H] = -\frac{1}{\rho} \quad . \tag{3.3b}$$

Solving (3.3) gives us

$$p_{xD} = -\frac{z}{\rho} + p_{x0} = -\frac{z}{\rho} + \sin\psi \quad . \tag{3.4}$$

Clearly, at the end of the half-bend the momentum p_{xD} vanishes. Therefore, we find for the total length of the half-bend along the z-axis

$$z_t = \rho \sin\psi \tag{3.5}$$

To obtain the approximate δ-dependent Hamiltonian from H, we must expand it in power of p_x around the design orbit and keep only terms which are of degree lower or equal to 2.

$$H = -\left[(1+\delta)^2 - (p_x + p_{xD})^2 - p_y^2\right]^{1/2} + \frac{x}{\rho} + \frac{x_D}{\rho}$$

$$= -n\left(1 - \frac{2p_x p_{xD}}{n^2} - \frac{p_x^2}{n^2} - \frac{p_y^2}{n^2}\right)^{1/2} + \frac{x}{\rho} + \frac{x_D}{\rho}$$

$$K \approx \frac{1}{2}\frac{(1+\delta)^2}{n^3}p_x^2 + \frac{1}{2}\frac{p_y^2}{n} - p_x p_{xD}\left(\frac{1}{n} - \frac{1}{\sqrt{1-p_{xD}^2}}\right) - n - \frac{p_\tau}{\beta\sqrt{1-p_{xD}^2}} \tag{3.6a}$$

$$n^2 = (1+\delta)^2 - p_{xD}^2 = 1 - p_{xD}^2 - \frac{2p_\tau}{\beta} + p_\tau^2 \quad . \tag{3.6b}$$

The last two terms are added so as to produce the correct **differential** time of flight. It is easy to show (using $[H(z),H(z')] = 0$) that K produces a drift-like transformation:

$$\bar{x} = x + \lambda p_x \pm \nu\,;\ \lambda = \int_0^{z_t} \frac{(1+\delta)^2}{n^3}\,dz\ ;\ \nu = \int_0^{z_t} p_{xD}\left(\frac{1}{n} - \frac{1}{\sqrt{1-p_{xD}^2}}\right)dz \tag{3.7a}$$

$$\bar{y} = y + \mu p_y\ ;\ \mu = \int_0^{z_t} \frac{dz}{n} \tag{3.7b}$$

$$\bar{\tau} = \tau + \frac{\partial}{\partial p_\tau}\left(\frac{\lambda p_x^2}{2} + \frac{\mu p_y^2}{2} \pm \nu p_x - \Delta\right);\ \Delta = \int_0^{z_t}\left(n + \frac{p_\tau}{\beta\sqrt{1-p_{xD}^2}}\right)dz \tag{3.7c}$$

The quantities λ, μ and Δ can all be evaluated. The results are:

$$\lambda = \frac{\rho \sin\psi}{\left((1+\delta)^2 - \sin^2\psi\right)^{1/2}} \tag{3.8a}$$

$$\mu = \rho \sin^{-1}\left(\frac{\sin\psi}{1+\delta}\right) \tag{3.8b}$$

$$\nu = -\rho\left[\sqrt{(1+\delta)^2 - \sin^2\psi} - \sqrt{\cos^2\psi} - \delta\right] \tag{3.8c}$$

$$\Delta = +\frac{\rho}{2}\left\{\sin\psi[(1+\delta)^2 - \sin^2\psi]^{1/2} + (1+\delta)^2 \sin^{-1}\left(\frac{\sin\psi}{1+\delta}\right)\right\} + \rho\frac{\psi p_\tau}{\beta} \quad . \tag{3.8d}$$

Here, the ± in front of ν refers to the entrance and exit half bend. Again, we emphasize the drift-like nature of a parallel face bend. The horizontal focussing of the sector bend vanished, and the vertical focussing, as we will see later, will appear as a result of a fringe field effect (solenoidal terms needed to obey Maxwell's equations).

4. Maxwellian Fringe Field Effects with the Hamiltonians *H* and *K*

Going back to Fig. 2, we are interested in the potential effect of Maxwell's equation on the dynamics between point 1 and 2.

