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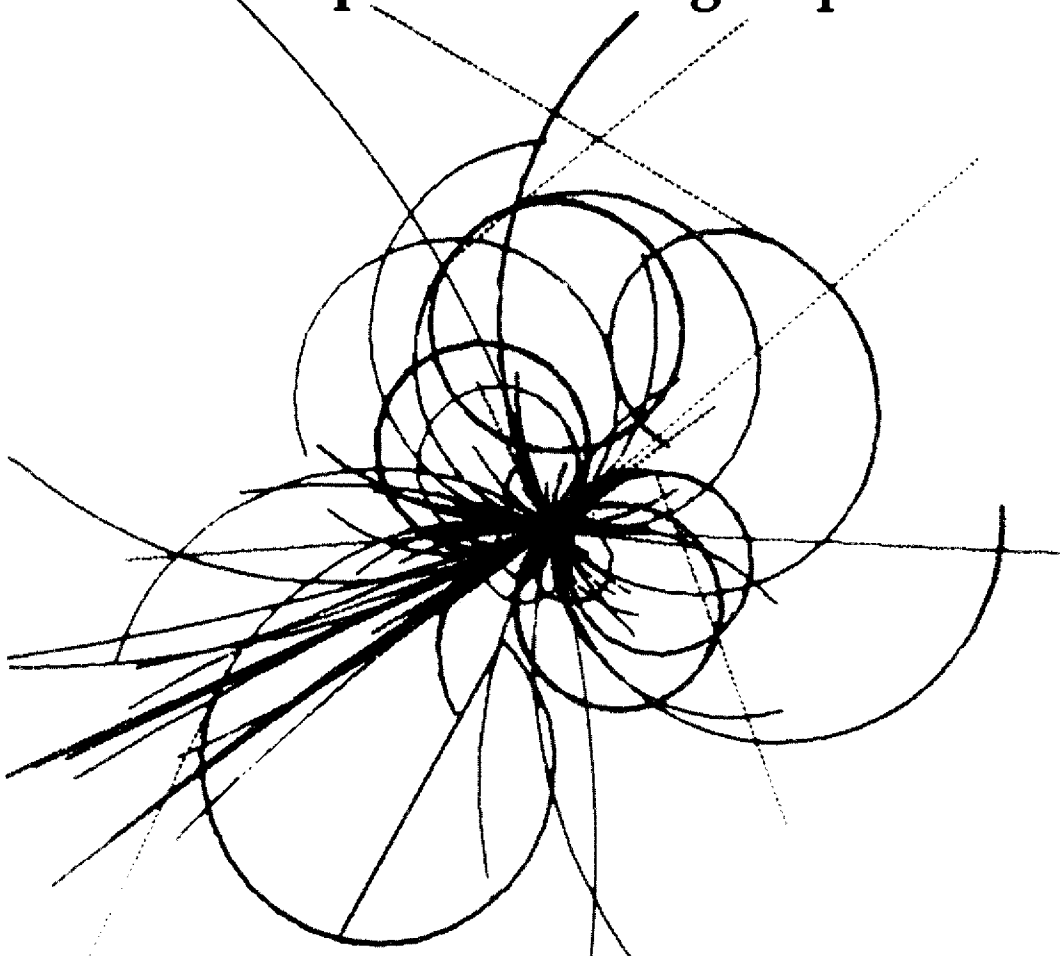

Hamiltonian-Free Perturbation Theory: The Concept of Phase Advance

Etienne Forest

SSC Central Design Group

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HAMILTONIAN-FREE PERTURBATION THEORY: THE CONCEPT OF PHASE ADVANCE

Etienne Forest

SSC Central Design Group*
c/o Lawrence Berkeley Laboratory
Berkeley, California 94720

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ABSTRACT

In this paper, we generalize the concept of phase aavance to non-linear systems. We show how non-linear perturbation theory can be made in a totally Hamiltonian free context. As examples, we give expressions for the computation of first and second order tune shifts, as well as for the canonical transformation responsibile for the first order distorsions of the invariants. Our reliance on Lie operators allows for a very simple implementation of these methods in the context of an algebraic manipulator.

Introduction

Traditionally, the use of Hamiltonians (or Newton's equations) have played a central role in single particle computations. Indeed, given a set of canonical coordinates $\vec{z} = (q_1, p_1, \ldots, q_n, p_n)$, the motion of the particle is governed by some Hamiltonian H

$$\frac{d\vec{z}}{d\tau} = [\vec{z}(\tau), H(\vec{z}(\tau); \tau)]. \tag{1.1}$$

Here, τ is some time-like variable which parametrizes the motion of the ray $z(\tau)$.

