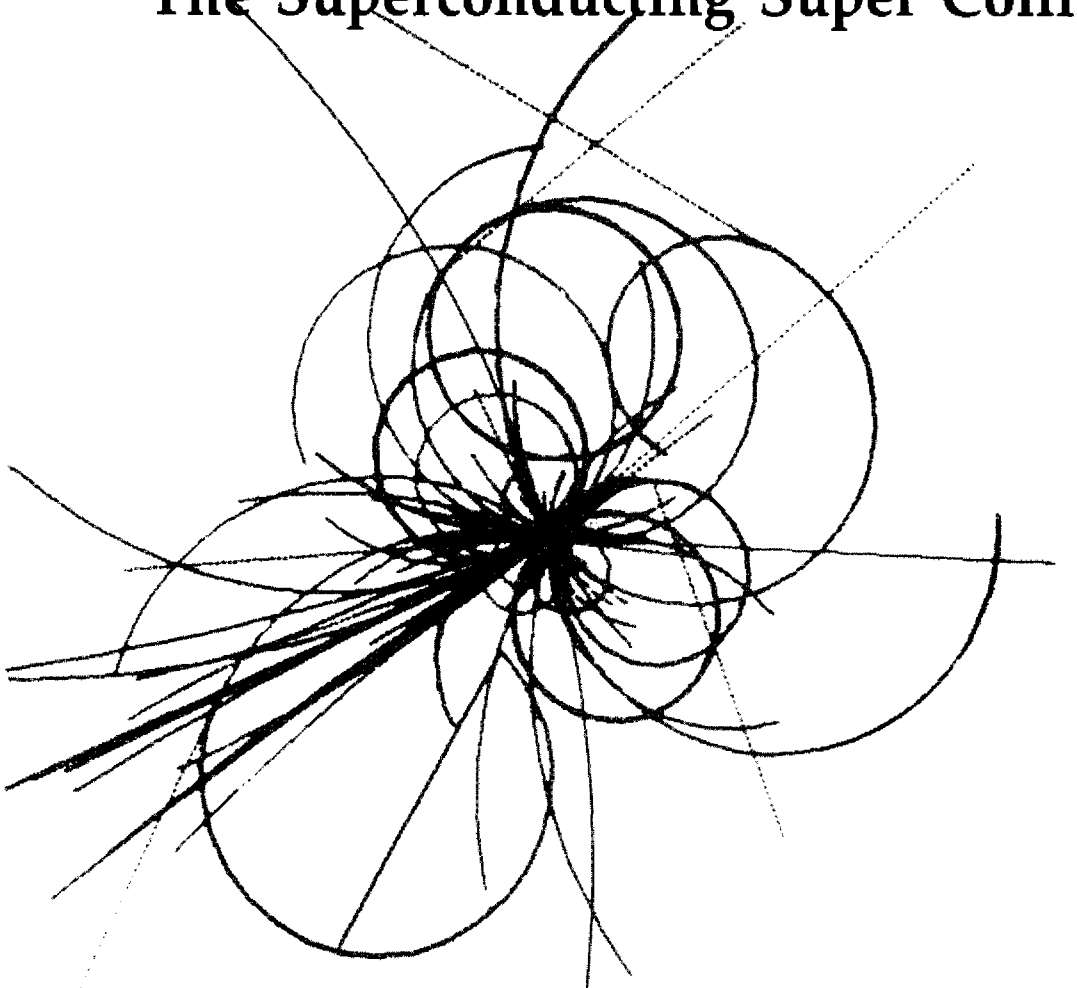
SSC-138

The Superconducting Super Collider



Canonical Integrators as Tracking Codes (or How to Integrate Perturbation Theory with Tracking)

Etienne Forest
SCC Central Design Group

September 1987

SSC-138

CANONICAL INTEGRATORS AS TRACKING CODES (OR HOW TO INTEGRATE PERTURBATION THEORY WITH TRACKING)

Etienne Forest
SSC Central Design Group,*
c/o Lawrence Berkeley Laboratory, 1 Cyclotron Road, Berkeley, CA 94720

September 1987

*Operated by Universities Research Association for the Department of Energy

FOREWORD – *WARNING*

This lecture is supposed to be on the topic of tracking codes. Tracking codes are software beings and therefore one might have expected me to discuss some of the existing codes. Well, in fact, I have avoided discussing or even mentioning any of the existing codes. For example, when the word "TEAPOT" appears in this lecture, the reader should automatically replace it by the locution "TEAPOT Philosophy." This lecture turns out to be on the topic of the ideal single particle code for a large circular machine. This code does not exist, although I will proudly say that the SSC Central Design Group has two mutant versions of this ideal code.

The purpose of this lecture is to set guidelines for critical judging of the abilities of existing codes and the future codes that some of you might design.

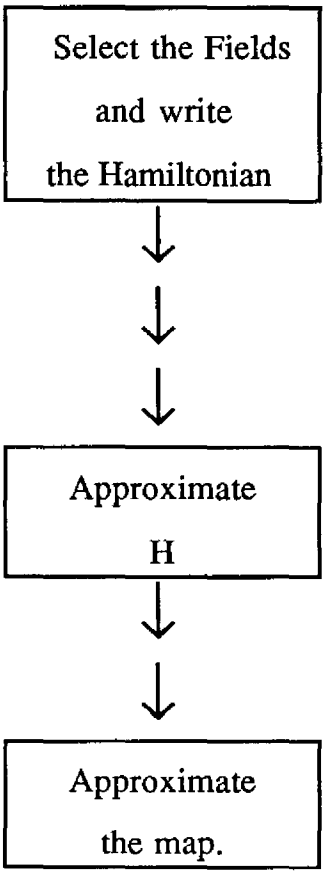
The initial chapter will consist of a description of the capabilities of an ideal code. This will narrow down the possible solutions. Finally, let me say that despite my avoidance of code names, I am perfectly willing to denigrate, attack and even clarify what existing codes do. Of course, I will do this in private (in front of no more than 1,000 persons).

Introduction – The Requirements

Our ideal monster should be able to satisfy the following requirements:

1) Adequate for **large circular** machine.
2) Can track **exactly** a particle within its model. (No approximation of its map).
3) Should include synchrotron oscillations **exactly**.
4) Should produce **exactly** Taylor expansions of its map around **any** trajectory. This can be used to do fast tracking or to perform some **exact** perturbation theory on the ring.

How do we start to construct this beast? Well let us look at the flow chart which describes the steps involved in designing the code.



Firstly, we must select the kind of fields our code will model. (No gravitational field necessary!) Typically, for a large machine this means ideal fields which are constant over a region of space and zero elsewhere (or delta function kicks).

Secondly, we must consider whether or not an approximation of the Hamiltonian H could lead to some simplification without invalidating the code completely. Incidentally, by approximation I do not mean something like a thin lens approximation. When one replaces a thick element by a thin one, one selects a different field. By approximation I mean something like expanding a square-root. It must be without "physical" interpretation to be called an approximation of H.

Finally, one can allow for an approximation of the exact map resulting from the field selection and the approximate H. This is forbidden by rules 2) and 3). However, our choices of fields and H, must be such that a subroutine of our code must allow for the production of a Taylor expanded map around any trajectory if needed.

It is clear from the previous discussion that one must always have in mind the three boxes of the flow chart and the four dogmatic criteria while designing my visionary code. (As well a glass of beer in hand!)

The chapters of this lecture describe the ingredients which are necessary to construct this ideal code. Hence, I will tell you the answer and if you don't like it, you can challenge or ignore it.

i) The fields selected will be that of an ideal combined function sector bend. Arbitrary entrance angle will have to be mimicked by thin quadrupoles. This is permitted by rule 1.

ii) The Hamiltonian H follows from i). However, a Hamiltonian K which approximates H in large machine will also be discussed.

iii) The code will have to be a canonical integrator. (3 solutions: one for H and two for K.) Canonical integrators can often be interpreted as a re-selection of the fields.

iv) The exact Taylor series map extraction will make one of the canonical integrator of K hard to use. (Thesis project there!)[1]

The items i) to iv) motivate the topics of the chapters to follow.

Chapter 1 Potpourri of things that ought to be known. (H, K and the ability to manipulate maps.)

Chapter 2 Canonical Integrators.

Chapter 3 Map extraction.

Chapter 4 The beginning of a perturbation theory based on maps alone. (Phase advance, Floquet ring . . .)