We start again with the vector potential A_z of a x-independent parallel face bend.

$$A_z = -x\, B(z) \quad . \tag{4.1}$$

It is easy to check that A_z does not obey Maxwell's equation. This can be corrected to order y^2

$$A_x = -\frac{1}{2}\frac{dB}{dz}\, y^2 \ldots \tag{4.2a}$$

$$A_z = -x\, B \quad . \tag{4.2b}$$

In fact, for a given profile of $B(z)$, one must solve the equation

$$\frac{\partial^2}{\partial y^2} A_x + \frac{\partial^2}{\partial z^2} A_x = \frac{-dB}{dz} \tag{4.3}$$

We substitute this result in the expression for H:

$$H = -\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2} - \frac{p_\tau}{\beta} - a_z \tag{4.4a}$$

$$a = \frac{qA}{p_0} \quad , \tag{4.4b}$$

We then expand H to leading order in a_x

$$H = H_{a_x=0} - \frac{p_x\, a_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} + \ldots \quad . \tag{4.5}$$

To first order in a_x, the map from $z = -\Delta$ to $z = \Delta$ is just given by:

$$\mathcal{M} = \exp\left(: \left[\int_{-\Delta}^{\Delta} ax\, dz \right] \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} : \right)$$

$$\mathcal{M} = \exp\left(: \left[\frac{q}{p_0} \int_{-\Delta}^{\Delta} \frac{dB}{dz}\, dz \right] \frac{-\frac{1}{2} y^2\, p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} : \right) + O(y^4) \ldots \tag{4.6}$$

The Lie generator of $\mathcal{M}$ is just given by

$$f = \mp \frac{1}{2\rho} \frac{y^2\, p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} \tag{4.7}$$

Consistent with the expansion in y^2, we can neglect p_y^2 completely in f

$$f = \mp \frac{1}{2\rho} \frac{y^2\, p_x}{\sqrt{(1+\delta)^2 - p_x^2}} \tag{4.7}$$

In that case, $\mathcal{M}$ can be evaluated exactly as a "kick"

$$\bar{x} = x \pm \frac{y^2}{2\rho} \frac{(1+\delta)^2}{\left[(1+\delta)^2 - p_x^2\right]^{3/2}} \tag{4.9a}$$

$$\bar{p}_x = p_x$$

$$\bar{y} = y \tag{4.9b}$$

$$\bar{p}_y = p_y \mp \frac{y}{\rho} \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2}} \tag{4.9c}$$

$$\bar{\tau} = \tau \mp \frac{1}{2\rho} \frac{y^2\, p_x\, (1+\delta)}{\left[(1+\delta)^2 - p_x^2\right]^{3/2}} \frac{d\delta}{dp_\tau} \tag{4.9d}$$

$$\bar{p}_\tau = p_\tau \tag{4.9e}$$

$$\frac{d\delta}{dp_\tau} = \frac{p_\tau - {}^1/\beta}{(1+\delta)} \tag{4.9f}$$

In the expressions for the fringe field, p_x is usually a large quantity. In fact, it is often useful to replace p_x by $p_x \pm \sin\psi$, where $\pm \sin\psi$ is the design entrance (or exit) angle.

In the context of the expanded Hamiltonian K, used in programs such as PATRIS or FASTRAC, one can simplify (4.9) further.

$$\bar{x} = x \tag{4.10a}$$

$$\bar{p}_x = p_x \tag{4.10b}$$

$$\bar{y} = y \tag{4.10c}$$

$$\bar{p}_y = p_y - \frac{y}{\rho} \frac{\sin\psi}{[(1+\delta)^2 - \sin^2\psi]^{1/2}} \tag{4.10d}$$

$$\bar{\tau} = \tau - \frac{1}{2\rho} \frac{y^2 \sin\psi\,(1+\delta)}{[(1+\delta)^2 - \sin^2\psi]^{3/2}} \frac{d\delta}{dp_\tau} \tag{4.10e}$$

$$\left\{ \begin{array}{l} p_x \text{ is replaced by } \pm \sin\psi. \\ \text{or } f = -\dfrac{1}{2\rho} y^2 \dfrac{\sin\psi}{[(1+\delta)^2 - \sin^2\psi]^{1/2}} \end{array} \right. \tag{4.10f}$$

This new expression is consistent with K. The Hamiltonian K is exact in δ, but quadratic in the betatron variables.