In accelerator studies, one is often interested in perturbing H at select locations by the introduction of multipole errors or rf cavities. To a great accuracy, these are localized perturbations (or kicks), hence the exact knowledge of H inbetween two of these "kicks" is totally irrelevant. The only things that matters are the maps between these kicks.

In the case of linear systems with mid-plane symmetry, the previous statements are familiar to most readers. For example, one is used to statements such as "two sextupoles π-apart cancel their geometric effect." In this paper we will show that this concept is generalizable to non-linear systems. In particular, we will show how the phase advance between two points is intimately connected to the chosen canonical transformation which puts the map into a normal form depending only on the action variables. In other words the formula

$$\Delta\varphi = \int_{\tau_1}^{\tau_2} \frac{1}{\beta}\, d\tau \tag{1.2}$$

is correct in systems with midplane symmetry because one goes to a normalized set of coordinates using

$$x_N = x/\sqrt{\beta} \tag{1.3a}$$

$$p_N = \sqrt{\beta}\ \ p + \frac{\alpha}{\sqrt{\beta}}\ \ x\ . \tag{1.3b}$$

If one were to use the following tranformation instead

$$x_N = \sqrt{\gamma}\ \ x + \frac{\alpha}{\sqrt{\gamma}}\ p \tag{1.4a}$$

$$p_N = p/\sqrt{\gamma} \tag{1.4b}$$

then the phase advance would be given by

$$\Delta\varphi = \int_{\tau_1}^{\tau_2} \frac{K}{\gamma}\, d\tau\ . \tag{1.5}$$

Here $K(\tau)$ is the focussing function of the lattice

$$(H = \frac{p^2 + K\ x^2}{2}),$$

and γ is the usual Twiss Parameter.

Between "matched" sections of a ring, such as cells, the definition of (1.2) and (1.5) can only differ by factors of 2π. It is easy to prove directly using Eq. (1.5) and the well known formulas for the derivatives of α, β, and γ

$$\Delta\varphi = \int_{\tau_1}^{\tau_2} \frac{K}{\gamma}\, d\tau$$

$$= \int_{\tau_1}^{\tau_2} \frac{K\beta}{\beta\gamma}\, d\tau$$

$$= \int_{\tau_1}^{\tau_2} \frac{1}{\beta}\, d\tau + \int_{\tau_1}^{\tau_2} \frac{\alpha'}{1+\alpha^2}\, d\tau$$

$$= \int_{\tau_2}^{\tau_2} \frac{1}{\beta}\, d\tau + \tan^{-1}\alpha \Big|_{\tau_1}^{\tau_2} \quad \text{Q.E.D.} \tag{1.6}$$

Hence, the phase advances are linked to the canonical transformations (1.3) or (1.4). The canonical transformations are themselves motivated by the kind of perturbations one applies to the ring; for multipole kicks the usual transformation (1.3) is easier to use. In this paper, we will formulate these concepts in a map formalism, abandoning the Hamiltonian completely.

In section 2, we show how the assumption of integrability leads to a non-linear phase advance. In section 3, we propose a definition of the canonical transformation for systems without midplane symmetry; our definition matches Edwards-Teng's definition as demonstrated in Appendix I. In section 3, we commit a small sin in writing differential equations for all our operators in flagrant violation of the spirit of the author. This is done to facilitate the connection to the previous work of Courant-Snyder[1] and Edwards-Teng.[2]

Finally, in section 4, we show how one can compute some quantities such as tune shifts and distortions arising from a perturbation of a non-linear ring. This is done in a totally Hamiltonian-free language using the concepts of section 2.