Chapter 1 – The Tools

1. The Hamiltonians of the Combined Sector Bend

Here, I will refer the reader to the treatment of Ruth presented in a previous summer school.[2] In scaled variables and parametrized in terms of s (the design path-length), this Hamiltonian is just:

$$H = \frac{x}{\rho} + \frac{x^2}{2\rho^2} - (1 + \frac{x}{\rho})\,[(1+\delta)^2 - p_x^2 - p_y^2]^{1/2} - \frac{p_\tau}{\beta} - \tilde{V} \tag{1.1a}$$

$$\tilde{V} = \frac{-q}{p_0 c} A_s\,;\ \ q = \text{electric charge} \tag{1.1b}$$

Here, p_x and p_y are scaled by p_0c, the design momentum. The momentum p_τ is $-\frac{\delta E}{p_0 c}$ and is canonically conjugate to the differential time $\tau\,[= c(T-T_0)]$. Finally β is $\frac{v_0}{c}$ and δ is just $(p-p_0)/p_0$. The expression $(1+\delta)^2$ can be written as:

$$(1+\delta)^2 = 1 - \frac{2p_\tau}{\beta} + p_\tau^2. \tag{1.2}$$

We will see that it is possible to write a canonical integrator (which amounts to a reinterpretation of the fields) for H.

Furthermore, it is possible to find a reasonable approximation of H. Consider the Hamiltonian K gotten by expanding the square root to second order in p_x and p_y :

$$K = -\frac{\delta x}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)} - \Delta\delta - \tilde{V} \tag{1.3a}$$

$$\Delta\delta = \delta - 1 + \frac{p_\tau}{\beta} \tag{1.3b}$$

This famous approximation is valid for two reasons. First, p_x^2 and p_y^2 are small compared to 1, and small compared to x^2 and y^2 in a typical accelerator (since the Twiss ß-functions are usually larger than the γ–functions).

Also, the term $\frac{x}{\rho}p_x^2$, potentially troublesome in small ring, can be ignored in a large ring where ρ is large.[3,4]

We will see that K is solvable in terms of two canonical integrators.

2. Approximate Maps and Their Representation[5,6]

In general, we will be confronted with a map $\mathcal{M}$ around some orbit. Of course, such a map can be represented by a Taylor series (Einstein summation convention):

$$\overline{\zeta}_a = \mathcal{M}\, \zeta_a^0 = R_{ab}\, \zeta_b^0 + T_{abc}\, \zeta_b^0\, \zeta_c^0 + W_{abcd}\, \zeta_b^0\, \zeta_c^0\, \zeta_d^0 + \ldots \tag{2.1}$$

Since the map is generated by a canonical integrator, it can be represented by a factored product of canonical maps. Each of these maps is generated by a Lie operator (except for the linear part for which we keep the matrix representation R):

$$\mathcal{M}(\vec{\zeta}) = \mathcal{R}(\vec{\zeta}) \exp(: f_3\, (\vec{\zeta}) :) \exp(: f_4\, (\vec{\zeta}) :) \ldots \tag{2.2a}$$

where, $$: f\, (\vec{\zeta}) :\ g\, (\vec{\zeta}) = [f,g]_{\vec{\zeta}} = \text{Poisson Bracket}, \tag{2.2b}$$

and, $$: f\, (\vec{\zeta}) :^0 = \text{Identity}\,. \tag{2.2c}$$

Canonical operators represented by Eq. (2.2a) act on functions of the argument of their Lie generators and are denoted by script letters.

This implies, for example, that Eq. (2.1) is represented in terms of Lie transformations as

$$\overline{\vec{\zeta}} = \mathcal{M}(\vec{\zeta})\, \vec{\zeta} \Big|_{\vec{\zeta} = \vec{\zeta}^0} \tag{2.3}$$

To simplify the notation, we will often write Eq. (2.3) as

$$\vec{\zeta} = \mathcal{M}\, (\vec{\zeta}^0)\, \vec{\zeta}^0 \tag{2.4}$$

Furthermore, it also implies that a product of two Lie transformations with the same argument in their generator act in the reverse order from the usual matrix multiplication. Before proving this, we will give two standard results involving Lie operators:

$$\exp(: f :) \exp(: g :) \exp(: -f :) = \exp(: \exp(: f :)\, g :), \tag{2.5a}$$

$$\exp(:f:)\, g(\vec{\zeta}) = g(\exp(:f:)\vec{\zeta})\ . \tag{2.5b}$$

In Eq. (2.5), it is assumed that both f and g are functions of $\vec{\zeta}$. Now, we are in a position to examine the problem of the reverse ordering!

Consider a two step process represented by the maps $\mathcal{N}$ and $\mathcal{L}$:

$$\vec{\zeta} \xrightarrow{\mathcal{N}(\vec{\zeta})} \vec{\xi} \xrightarrow{\mathcal{L}(\vec{\xi})} \vec{\eta} \tag{2.6a}$$

$$\mathcal{N}(\vec{\zeta}) = \exp(:f(\vec{\zeta}):) \tag{2.6b}$$

$$\mathcal{L}(\vec{\xi}) = \exp(:g(\vec{\xi}):) \tag{2.6c}$$

We must have

$$\vec{\eta}(\vec{\xi}) = \mathcal{L}(\vec{\xi})\,\vec{\xi}$$

$$\begin{aligned}
\vec{\eta}(\vec{\zeta}) &= \mathcal{L}(\vec{\xi})\,\mathcal{N}(\vec{\zeta})\,\vec{\zeta} \\
&= \mathcal{L}(\mathcal{N}(\vec{\zeta})\vec{\zeta})\,\mathcal{N}(\vec{\zeta})\,\vec{\zeta} \\
&= \mathcal{N}(\vec{\zeta})\,\mathcal{L}(\vec{\zeta})\,\mathcal{N}^{-1}(\vec{\zeta})\,\mathcal{N}(\vec{\zeta})\,\vec{\zeta} \\
&= \mathcal{N}(\vec{\zeta})\,\mathcal{L}(\vec{\zeta})\ \vec{\zeta} \qquad \text{Q.E.D.}
\end{aligned} \tag{2.7}$$

In Eq. (2.7), we have made use of (2.5a) and (2.5b). The necessity of using only one argument comes from the required need to concatenate maps.[6] The Lie operators must have the same arguments if Poisson brackets are to be evaluated.

As far as linear operators are concerned, we will tend to use their matrix representation:

$$\mathcal{R}(\vec{\zeta})\,\zeta_a = R_{ab}\,\zeta_b \tag{2.8}$$

This concludes this short chapter. For more material on Lie operators, the reader is invited to consult the references.

Chapter 2 – Canonical Integration Techniques[7,8]

Consider the time evolution of an operator $\mathcal{M}$. We will assume that it is controlled by some Hamiltonian operator[9] $:H: = -:A: - :B:$ independent of s:

$$\frac{d\mathcal{M}}{ds} = \mathcal{M}\ :-H: = \mathcal{M}\ (:A: + :B:) \tag{1}$$

Formally the solution of Eq. (1) for the operator from s = 0 to s is just:

$$\mathcal{M}(s) = \exp(s:-H:) = \exp(s:A: + s:B:) \tag{2}$$

The problem encountered in solving the differential equations of a tracking code is the impossibility of evaluating the operator $\mathcal{M}$ as given by (2) exactly.

Quite often, however, one can evaluate exactly the simpler operators $\mathcal{A}(s)$ and $\mathcal{B}(s)$ given by

$$\mathcal{A}(s) = \exp(s:A:) \tag{3a}$$

$$\mathcal{B}(s) = \exp(s:B:) \tag{3b}$$

One possible solution is to represent $\mathcal{M}$ as

$$\mathcal{M}(s) = \mathcal{A}(s)\,\mathcal{B}(s) + O(s^2) \tag{4}$$

A clearer representation consists in symmetrizing the approximate expression.

$$\mathcal{M}(s) = \mathcal{A}(\tfrac{s}{2})\,\mathcal{B}(s)\,\mathcal{A}(\tfrac{s}{2}) + O(s^3) \tag{5}$$

In fact, one can derive even more complicated schemes with the help of the Campbell Bake-Hausdorf formula.[5,6,8]

$$\mathcal{M}(s) = \mathcal{A}\left(\frac{s}{2(2-\beta)}\right)\mathcal{B}\left(\frac{s}{2-\beta}\right)\mathcal{A}\left(\frac{s(1-\beta)}{2(2-\beta)}\right)\mathcal{B}\left(\frac{-s\beta}{2(2-\beta)}\right)\mathcal{A}\left(\frac{s(1-\beta)}{2(2-\beta)}\right)\mathcal{B}\left(\frac{s}{2-\beta}\right)\mathcal{A}\left(\frac{s}{2(2-\beta)}\right) + O(s^5) \tag{6a}$$

$$\beta = 2^{1/3} \tag{6b}$$

The last canonical integrator is fifth order in "s". To illustrate these techniques, we will use the exactly solvable harmonic oscillator.