Appendix I
The Matrix of the General Bend

As pointed out in Sec. 1, we need to compute the sector bend first. This is very easy using the expanded Hamiltonian K of a sector bend.

$$K = -\frac{\delta x}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} \tag{A1}$$

A simple exercise leads to the results

$$\bar{x} = \cos\tilde{\alpha}\, x + \frac{\sin\tilde{\alpha}}{\omega(1+\delta)} p_x - \rho(\cos\tilde{\alpha} - 1)\,\delta \tag{A2a}$$

$$\bar{p}_x = -\omega(1+\delta)\sin\tilde{\alpha}\, x + \cos\tilde{\alpha}\, p_x + \rho\omega(1+\delta)\sin\tilde{\alpha}\,\delta \tag{A2b}$$

$$\bar{y} = y + \frac{s}{1+\delta} p_y \;;\; s = \text{total path length.} \tag{A2c}$$

$$\bar{p}_y = p_y \tag{A2d}$$

$$\tilde{\alpha} = \frac{\alpha}{\sqrt{1+\delta}} \;;\; \alpha = \text{bend angle} = \frac{s}{\rho} \tag{A2e}$$

$$\omega = \sqrt{\frac{1}{\rho^2(1+\delta)}} \tag{A2f}$$

Following our discussion, the general bend can be written as follows

$$G(\alpha,\psi,\phi) = \mathrm{rot}^{-1}(-\phi)\, F(\phi)\, H_-(\phi)\, \mathrm{sector}(\alpha-\phi-\psi)\, H_+(\psi)\, F(\psi)\, \mathrm{rot}(\psi) \tag{A3}$$

Here, rot(ψ) is the change of coordinate given by (2.2), $F(\psi)$ is the fringe matrix (4.10), $H_\pm(\phi)$ is the half bend given by (3.7) and (3.8), and finally sector(α) is just (A2).

Needless to say that the quadrupoles added to sector bends to create arbitrary bends are over-simplifications. In fact, it is easy to show using (A3) that for small bending angles ψ, ϕ and α, the pseudo-quadrupoles to be added to a sector bend are

$$G(\alpha,\psi,\phi) \approx Q(\phi)\, \mathrm{sector}(\alpha-\phi-\psi)\, Q(\psi) + O(2)\ldots \tag{A4}$$

$$Q(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-\tan\phi}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan\phi}{(1+\delta)\rho} & 1 \end{bmatrix} \quad . \tag{A5}$$

One can see that $Q(\phi)$ is not a real quadrupole since it depends on δ in the vertical plane. This is not surprising, since the horizontal and vertical focussing have totally different origins. In the horizontal plane, the focussing is completely a geometrical effect due to the presence or absence of a piece of bend. For a small angle, we expect its δ-dependence to be that of the second term of (A2b). In the vertical plane, the presence of focussing is due to the solenoidal field, due to Maxwell's equations. Such effects always have a $(1 + \delta)$ term in the denominator, since they came from the $\frac{(p_x - a_x)^2}{2(1+\delta)}$ drift term in the Hamiltonian. Like the drift in canonical variables, they must have a $(1 + \delta)$ dependence.

In general, one can attempt to write the map of the parallel face bend as

$$\mathcal{P} = \mathcal{E}\,\mathcal{S}\,\mathcal{E} \quad , \tag{A6}$$

where $\mathcal{S}$ is the sector bend and $\mathcal{E}$ is some "edge" map. However, Healy has shown that such a representation is unphysical, and as a result, $\mathcal{E}$ has some "resonance" type divergences at some bend angles. Therefore, the quadrupole edge trick cannot be extended in the exact treatment of a bend even in the restricted δ-dependent quadratic theory of FASTRAC and PATRIS.