Before going further, we would like to summarize some properties of maps and some notations.[3]

$$[f,g] = \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial g}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial g}{\partial \vec{q}} \tag{1.7}$$

$$:f:g = [f,g] \tag{1.8}$$

$$\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} \tag{1.9a}$$

$$:f:^0 = \mathscr{I} = \text{identity map.} \tag{1.9b}$$

$$\exp(:f:)\ \exp(:g:)\ \exp(:-f:) = \exp(:\exp(:f:)g:) \tag{1.10}$$

$$\exp(:f:)\ g(\vec{z}) = g(\exp(:f:)\vec{z}) \tag{1.11}$$

$$\vec{z} = (q_1,\ p_1,\ q_2,\ p_2,\ \ldots,\ q_n,\ p_n) \tag{1.12}$$

Script letters ($\mathscr{M}$) represent mappings while the usual capitals (M) will denote the matrix representation of $\mathscr{M}$ if $\mathscr{M}$ happens to be a linear map.

$$\mathscr{M}\, z_i = \sum_{j=1}^{n} M_{ij} z_j = M_{ij} z_j \tag{1.13}$$

Finally, Lie operators are concatenated in the reverse order from matrix multiplication. This is proven below using some of the properties listed above. Consider a two step process

$$\vec{x}_0 \xrightarrow{\mathscr{N}} \vec{\xi} \xrightarrow{\mathscr{L}} \vec{z} \tag{1.14a}$$

$$\mathscr{N}(\vec{z}_0) = \exp(:f(\vec{z}_0):) \tag{1.14b}$$

$$\mathscr{L}(\vec{\xi}) = \exp(:g(\vec{\xi}):) \tag{1.14c}$$

So, we must have

$$\begin{aligned}
\vec{z} &= \mathscr{L}(\vec{\xi})\ \vec{\xi} \\
&= \mathscr{L}(\vec{\xi})\ \mathscr{N}(\vec{z}_0)\ \vec{z}_0 \\
&= \mathscr{L}(\ \mathscr{N}(\vec{z}_0)\ \vec{z}_0)\quad \mathscr{N}(\vec{z}_0)\ \vec{z}_0 \\
&= \mathscr{N}(\vec{z}_0)\ \mathscr{L}(\vec{z}_0)\quad \mathscr{N}(\vec{z}_0)^{-1}\ \mathscr{N}(z_0)\ \vec{z}_0 \qquad , \\
&= \mathscr{N}(\vec{z}_0)\ \mathscr{L}(\vec{z}_0)\ \vec{z}_0 \ \text{Q.E.D.}
\end{aligned} \tag{1.15}$$

Now, we are ready for the generalization of some familiar concepts.

2. Non-Linear Floquet Variables and the Associated Phase Advances

Consider two positions, a and b, on a closed orbit as shown in Fig. 1. Let us write the mappings around the ring as $\mathscr{M}_a$ and $\mathscr{M}_b$. Also, let us assume that the mapping from a to b is $\mathscr{N}_{ab}$. Finally, we assume that the motion is integrable, that is

$$\exists\ \mathscr{R} \text{ unique } \exists\ \mathscr{A}_K \text{ such that } \mathscr{A}_K\ \mathscr{M}_K\ \mathscr{A}_K^{-1} = \mathscr{R}\ ,\ K = a,b \tag{2.1a}$$

$$\mathscr{R} = \exp(:-\vec{\mu}\cdot\vec{J} - F(\vec{J}):) \tag{2.1b}$$

$$F(\vec{J}) = \text{detuning term} \tag{2.1c}$$

$$J_i = \frac{q_i^2 + p_i^2}{2} \qquad i = 1,\ n\ . \tag{2.1d}$$

The invariants of the motion I_i are simply given by

$$I_{K,i} = \mathscr{A}_K^{-1} J_i \qquad i = 1,\ n;\ K = a,b, \tag{1.2a}$$

$$\mathscr{M}_K = \exp(:-\vec{\mu}\cdot\vec{I} - F(\vec{I}):)\ . \tag{2.2b}$$

The phase advance $\vec{\phi}$ between a and b will have its definition rooted in the choice of $\mathscr{A}_a$ and $\mathscr{A}_b$. To prove this, we explore the relationship between $\mathscr{A}_a$, $\mathscr{A}_b$ and, $\mathscr{N}_{ab}$.

Using Eq. (2.1a), we solve for $\mathscr{M}_b$ in terms of $\mathscr{R}$ and $\mathscr{A}_b$.

$$\mathscr{M}_b = \mathscr{A}_b^{-1}\mathscr{R}\ \mathscr{A}_b\ . \tag{2.3}$$

We can substitute for $\mathscr{R}$ its expression in terms of $\mathscr{A}_a$ and $\mathscr{M}_a$

$$\mathscr{M}_b = \mathscr{A}_b^{-1}\ \mathscr{A}_a\ \mathscr{M}_a\ \mathscr{A}_a^{-1}\ \mathscr{A}_b\ . \tag{2.4}$$