We consider the operator :H: given by

$$:H: = :\frac{q^2}{2}: + :\frac{p^2}{2}: \tag{7}$$

From Eq. (7), we find the operators :A: and :B: to be

$$:A: = :\frac{-q^2}{2}: \quad \text{and} \quad :B: = :\frac{-p^2}{2}: \tag{8}$$

The operator $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are exactly solvable and in this case are linear operators.

$$\mathcal{A}(s)\begin{bmatrix} q \\ p \end{bmatrix} = M_A\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix} \tag{9a}$$

$$\mathcal{B}(s)\begin{bmatrix} q \\ p \end{bmatrix} = M_B\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix} \quad . \tag{9b}$$

Of course, the operator $\mathcal{M}(s)$ can also be found exactly. It is just a rotation of angle s:

$$\mathcal{M}(s)\begin{bmatrix} q \\ p \end{bmatrix} = M\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}. \tag{10}$$

As an exercise, the reader can check on this example the accuracy of the formulae (4), (5), and (6). Remembering that the operators act in opposite order from matrix multiplication, one gets the results:

$$M_{(4)} = \begin{bmatrix} 1-s^2 & s \\ -s & 1 \end{bmatrix} = M + O(s^2) \tag{11}$$

$$M_{(5)} = \begin{bmatrix} 1-\frac{s^2}{2} & s \\ s-\frac{s^3}{4} & 1-\frac{s^2}{2} \end{bmatrix} = M + O(s^3) \tag{12}$$

$$M_{(6)} = \begin{bmatrix} a & b \\ c & a \end{bmatrix} = M + O(s^5)\,, \tag{13}$$

where, for example, a and b given by

$$\begin{cases} a = 1 - \frac{s^2}{2} + \frac{s^4}{24} - \left(\frac{62\beta^2 - 521\beta + 558}{105408\beta^2 - 478656\beta + 435744}\right) s^6 = \cos(s) + O(s^5) \\ b = s - \frac{s^3}{6} + \frac{(199\beta^2 - 162\beta - 520)\, s^5}{40176\beta^2 + 8928\beta - 74024} = \sin(s) + O(s^5)\,. \end{cases} \tag{14a}$$

or,

$$\begin{cases} a \approx 1 - \frac{s^2}{2} + \frac{s^4}{24} - 2.07\ 10^{-3}\, s^6 \\ b \approx s - \frac{s^3}{6} + 1.45\ 10^{-2}\, s^5\,. \end{cases} \tag{14b}$$

As one sees, the results are as advertised. Canonical integration techniques can be derived for a wide class of problems and with different goals in mind. They preserved the symplectic properties of the phase-space flow and they are composed of a succession of simple operations such as drifts and multipole kicks.

The applications we have in mind are related to tracking codes. Let us take the approximate Hamiltonian K for ideal small curvature magnets:

$$K = \frac{-\delta x}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} - \Delta\delta - \tilde{V} \tag{15a}$$

$$1 + \delta = \sqrt{1 - \frac{2p_\tau}{\beta} + p_\tau^2} \tag{15b}$$

$$\Delta\delta = \delta - 1 + \frac{p_\tau}{\beta} \tag{15c}$$

$$\tilde{V}(x,y) = \frac{-q}{p_0 c} A_s\ . \tag{15d}$$

For non-quadratic $\tilde{V}$, it is impossible to evaluate the map $\mathcal{M}(s)$ produced by :K:. How can we define the operators :A: and :B: of the canonical integrator corresponding to K?

Answer:

$$A = -\frac{p_x^2}{2(1+\delta)} - \frac{p_y^2}{2(1+\delta)} + \Delta\delta \tag{16a}$$

$$B = \frac{\delta x}{\rho} - \frac{x^2}{2\rho^2} + \tilde{V} \tag{16b}$$

Exercise 1

Evaluate $\mathcal{A}(s)$ and $\mathcal{B}(s)$ corresponding to Eqs. (16a) and (16b) and give them a physical (pictorial) meaning.

Answers

$$\mathcal{A}(s)\, x = x + \frac{s p_x}{1+\delta} \tag{17a}$$

$$\mathcal{A}(s)\, y = y + \frac{s p_y}{1+\delta} \tag{17b}$$

$$\begin{aligned}\mathcal{A}(s)\, \tau &= \tau + \frac{s\vec{p}^2}{2(1+\delta)^2}\frac{d\delta}{dp_\tau} - \frac{s\, d(\Delta\delta)}{dp_\tau} \\ &= \tau + \frac{s\vec{p}^2}{2(1+\delta)^3}\left(\frac{1}{\beta} - p_\tau\right) - s\,\frac{(\delta/\beta + p_\tau)}{1+\delta}.\end{aligned} \tag{17c}$$

$$\mathcal{A}(s)\begin{bmatrix} p_x \\ p_y \\ p_\tau \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_\tau \end{bmatrix}. \tag{17d}$$

$$\mathcal{B}(s)\, p_x = p_x + \frac{s\delta}{\rho} - \frac{sx}{\rho} + s\frac{\partial\tilde{V}}{\partial x} \tag{18a}$$

$$\mathcal{B}(s)\, p_y = p_y + s\frac{\partial\tilde{V}}{\partial y} \tag{18b}$$

$$\mathcal{B}(s)\, \tau = \tau - s\frac{x}{\rho}\frac{d\delta}{dp_\tau} = \tau - s\frac{x}{\rho}\frac{(1/\beta + p_\tau)}{(1+\delta)} \tag{18c}$$

The operator $\mathcal{A}$ represents a drift. The operator $\mathcal{B}$ is a bit more complicated. Its $\tilde{V} - \frac{x^2}{2\rho^2}$ part consists of the multipole component $\tilde{V}$ and the focusing part of the bend. The $\frac{\delta x}{\rho}$ part of $\mathcal{B}$ is the left-over of a planar $x - s$ rotation after the exact Hamiltonian H is simplified into K. This term introduces dispersion in the bends. (See Exercise 3 for the exact Hamiltonian H.)

Exercise 2

A lot of the so-called kick codes[10] using K (or an erroneous version of it) restrict themselves to the four dimensional phase-space (x, p_x, y, p_y). For these codes, the exact mapping through quadrupoles and dipoles are given by δ–dependent matrices. Find the operators :A: and :B: for these codes. Convince yourself that the time of flight is also exactly computable and is given by a quadratic polynomial in the transverse betatron variables. (Hint: Think of a generating function representing the matrices.)

<u>Answer</u>

First of all, let us rewrite K as follows:

$$K = -A - B \tag{19a}$$

$$A = \frac{\delta x}{\rho} - \frac{x^2}{2\rho^2} + \tilde{V}_2(x,y) + \frac{p_x^2 + p_y^2}{2(1+\delta)} + \Delta\delta_2 \tag{19b}$$

$$B = \tilde{V} - \tilde{V}_2 + \Delta\delta - \Delta\delta_2 \tag{19c}$$

The quantity $\tilde{V}_2$ is the quadratic part of $\tilde{V}$. The terms $\Delta\delta$ and $\Delta\delta_2$ in K can be ignored since they produce only a time of flight term which is ignored in a four dimensional treatment.

The famous statement "quadrupoles and dipoles" are linear originates from the map $\mathcal{A}(s)$ produced by :A:. One notices that A is at most quadratic in the transverse variables. Therefore, it will lead to a linear map of transverse phase space.

Exercise 3

As discussed previously, the Hamiltonian K is an approximation of the true ideal element Hamiltonian H:

$$H = \frac{x}{\rho} + \frac{x^2}{2\rho^2} - (1 + \frac{x}{\rho})\,((1+\delta)^2 - p_x^2 - p_y^2)^{1/2} - \frac{p_\tau}{\beta} - \tilde{V} \tag{20}$$

Split H into 3 exactly solvable operations. Give them some "physical" interpretation and solve them. In addition give a possible second order integrator.

<u>Answer</u>

$$H = -A - B - C \tag{21a}$$

$$A = ((1+\delta)^2 - p_x^2 - p_y^2)^{1/2} - \frac{p_\tau}{\beta} \tag{21b}$$

$$B = \frac{x}{\rho}((1+\delta)^2 - p_x^2 - p_y^2)^{1/2} \tag{21c}$$

$$C = -\frac{x^2}{2\rho^2} + \tilde{V} - \frac{x}{\rho} \tag{21d}$$

The operator $\mathcal{A}(s)$ is just a drift of length s. The operator $\mathcal{B}(s)$ is a drift in cylindrical coordinates, it represents a rotation of angle $\frac{s}{\rho}$ in the x-s plane. Finally, $\mathcal{C}(s)$ (= exp(:sC:)) is a kick containing the multipole content of the bend $\tilde{V}$, the focusing power of the sector bend $\frac{-x^2}{2\rho^2}$ and the actual dipole kick $\frac{x}{\rho}$.

The only difficult term to evaluate is $\mathcal{B}(s)$. We proceed using the differential equation associated with $\mathcal{B}(s)$. Consider a function of phase space $g(\vec{z}(s))$ where $\vec{z} = (x, p_x, y, p_y, \tau, p_\tau)$:

$$\frac{d}{ds} g(\vec{z}(s)) = \frac{d}{ds} \exp(:sB:)\, g(\vec{z}(0))$$

$$= :B: \mathcal{B}(s)\, g(\vec{z}(0))$$

$$= [B, g(\vec{z}(s))] \tag{22}$$

Equation (22) is essentially Hamilton's equation of motion for the Hamiltonian -B. Replacing $g(\vec{z})$ by the various components of $\vec{z}$, we obtain:

$$\frac{dx}{ds} = \frac{x}{\rho} \frac{p_x}{(f^2 - p_x^2)^{1/2}} \tag{23a}$$

$$\frac{dp_x}{ds} = \frac{(f^2 - p_x^2)^{1/2}}{\rho} \tag{23b}$$

$$\frac{dy}{ds} = \frac{x}{\rho} \frac{p_y}{(f^2 - p_x^2)^{1/2}} \tag{23a}$$

$$\frac{dp_y}{ds} = 0 \tag{23d}$$

$$\frac{d\tau}{ds} = \frac{x}{\rho} \frac{1/\beta - p_\tau}{(f^2 - p_x^2)^{1/2}} \tag{23e}$$

$$\frac{dp_\tau}{ds} = 0 \tag{23f}$$

$$f^2 = (1 + \delta)^2 - p_y^2 = 1 - \frac{2p_\tau}{\beta} + p_\tau^2 - p_y^2 \ . \tag{23g}$$

Starting with (23b), we get:

$$p_x(s) = \cos(\frac{s}{\rho})\, p_x(0) + \sin(\frac{s}{\rho})\, p_s(0) \tag{24a}$$

$$p_s = \sqrt{f^2 - p_x^2} \tag{24b}$$

Substituting in (23a), we obtain for x(s):

$$x(s) = x(0) \frac{p_s(0)}{p_s(s)} = \frac{x(0)}{\cos(\frac{s}{\rho})(1 - \frac{p_x(0)}{p_s(0)} \tan(\frac{s}{\rho}))} \tag{25}$$

Similarly, substituting (25) into (23c), we get

$$y(s) = y(0) + \frac{x(0)\, p_s(0)\, p_y(0)}{\rho} \int_0^s p_s(s')^{-2}\, ds'$$

$$= y(0) + \frac{x(0)\, p_y(0) \sin(\frac{s}{\rho})}{p_s(s)} = y(0) + \frac{x(0)\, p_y(0) \tan(\frac{s}{\rho})}{p_s(0)\,(1 - \frac{p_x(0)}{p_s(0)} \tan(\frac{s}{\rho}))} \quad . \tag{25}$$

The only computation left involves the time of flight τ. Quite clearly, the answer looks very similar to y(s):

$$\tau(s) = \tau(0) + (\frac{1}{\beta} - p_\tau) \frac{x(0) \sin(\frac{s}{\rho})}{p_s(s)} \quad . \tag{27}$$

A final comment about $\mathcal{B}(s)$:

> $\mathcal{B}(s)$ is a drift in cylindrical coordinates, therefore the differential equations were not necessary. By simple geometry, one can reconstruct Eqs. (24) through (27).

What about a quadratic integrator? One possible choice used in the program TEAPOT[4] goes as follows:

$$\mathcal{M}(s) = \mathcal{A}(\tfrac{s}{2})\, \mathcal{B}(\tfrac{s}{2})\, \mathcal{C}(\tfrac{s}{2})\, \mathcal{B}(\tfrac{s}{2})\, \mathcal{A}(\tfrac{s}{2}) \tag{28}$$

Clearly, a fifth order integrator is messy with three operators to play with. (I had not the courage to look at it.)

A few readers have probably noticed that A and B can be combined. Why?

Conclusion

We have demonstrated in this chapter how two modelings of the equations of motions for ideals elements (combined sector bends) can be broken into exactly solvable pieces, on which the theory of canonical integration can be applied.

In Exercise 1, we look at a solution for the approximate Hamiltonian K which is fully six-dimensional. In Exercise 2, we look at the restricted four-dimensional problem

and pointed out that the six-dimensional extension is also solvable exactly in terms of quadratic generating functions.

Finally, we looked at the exact ideal sector bends Hamiltonian and found an exactly solvable division in terms of three operators. It should become clear in the next chapter that the map generating algorithm is quite hard to implement in 6 dimensions for the Hamiltonian K of Exercise 2. (although possible)

Chapter 3 – Extraction of a Consistent Approximate Map[11]

1. The Idea

We have restricted our code to canonical integrators. In other words, we have an exact analytical expression for every step of the tracking. In general, we have seen that the ith step in the tracking can be expressed as:

$$\vec{z}_p^{\,i+1} = \exp(:K^i(\vec{z}^{\,i}):)\,\vec{z}^{\,i}\Big|_{\vec{z}^{\,i}=\vec{z}_p^{\,i}} \tag{1.1c}$$

$$= \vec{F}^i(\vec{z}_p^{\,i}) \;;\; \vec{z}_p^{\,i} = \text{some particular orbit.} \tag{1.1b}$$

The function $\vec{F}^i(\vec{z}_p^{\,i})$ has an analytic expression known to us. Now, we attempt to follow an arbitrary ray $\vec{z}^{\,i}$, "close" to $\vec{z}_p^{\,i}$:

$$\vec{z}^{\,i} = \vec{z}_p^{\,i} + \vec{\zeta}^{\,i} \tag{1.2a}$$

$$\vec{z}^{\,i+1} = \vec{F}^i(\vec{z}^{\,i}) \tag{1.2b}$$

To proceed further, we expand $\vec{F}^i(\vec{z}^{\,i})$ in a Taylor series in $\vec{\zeta}^{\,i}$:

$$\vec{z}_p^{\,i+1} + \vec{\zeta}^{\,i+1} = \vec{F}^i(\vec{z}_p^{\,i} + \vec{\zeta}^{\,i}) \tag{1.3a}$$

or,

$$\vec{\zeta}^{\,i+1} = \sum_{m=1}^{\infty} \vec{F}_m^i(\vec{\zeta}^{\,i} \;;\; \vec{z}_p^{\,i}) \quad , \tag{1.3b}$$

where $\vec{F}_m^i(\vec{\zeta}^{\,i})$ is a polynomial of degree m in $\vec{\zeta}^{\,i}$.

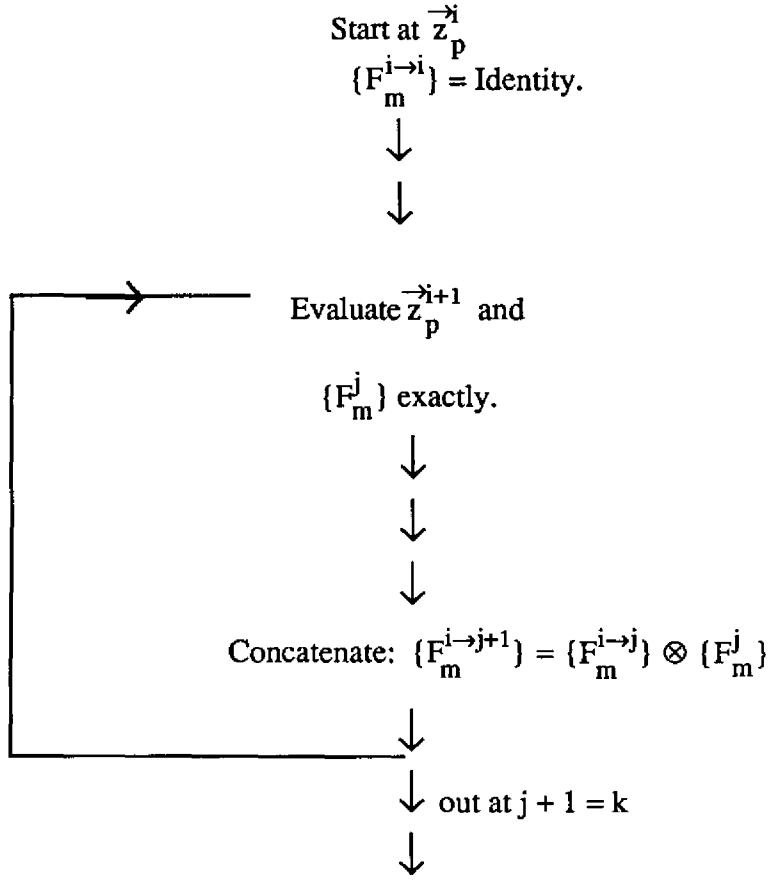
Clearly all the $\vec{F}_m^i$ are computable exactly, since $\vec{F}^i(\vec{z}^{\,i})$ which is just $\vec{F}_0^i$ is known analytically. In the optics literature, $\vec{F}_m^i(\vec{\zeta}^{\,i})$ is often known under the name "m^{th} order (transport) matrix."