However from the definition of $\mathscr{N}_{ab}$, we also have

$$\mathscr{M}_b = \mathscr{N}_{ab}^{-1}\mathscr{M}_a\ \mathscr{N}_{ab}\ . \tag{2.5}$$

From Eq. (2.4) and (2.5), we derive the identity

$$\mathscr{M}_a = \mathscr{B} \quad \mathscr{M}_a \; \mathscr{B}^{-1} \tag{2.6a}$$

$$\mathscr{B} = \quad \mathscr{N}_{ab} \; \mathscr{A}_b^{-1} \quad \mathscr{A}_a \, . \tag{2.6b}$$

Exploiting (2.6a), we rewrite it as

$$\mathscr{M}_a = \quad \mathscr{B} \; \exp(:-\vec{\mu}\cdot\vec{I} - F(\vec{I}):) \, \mathscr{B}^{-1}$$

$$= \exp(:-\vec{\mu}\cdot(\mathscr{B}\,\vec{I}) - F(\mathscr{B}\,\vec{I}):) \; . \tag{2.7}$$

Comparing Eq. (2.7) and (2.2b), we conclude that

$$\mathscr{B}\,\vec{I} = \vec{I} \quad . \tag{2.8}$$

From this, we can write down the most general form for $\mathscr{B}$

$$\mathscr{B}(\vec{I}) = \exp(:\Gamma(\vec{I}):) \tag{2.9a}$$

$$\vec{\varphi}(\vec{I}) = -\frac{d\Gamma}{d\vec{I}} \; . \tag{2.9b}$$

The quantity $\vec{\phi}(\vec{I})$ is the non-linear phase advance between the points a and b. In fact, this is easy to show by rewriting $\mathscr{N}_{ab}$ in terms of $\mathscr{B}$. Using (2.6b) and (2.2a), we must have

$$\mathscr{N}_{ab} = \mathscr{B}(\vec{I}) \quad \mathscr{A}_a^{-1} \quad \mathscr{A}_b$$

$$= \mathscr{A}_a^{-1} \, \mathscr{A}_a \quad \mathscr{B} \quad \mathscr{A}_a^{-1} \quad \mathscr{A}_b$$

$$= \mathscr{A}_a^{-1} \; \exp(:\Gamma(\mathscr{A}_a \; \vec{I}):) \; \mathscr{A}_b$$

$$\mathscr{N}_{ab} = \mathscr{A}_a^{-1} \mathscr{B}(\vec{J}) \quad \mathscr{A}_b \; . \tag{2.10}$$

The interpretation of Eq. (2.10) is simple and depicted in Fig. 2. Remembering that the operators in (2.10) are in the reverse order from the usual matrix multiplication, we conclude that a ray at location "a" is taken into non-linear Floquet variables by $\mathscr{A}_a^{-1}$, then it is propagated by a simple

rotation of angle $\vec{\varphi}(\vec{J})$ by $\mathscr{B}(J)$and, finally, it is brought back to the ring at location b by $\mathscr{A}_b$.

Again, we need to emphasize that the definition of $\vec{\phi}$ is not unique. The phase advance between two points depends on the insertion and extraction mechanism of $\mathscr{A}_a^{-1}$ and $\mathscr{A}_b$ respectively.[4]

In the next section we illustrate this on the linear case where the concept of phase advance is familiar (Courant-Snyder in 2-dimensions and Edwards-Teng in 4-dimensions). However we would like to emphasize one particular sub-set of possible choices for $\mathscr{A}_K$.

Consider the following statement:

THEOREM: If two transformations $\mathscr{A}_s$ and $\mathscr{A}_s'$ are chosen such that they depend <u>only</u> on the mapping $\mathscr{M}_s$, then the two resulting phase advances will have the same fractional tunes through a matched section.

This was displayed explicitly on the examples (1.3) and (1.4). For that reason, we will always present examples of $\mathscr{A}_s$ which are internally defined. (i.e., depending on $\mathscr{M}_s$ alone).