If our kick code is provided with a concatenator for such matrices, then the total map for a section of the ring will involve the product over i of all the $\vec{F}_m^i (m < m_{max})$.

Here, m_{max} is the maximum order of these matrices. For example, if only a linear approximation of the map is needed (as required in a Newton search for the closed orbit), then the entire process will involve a first derivative of $\vec{F}^i(\vec{z}^i)$ evaluated at $\vec{z}^i_p$ and a linear matrix multiplier.

Notice that our matrix is accurate to order m_{max} in $\vec{\zeta}^i$ and to all order in $\vec{z}^i_p$, quite unlike Taylor series codes (Lie algebraic or "matrix" codes) which are only accurate to order m_{max} around the design orbit $\vec{z} = 0$.

We can present a flow chart of a computer evaluation of $\{\vec{F}^i_m\}$.



Typically, there are two possible ways to concatenate the matrices $\{F^{i\to k}_m\}$ and the method chosen depends on the representation of the map $\{F^{i\to k}_m\}$. In the next section, we examine these two ways.

2. The Representation of $\{F_m^{i\to k}\}$

The most straightforward way to represent our expanded map is based on the Taylor series representation. One expands (1.3b) as follows:

$$\zeta_a^k = R_{ab}^{i\to k}\,\zeta_b^i + T_{abc}^{i\to k}\,\zeta_b^i\,\zeta_c^i + \ldots \tag{2.1}$$

The linear part is represented by a matrix $R^{i\to k}$ and the non-linear contribution by the so-called non-linear "matrices" $T^{i\to k}$, etc. . .

A truncation at the level of the "T-matrix" as it is done in (2.1), would lead to a second order approximation of the map around the trajectory $\vec{z}_p$ of Eq. (1.2).

Exercise 1

Mentally construct a tracking code consisting of only two possible elements:

$$\text{i) rotation:} \quad \Phi(\mu) = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix} \tag{2.2a}$$

$$\text{ii) sextupole kick:} \quad \begin{aligned} q^{i+1} &= q^i \\ p^{i+1} &= p^i + \alpha\,[q^i]^2 \end{aligned} \tag{2.2b}$$

Derive the $R^{i\to i+1}$ and $T^{i\to i+1}$ matrices necessary to produce a map producing algorithm for this fictitious code.

<u>Answer</u>

i) For Φ, we simply have $R = \Phi$ and $T = 0$. The map does not depend on the local orbit $\vec{z}_p = (q_p, p_p)$.

ii) For the sextupole, we expand (2.2b) in terms of q_p and p_p.

$$q_p^{i+1} + \zeta_1^{i+1} = q_p^i + \zeta_1^i$$

$$p_p^{i+1} + \zeta_2^{i+1} = p_p^i + \zeta_2^i + \alpha[\zeta_1^i]^2 + 2\alpha\zeta_1^i q_p^i + \alpha[q_p^i]^2$$

$$\Rightarrow R = \begin{bmatrix} 1 & 0 \\ 2\alpha q_p^i & 1 \end{bmatrix} \; ; \; T_{211} = \alpha \; . \tag{2.3}$$

Is the "matrix" representation the only acceptable Taylor series representation? As discussed in the introductory chapter, we have seen that Hamiltonian (symplectic) maps can be represented by Lie operators. These operators also have well defined rules for concatenation.[6,9,11] Furthermore, the factorized representation of a map has well behaved Lie generators if the "matrix" representation is itself well-behaved. Therefore we will examine the factored Lie representation because it is well behaved, because concatenation soft-ware is already available to 5th order in $\vec{\zeta}^i$ and more importantly, high order perturbation theory can be performed on the Lie representation of a Taylor series map.

In the Lie algebraic language, one possible representation of Eq. (2.1) is given by:

$$\vec{\zeta}^k = \mathcal{R}^{i\to k} \exp(:f_3(\vec{\zeta}):)\, \vec{\zeta} \Big|_{\vec{\zeta} = \vec{\zeta}^i} \tag{2.4c}$$

Exercise 2

Find R and f_3 for Eq. (2.3) of Exercise 1.

<u>Answer</u>

R is of course identical to that of Exercise 1, and f_3 is given by:

$$f_3 = \frac{\alpha}{3}\, \vec{\zeta}_1^{\,3} \; . \tag{2.5}$$

Is there a fundamental difference between the "matrix" representation (2.1) and the Lie generator representation (2.4)? The answer is a unequivocal **no**. Both representations are "Taylor series" expansions. For example, if one truncates at the level of the (R,T) "matrices" the information retained is identical to what is buried in the (R,f_3) pair. In fact,

one can go freely between the (R,f_3) and the (R,T) representation. The following exercises illustrate this mathematical tautology.

Exercise 3

Compute f_3 from the R and T matrices. Show that the uniqueness of the connection between (R,T) and (R,f_3) is insured by the symplectic condition on the Jacobian of the map.

<u>Answer</u> [Skip Eq. (2.18) if not interested.]

Computation of f_3.

We start by assuming a Taylor series representation for a map $\mathcal{M}$. Calling $\vec{\zeta}$ the initial value of the variables:

$$\vec{\zeta}^f = \mathcal{M}\vec{\zeta} = R\,\vec{\zeta} + T\,\vec{\zeta}\,\vec{\zeta} + \ldots \tag{2.6a}$$

or,

$$\zeta_i^f = R_{ij}\,\zeta_j + T_{ijk}\,\zeta_j\,\zeta_k + \ldots \tag{2.6b}$$

Here we use the Einstein summation convention. Equation (2.6) is to be compared with the Lie algebraic representation:

$$\vec{\zeta}^f = \mathcal{M}\vec{\zeta} = \mathcal{R}\,e^{:f_3:}\,\vec{\zeta} = \mathcal{R}\,(\vec{\zeta} + [f_3,\vec{\zeta}]\ \ldots) \tag{2.7}$$

Equating (2.6) and (2.7) gives the following equation:

$$\mathcal{R}\,\vec{\zeta} + \mathcal{R}\,[f_3,\vec{\zeta}] = R\,\vec{\zeta} + T\,\vec{\zeta}\,\vec{\zeta} \tag{2.8}$$

$$\vec{\zeta} + [f_3,\vec{\zeta}] = \mathcal{R}^{-1}\,R\,\vec{\zeta} + \mathcal{R}^{-1}\,T\,\vec{\zeta}\,\vec{\zeta}$$

$$= R\,(\mathcal{R}^{-1}\,\vec{\zeta}) + T\,(\mathcal{R}^{-1}\,\vec{\zeta})\,(\mathcal{R}^{-1}\,\vec{\zeta})$$

$$= R\,R^{-1}\,\vec{\zeta} + T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\vec{\zeta})$$

$$= \vec{\zeta} + T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\vec{\zeta})$$

from (2.8), we can derive an expression for f_3:

$$[f_3, \vec{\zeta}] = T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\vec{\zeta}) \tag{2.9a}$$

$$-J\,\nabla f_3 = T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\zeta)\,, \tag{2.9b}$$

or, in component form:

$$\frac{\partial f_3}{\partial \zeta_a} J_{ai} = T_{ijk}\,R^{-1}_{j\ell}\,\zeta_\ell\,R^{-1}_{km}\,\zeta_m \tag{2.9c}$$

Multiplying (2.9b) by the symplectic form J gives ∇f_3:

$$\nabla f_3 = J\,T(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\zeta) \tag{2.10a}$$

or,

$$\frac{\partial f_3}{\partial \zeta_i} = J_{ia}\,T_{ajk}\,R^{-1}_{j\ell}\,\zeta_\ell\,R^{-1}_{km}\,\zeta_m \tag{2.10b}$$

Finally, we obtain f_3:

$$f_3 = \int_{\substack{0 \\ \text{over some path}}}^{\vec{\zeta}} d\,\vec{\zeta}'\,J\,T\,(R^{-1}\,\vec{\zeta}')\,(R^{-1}\,\vec{\zeta}') \tag{2.11}$$

One very easy path to choose is the "diagonal" between 0 and $\vec{\zeta}$:

$$\vec{\zeta}' = \alpha\,\vec{\zeta} \quad 0 \le \alpha \le 1$$

$$d\,\vec{\zeta}' = \vec{\zeta}\,d\alpha$$

$$f_3 = \int_0^1 \alpha^2\,d\alpha\,\vec{\zeta} \cdot J\,T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\vec{\zeta})$$

$$= \frac{1}{3}\,\vec{\zeta} \cdot J\,T\,(R^{-1}\,\vec{\zeta})\,(R^{-1}\,\vec{\zeta})\,, \tag{2.12a}$$

or, in component form,

$$f_3 = \frac{1}{3}\,\vec{\zeta}\,J_{ia}\,T_{ajk}\,R^{-1}_{j\ell}\,R^{-1}_{km}\,\zeta_m\,\zeta_\ell\,\zeta_m \tag{2.12b}$$

Equation (2.11) seems to indicate that given a symplectic matrix R, we can always find a polynomial f_3 reproducing the effect of T. We know that this is only true if R and T are the first and second order contribution to the Taylor series of a symplectic map. Where is the catch? It resides in the assumption that in (2.10) "∇f_3" is an exact derivative.

If we require ∇f_3 to be an exact derivative, we will obtain a condition identical to the symplectic condition. Proof:

$$\frac{\partial^2 f_3}{\partial \zeta_a \partial \zeta_b} - \frac{\partial^2 f_3}{\partial \zeta_b \partial \zeta_a} = 0$$

$$(J_{ai}\, T_{ijk}\, R^{-1}_{j\ell}\, R^{-1}_{kb} - J_{bi}\, T_{ijk}\, R^{-1}_{j\ell}\, R^{-1}_{ka})\, \zeta_\ell = 0$$

but, it must be true for all ζ_ℓ,

$$J_{ai}\, T_{ijk}\, R^{-1}_{j\ell}\, R^{-1}_{kb} - J_{bi}\, T_{ijk}\, R^{-1}_{j\ell}\, R^{-1}_{kb} = 0$$

multiplying by R_{bf} and R_{as}, and summing,

$$J_{ai}\, T_{ijf}\, R_{as} - J_{bi}\, T_{ijs}\, R_{bf} = 0$$

or,

$$J_{ia}\, T_{ifj}\, R_{as} + J_{bi}\, T_{isj}\, R_{bf} = 0. \tag{2.13}$$

We can show that the symplectic condition applied on the Jacobian of $\mathcal{M}$ leads to Eq. (2.13). Notice that in deriving (2.12) we implicitly assumed that R was a symplectic matrix. The real question is whether or not the T_{ijk} matrix is consistent with the symplectic condition given a symplectic R. Let us check it using the Jacobian of $\mathcal{M}$.

$$M_{ij} = \frac{\partial \zeta^f_i}{\partial \zeta_j} = R_{ij} + 2\, T_{ijk}\, \zeta_k \ . \tag{2.14}$$

We substitute (2.14) in the symplectic condition:

$$\tilde{M}\, J\, M = J$$

or,

$$M_{ji}\, J_{jm}\, M_{ms} = J_{is}$$

$$(R_{ji} + 2T_{jik}\,\zeta_k)\, J_{jm}\, (R_{ms} + 2T_{msn}\,\zeta_n) = J_{is} \ . \tag{2.15}$$

Equation (2.15) is complete to first order in $\vec{\zeta}$. Retaining the zeroth and first order terms produces two equations

zeroth order $$R_{ji}\, J_{jm}\, R_{ms} = J_{is} \tag{2.16a}$$

first order $$R_{ji}\, T_{jm}\, T_{msn}\, \zeta_n + T_{jik}\, \zeta_k\, J_{jm}\, R_{ms} = 0 \tag{2.16b}$$

By assumption Eq. (2.16a) is satisfied. On the other hand, (2.16b) is a new condition involving T_{ijk}. Notice that (2.16) is a first degree polynomial in $\vec{\zeta}$ which is zero for all $\vec{\zeta}$. This implies that the coefficients of each ζ_b must vanish separately.

$$R_{ji}\, T_{jm}\, T_{msb} + T_{jib}\, J_{jm}\, R_{ms} = 0 \ . \tag{2.17}$$

If we play with the dummy indices of (2.17) we can turn it into (2.13). We apply the following change of nomenclature on (2.17)

First Term	Second Term
$j \to b$	$j \to i$
$m \to i$	$m \to a$
$i \to f$	$i \to f$
$s \to s$	$s \to s$
$b \to j$	$b \to j$

The transformation on (j,m) is allowed to be different in both terms, since j and m are summed over. The three remaining indices must transform identically. One can check that (2.17) looks exactly like (2.13) once we perform the renaming operation.

As expected, the uniqueness of f_3 is equivalent to the symplectic condition. Or, we can say that the path chosen in (2.11) is indeed irrelevant if the symplectic condition is satisfied. Indeed, if we produce a polynomial f_3 from a symplectic matrix R and some second

order matrix T, and then reverse the process to produce back the T matrix, this T matrix will agree with the original one if, and only if, the original T obeys (2.13).

Conclusion

We hope that the reader is now convinced of the mathematical equivalence of various ways to write a Taylor expanded map. The choice of representation used inside the tracking code is irrelevant since one can transform freely between representations.

At the end, however, a symplectic (Lie generator) representation is ideal for the perturbation theory of a circular ring. We will glance at this in the next chapter which is devoted to the concept of a Floquet ring.

Chapter 4 – Floquet Ring and Phase Advance

1. Old Stuff Revisited

Traditionally, the use of Hamiltonians (or Newton's equations) have played a central role in single particle computations. Indeed, given a set of canonical coordinates $\vec{z} = (q_1, p_1, \ldots, q_n, p_n)$, the motion of the particle is governed by some Hamiltonian H

$$\frac{d\vec{z}}{ds} = [\vec{z}, H(\vec{z}; s)] \Bigg|_{\vec{z} = \vec{z}(s)} \tag{1.1}$$

Here, s is some time-like variable which parametrizes the motion of the ray $\vec{z}(s)$.

In accelerator studies, one is often interested in perturbing H at selected locations by the introduction of multipole errors or rf cavities. To a great accuracy, these are localized perturbations (or kicks), hence the exact knowledge of H in between two of these "kicks" is totally irrelevant. The only things that matter are the maps between these kicks.

In the context of tracking codes, we have already expressed our preference for canonical integrators which use kicks or otherwise exact analytical expressions for the various parts of the ring. Hence, it is only natural to adopt the perturbation theory to the concept of canonical integrators. Central to a Hamiltonian-free perturbation theory is the concept of the Floquet ring and its associated phase advances.

Why is a Hamiltonian free formulation ideal for canonical integrators?

We have seen how it is possible in a routine manner to extract a well defined approximation of the map between any two points along the true closed orbit.

The localized nature of potential perturbations and the simplicity of map extraction lead us to consider a map oriented perturbation theory and to abandon, in the context of tracking codes, the continuous description favored by Hamiltonians.

In the case of linear systems with mid-plane symmetry, the previous statements are familiar to most readers. For example, one is used to statements such as "two sextupoles π–apart cancel their geometric effect." In this chapter we will show that this concept is

generalizable to non-linear systems. In particular, we will show how the phase advance between two points is intimately connected to the chosen canonical transformation which puts the map into a normal form depending only on the action variables. In other words, the formula

$$\Delta\varphi = \int_{s_1}^{s_2} \frac{1}{\beta}\, ds \tag{1.2}$$

is correct in systems with midplane symmetry because one goes to a normalized set of coordinates, or in other words, to a Floquet ring by the transformation

$$x_N = \frac{x}{\sqrt{\beta}} \tag{1.3a}$$

$$p_N = \sqrt{\beta}\, p + \frac{\alpha x}{\sqrt{\beta}} \ . \tag{1.3b}$$

If one were to use the following transformation instead

$$x_N = \sqrt{\gamma}\, x + \frac{\alpha}{\sqrt{\gamma}}\, p \ . \tag{1.4a}$$

$$p_N = \frac{p}{\sqrt{\gamma}} \tag{1.4b}$$

then the phase advance would be given by

$$\Delta\varphi = \int_{s_1}^{s_2} \frac{K}{\gamma}\, ds \ . \tag{1.5}$$

Here K(s) is the focusing function of the lattice

$$\left(H = \frac{p^2 + K\, x^2}{2} \right) \ .$$

and γ is the usual Twiss parameter. Hence the phase advances are linked to the canonical transformations (1.3) or (1.4).

Between "matched" sections of a ring, such as cells, the definition of (1.2) and (1.5) can only differ by factors of 2π. It is easy to prove directly using Eq. (1.5) and the well-known formulas for the derivatives of α, β, and γ.

$$\begin{aligned}
\Delta\varphi &= \int_{s_1}^{s_2} \frac{K}{\gamma} ds \\
&= \int_{s_1}^{s_2} \frac{K\beta}{\beta\gamma} ds \\
&= \int_{s_1}^{s_2} \frac{1}{\beta} ds + \int_{s_1}^{s_2} \frac{\alpha'}{1 + \alpha^2} ds \\
&= \int_{s_2}^{s_2} \frac{1}{\beta} ds + \tan^{-1}\alpha \Big|_{s_1}^{s_2} \qquad \text{Q.E.D.} \qquad (1.6)
\end{aligned}$$

If the lattice is matched at s_1 and s_2, it follows that $\alpha(s_1) = \alpha(s_2)$ and the two phase advances are equal modulo an integer. The canonical transformations are themselves motivated by the kind of perturbations one applies to the ring; for multipole kicks the usual transformation (1.3) is easier to use. In this chapter, we will formulate these concepts in a map formalism, abandoning the Hamiltonian approach.

We will show how the assumption of integrability leads to a non-linear phase advance. Finally, we give a flow chart of the actual implementation of a phase advance calculation suited for a canonical integrator code.

2. Non-linear Floquet Variables and the Associated Phase Advances

Let us assume that we have a map $\mathcal{M}_s$ representing one turn around a circular machine from s back to s. We will assume that $\mathcal{M}_s$ is integrable, that is:

There exist $\mathcal{R}$ unique and $\mathcal{A}_s$ such that $\mathcal{A}_s \mathcal{M}_s \mathcal{A}_s^{-1} = \mathcal{R}$ for all positions s around the ring, and $\mathcal{R}$ is given by the formulae:

$$\mathcal{R} = \exp(: -\vec{\mu} \cdot \vec{J}_s - F(\vec{J}_s) :) \tag{2.1a}$$

$$F(\vec{J}_s) = \text{Detuning term} \tag{2.1b}$$

$$q_i = \sqrt{2J_i}\cos\phi_i \tag{2.1c}$$

$$p_i = -\sqrt{2J_i}\sin\phi_i \tag{2.1d}$$

Using $\mathcal{A}_s$, one can define a new set of action-angle variables:

$$\vec{\psi}_s = \mathcal{A}_s^{-1}\ \vec{\phi}_s \tag{2.2a}$$

$$\vec{I}_s = \mathcal{A}_s^{-1}\ \vec{J}_s \tag{2.2b}$$

$$\mathcal{M}_s = \exp(: -\mu \cdot \vec{I}_s - F(\vec{I}_s) :) \tag{2.2c}$$

The invariants $\vec{I}_s$ have a different functional expression in terms of $\vec{q}_s$ and $\vec{p}_s$ at different s. (Think of the dependence of the Twiss parameter α, β and γ on s.) However, if evaluated for a given particle along its trajectory, $\vec{I}_s$ remains a constant.

As seen in Eq. (2.2a), the map $\mathcal{M}_s$ has a unique functional dependence on $\vec{I}_s$, independent of the location s.

Before going further, we present here a few exercises to clarify some of the concept needed in this discussion.

Interlude on the Uniqueness of $\mathcal{R}$

Exercise 1

Under what conditions is $\mathcal{R}$ unique? In other words, what are the properties of $\mathcal{A}$ which guarantee that $\mathcal{R}$ is unique independent of the choice of $\mathcal{A}$, and independent of the point of observation.

Answer

The map $\mathcal{A}$ must produce an analytic transformation around the origin, and then $\mathcal{R}$ is unique within a permutation of the eigen-planes.

Exercise 2

Modify the statements of lines (2.1) and the results of Exercise 1, to include the case of no synchrotron oscillations. (coasting beam)

Answer

There exists $\mathcal{R}$ unique and $\mathcal{A}_s$ such that $\mathcal{A}_s \mathcal{M}_s \mathcal{A}_s^{-1} = \mathcal{R}$

$$\mathcal{R} = \exp(: -\vec{\mu} \bullet \vec{J}_s - \frac{\tilde{\alpha}}{2} p_\tau^2 + F(\vec{J}_s, p_\tau) :) \tag{2.3a}$$

$$\tilde{\alpha} = \text{Momentum compaction} \tag{2.3b}$$

$$p_\tau = \frac{-\delta E}{p_0 c} \tag{2.3c}$$

$$\vec{J}_s = (J_{s,1}, J_{s,2}) \Rightarrow \text{Transverse space} \quad . \tag{2.3d}$$

The uniqueness of $\mathcal{R}$ is insured by the analyticity of $\mathcal{A}$ and the time (τ) independence of its generator.

Exercise 3

Define the action angle variables of $\mathcal{R}$.

Answer

$$\begin{cases} q_i = \sqrt{2J_i}\ \cos(\phi_i)\ , \ i = 1,3 \ \text{(synchrotron oscillations)} \\ p_i = \sqrt{2J_i}\ \sin(\phi_i)\ , \ i = 1,2 \ \text{(coasting beam)} \ . \end{cases} \tag{2.4a}$$

$$\begin{cases} J_3 = p_\tau\ , & \text{for coasting beam} \\ \phi_3 = \tau \end{cases} \tag{2.4b}$$

Exercise 4

Starting with a map of the form $\exp(:\Gamma(\vec{J}):)$, modify the function $\Gamma(\vec{J})$ through an illegal transformation as defined in the previous exercises.

<u>Possible Answer</u>

For example, consider a phase advance in the x–y space

$$\mathcal{R} = \exp(: -\vec{\mu} \cdot \vec{J} :) \ ; \ \Gamma(\vec{J}) = \vec{\mu} \cdot \vec{J} = \mu_x J_x + \mu_y J_y \tag{2.5}$$

Now, consider the transformation $\mathcal{A} = \exp(: \gamma \phi_y J_x :)$. We can easily evaluate the operator $\mathcal{A}$:

$$\begin{aligned} \mathcal{A}\, \phi_x &= \phi_x - \gamma \phi_y \\ \mathcal{A}\, J_x &= J_x \\ \mathcal{A}\, \phi_y &= \phi_y \\ \mathcal{A}\, J_y &= J_y + \gamma J_x \end{aligned} \tag{2.6}$$

Finally, we compute the transformed operator $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} = \mathcal{A}\, \mathcal{M}\, \mathcal{A}^{-1} &= \exp(:\Gamma(\mathcal{A}\vec{J}):) \\ &= \exp(: -(\mu_x + \gamma\mu_y)\, J_x - \mu_y\, J_y :) \ . \end{aligned} \tag{2.7}$$

The form of Γ has been altered. However, it is quite clear that the expressions for ϕ_x and ϕ_y are not analytic around the origin $(\vec{q}, \vec{p}) = 0$.

End of the Interlude

Now that we have a clearer idea of the origin of the unique nature of $\mathcal{R}$, let us go back to the discussion leading to the phase advance. Consider two positions, a and b, on a closed orbit and their one turn maps $\mathcal{M}_a$ and $\mathcal{M}_b$ as well as $\mathcal{N}_{ab}$ the map between a and b.

The phase advance $\vec{\varphi}_{ab}$ between s = a and s = b will have its definition rooted in the choice of $\mathcal{A}_a$ and $\mathcal{A}_b$. To prove this, we explore the relationship between $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{N}_{ab}$. (From now on all the operators act on the phase space at s = a unless explicitly stated otherwise.) From the definition of $\mathcal{N}_{ab}$, we get:

$$\mathcal{M}_a = \mathcal{N}_{ab}\,\mathcal{M}_b\,\mathcal{N}_{ab}^{-1} \ . \tag{2.8}$$

Using the integrability condition, we replace $\mathcal{M}_b$ by its expression in terms of $\mathcal{R}$:

$$\mathcal{M}_a = \mathcal{N}_{ab}\,\mathcal{A}_b^{-1}\,\mathcal{R}\,\mathcal{A}_b\,\mathcal{N}_{ab} \ . \tag{2.9}$$

Finally, we replace $\mathcal{R}$ by its expression in terms of $\mathcal{M}_a$:

$$\begin{aligned}
\mathcal{M}_a &= \mathcal{N}_{ab}\,\mathcal{A}_b^{-1}\,\mathcal{A}_a\,\mathcal{M}_a\,\mathcal{A}_a^{-1}\,\mathcal{A}_b\,\mathcal{N}_{ab} \\
&= \mathcal{B}\,\mathcal{M}_a\,\mathcal{B}^{-1} \\
&= \mathcal{B}\,\exp(:-\vec{\mu}\cdot\vec{I}_a - F(\vec{I}_a):)\,\mathcal{B}^{-1} \\
&= \exp(:-\vec{\mu}\cdot(\mathcal{B}\vec{I}_a) - F(\mathcal{B}\vec{I}_a):)
\end{aligned} \tag{2.10}$$

From Eq. (2.10), we conclude that $\mathcal{B}$ must have $\vec{I}_a$ as its invariant, hence $\mathcal{B}$ can be written as:

$$\mathcal{B} = (:\Gamma_{ab}(\vec{I}_a):) \tag{2.11a}$$

$$\vec{\varphi}_{ab} = \frac{-\partial\Gamma_{ab}}{\partial\vec{I}_a} \tag{2.11b}$$

The quantity $\vec{\varphi}_{ab}$ is the phase advance between the points a and b. In fact, this is easy to show by rewriting $\mathcal{N}_{ab}$ in terms of $\mathcal{B}$. Using Eq. (2.10):

$$\begin{aligned}
\mathcal{N}_{ab} &= \mathcal{B}(\vec{I}_a)\,\mathcal{A}_a^{-1}\,\mathcal{A}_b \\
&= \mathcal{A}_a^{-1}\,\mathcal{A}_a\,\mathcal{B}(\vec{I}_a)\,\mathcal{A}_a^{-1}\,\mathcal{A}_b
\end{aligned}$$

$$= \mathcal{A}_a^{-1}\, \mathcal{B}(\mathcal{A}_a \vec{I}_a)\, \mathcal{A}_b$$

$$= \mathcal{A}_a^{-1}\, \mathcal{B}(\vec{J}_a)\, \mathcal{A}_b \tag{2.12}$$

The interpretation of (2.12) is simple. Remembering that the operators act on rays in the reverse order from ordinary matrix multiplication, we conclude that transporting a ray from a to b can be seen as extracting the ray at a with $\mathcal{A}_a^{-1}$, phase advancing it with $\mathcal{B}$ in a Floquet ring and re-inserting the ray back in real space with $\mathcal{A}_b$. In the Floquet ring, the ray is just rotated by $\vec{\varphi}_{ab}$.

Again, we need to emphasize that $\vec{\varphi}_{ab}$ is not unique. The Floquet ring depends on the insertion map $\mathcal{A}_s$.. Of all the possible $\mathcal{A}_s$ which are available to define our Floquet ring, we will restrict them to internally defined $\mathcal{A}_s$. That means that our rule for selecting $\mathcal{A}_s$ depends only on $\mathcal{M}_s$.[12] Automatically, this insures that matched locations ($\mathcal{M}_s = \mathcal{M}_{s'}$) have a phase advance $\vec{\varphi}_{ss'}$, independent of the choice of $\mathcal{A}_s$ and $\mathcal{A}_{s'}$.

We conclude this chapter with a flow chart implementation of our ideas into the ideal tracking code. We also suggest a possible definition of the linear phase advance.

Find the closed orbit

$\vec{z}_0$ and the map $\mathcal{M}_0$

$\downarrow$

Compute a possible $\mathcal{A}_0^t$ ("t" for tentative)

using $\mathcal{M}_0$, with a definite ordering of the eigen-planes.

$\downarrow$

$\downarrow$

From $i = 1$, $i_{total} = 0$

$\downarrow$

Get $\mathcal{N}_{i-1\ i}$

$\downarrow$

Compute $\mathcal{A}_i^t = \mathcal{A}_{i-1}\, \mathcal{N}_{i-1\ i}$

$\downarrow$

Modify $\mathcal{A}_i^t$ to satisfy the particular conditions imposed

on $\mathcal{A}_i$. (For example Teng-Edwards; see the ensuing discussion.)

$\downarrow$

Compute $\mathcal{B}_{i-1\ i} = \mathcal{A}_{i-1}\, \mathcal{N}_{i-1\ i}\, \mathcal{A}_i^{-1}$ and extract the phase $\vec{\varphi}_{i-1\ i}$

$\downarrow$

exit when $i = i_{total} = 0$

The flowchart can be applied to a linear or non-linear map. Routinely, however, tracking codes are expected to provide a linear phase advance. In the linear problem, all the maps have a matrix representation. Let us impose a sensible constraint on the matrix A(s) of the operator $\mathcal{A}_s$. These will be referred to as the Teng-Edwards conditions. They are:

$$A_{2i-1\ 2i-1} > 0 \tag{2.13a}$$

$$A_{2i-1\ 2i} = 0 \qquad i = 1,n \quad . \tag{2.13b}$$

These conditions specify A(s) up to permutations of the invariant subspaces of the matrix R. Once we assign each tune to a definite block of R, the conditions (2.13) will guarantee the uniqueness of A(s). In fact, for this choice, the following is true:

suppose that i) the motion is generated by

$$H(\vec{z}; s) = \frac{1}{2}\sum S_{ij}\, z_i\, z_j \tag{2.14a}$$

and ii) (2.13) is satisfied

then, it can be shown that:

$$\frac{d\varphi_k}{ds} = \sum_i \frac{S_{2k\,i}\, A_{i\,2k}}{A_{2k-1\,2k-1}};\ k^{th}\ \text{plane}\ . \tag{2.14b}$$

Although Eq. (2.14b) is never needed in a canonical integrator code, it allows us to connect our treatment to the more standard approaches.

Why is the Teng-Edwards choice for A(s) sensible? First, it is worth noticing that in the case of midplane symmetry, it reduces to the usual Courant-Snyder choice. Then why is the Courant-Snyder definition desirable? This can be answered by looking at the eventual perturbations of an ideal linear ring. Most perturbative fields, can be represented by a vector potential $\tilde{V}_s$ (x, y) as we have seen in the chapter on canonical integrators. Hence, we would like a definition of A(s) which affects x and y as little as possible. Clearly, this Courant-Snyder choice is best since it only scales the transverse positions. In fact, in this Floquet ring, the effective vector potential is given by $\mathcal{A}_s\, \tilde{V}(x,y)$ $(= \tilde{V}_s(\sqrt{\beta_x}\ x,\ \sqrt{\beta_y}\ y)$.

Conclusion

We have shown how the concept of phase advance is intimately linked to the idea of a Floquet ring. Furthermore, we have seen how these ideas are generalizable to non-linear systems. Indeed, in a circular machine it is possible to construct a perturbation theory based on maps alone. Such a peturbation theory is limited only by the order of the map concatenators available to the codes. At this moment, we have successfully implemented a

5th order (sixth-order in the Lie generator) in one of the programs used at SSC.[13] (In collaboration with Beat Leemann and Filippo Neri.)[14,15]

Finally, we would like to point out that the ideas of Floquet ring and of perturbation theory on maps can be extended to spin systems as demonstrated by Yokoya.[16]

Acknowledgements

I would like to express my gratitude to Beat Leemann, Filippo Neri and Alex Dragt, who have helped me put the ideas of this paper into software. I thank Lindsay Schachinger, Richard Talman, and Albin Wrulich for giving me a copy of their tracking codes (TEAPOT and RACETRACK) on which I implemented my ideas. Also, I thank Alex Chao and Melvin Month for their support. Finally, I am grateful to Donna Matthews for preparing so professionally the final form of this paper.

References

1. F. Schmidt: a) DESY-HERA 86-10, "Simulationsrechnungen für DESY III"

   b) PhD Thesis (to be published)

   c) "Erweiterung des Racetrack Codes auf eine 6-dimensionalen symplektischen Formalismus," to be published.

2. R. D. Ruth, "Single-Particle Dynamics in Circular Accelerators," AIP Conference Proceedings 153, Vol.1, p. 166.

3. A. J. Dragt, "Exact Numerical Calculation of Chromaticity in Small Rings," Particle Accelerators 12, 1980, p. 205.

4. L. Schachinger and R. Talman, "TEAPOT, A Thin Element Program for Optics and Tracking," to be published in Particle Accelerators, see Section 7.

5. A. J. Dragt and J. M. Finn, "Lie Series and Invariants for Analytic Symplectic Maps," Journal of Mathematical Physics 17, 2215 (1976).

6. A. J. Dragt, AIP Conference Proceedings 1987, P. 147 (1981).

7. R. D. Ruth, "A Canonical Integration Technique," IEEE Trans. Nuclear Science, NS-30, p. 2669 (1983).

8. F. Neri, "Lie Algebras and Canonical Integration," unpublished, University of Maryland, (1987).

9. A. J. Dragt and E. Forest, "Computation of Non-linear Behavior of Hamiltonian Systems using Lie Algebraic Methods," Journal of Mathematical Physics 24 (12), p. 2734 (1983).

10. Beat Leemann and the author have implemented the two versions of K into two codes which they call respectively THINTRAC and FASTRAC.

11. E. Forest, "Lie Algebraic Maps and Invariants Produced by Tracking Codes," to be published in Particle Accelerators.

12. A. J. Dragt, "Non-linear Lattice Functions," Proceedings of the 1984 Study on the Design and Utilization of the Superconducting Super Collider, p. 369.

This paper was the first to point out the existence of non-linear lattice functions. However, the author did not recognize the importance of precisely defining $\mathcal{A}_s$. Furthermore, his choice for $\mathcal{A}_s$ would not be internal since it depended on the design trajectory path length.

13. One has to acknowledge the work of Filippo Neri in providing extremely reliable tools for the analysis of 5th order maps. (In the context of the code MARYLIE)

14. E. Forest, "Hamiltonian-Free Perturbation Theory: The Concept of Phase Advance," SSC-111, 1987. (A paper in preparation.)

15. See also Reference 11.

16. K. Yokoya, "Calculation of the Equilibrium Polarization of Stored Electron Beams Using Lie Algebras," KEK Preprint 86-90, 1986.