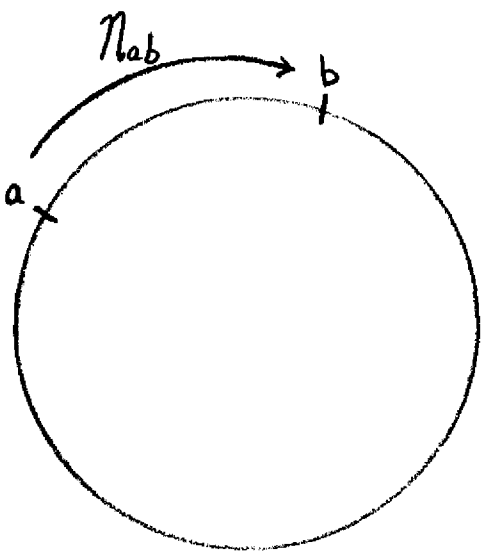

Fig. 1

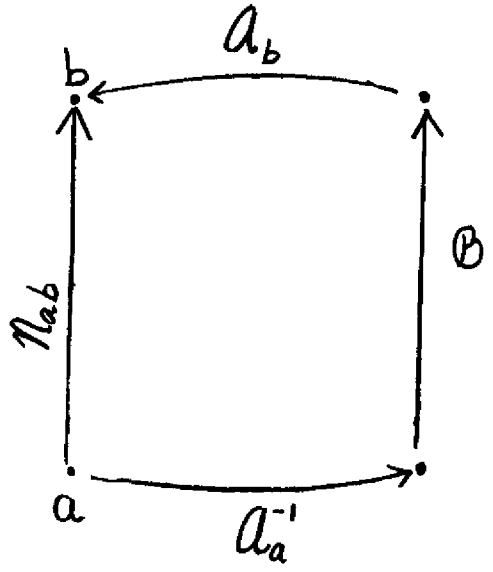

Fig. 2

3. The Phase Advance of Linear System: A Simple Example.

In a linear system all the transformations and operators of section 2 have a unique matrix representation, i.e., given a transformtion $\mathscr{S}\vec{z}$, we have

$$\mathscr{S} z_i = \sum_j T_{ij} z_j \ . \tag{3.1}$$

In particular, the mapping $\mathscr{R}$ of (2.1b) has the form.

$$\mathscr{R} = \exp(:-\vec{\mu}\cdot\vec{J}:) \tag{3.2a}$$

$$R = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{pmatrix} , \ R_i = \begin{pmatrix} \cos\mu_i & \sin\mu_i \\ -\sin\mu_i & \cos\mu_i \end{pmatrix}. \tag{3.2b}$$

For the linear system we select the operator $\mathscr{A}$ (dropping the subscript referring to the location) in anticipation of future non-linear insertions inside our ring. Most of the time, we introduce kicks which arise from multipoles and rf cavities. They are characterized by a simple Lie operator $\tilde{A}(\vec{q})$. Going across a kick is given by

$$\vec{z}^f = \exp(:\tilde{A}:)\vec{z} \tag{3.3a}$$

or,

$$\vec{q}^f = \vec{q} - \frac{\partial\tilde{A}}{\partial\vec{p}} = \vec{q} \tag{3.3b}$$

$$\vec{p}^f = \vec{p} + \frac{\partial\tilde{A}}{\partial\vec{q}} \ . \tag{3.3c}$$

Considering that we might later investigate the response of our linear system under the perturbative effect of $\tilde{A}(\vec{q})$, we would like to choose a definition of $\mathscr{A}$ which modifies $\tilde{A}(\vec{q})$ as little as possible.

In the case of a system with uncoupled degrees of freedom, this was done by Courant-Snyder. In this case, $\mathscr{A}$ has the form[3]

$$\mathscr{A} = \exp(:-\vec{\mu}\cdot\vec{I}:) \tag{3.4a}$$

$$I_i = \gamma_i q_i^2 + 2\alpha_i q_i p_i + \beta_i p_i^2, \ 1 + \alpha_i^2 = \beta_i\gamma_i \ . \tag{3.4b}$$

The transformation $\mathscr{A}$ which is least destructive to $\vec{q}$ [and as a result to $\tilde{A}(\vec{q})$] is given in each degrees of freedom by the formulas

$$\mathscr{A} q_i = \sqrt{\beta_i}\, q_i \tag{3.5a}$$

$$\mathscr{A} p_i = \frac{1}{\sqrt{\beta_i}} \left(p_i - \frac{\alpha_i}{\sqrt{\beta_i}}\, q_i\right) . \tag{3.5b}$$

With this choice of $\mathscr{A}$, the component q_i is only modified by a scaling of magnitude $\sqrt{\beta_i}$.

As an example of the treatment of fully coupled systems, we can adopt the same philosophy. We chose $\mathscr{A}$ on the basis of the following criteria

$$A_{2i-1\ 2i-1} > 0 \tag{3.6a}$$

$$A_{2i-1\ 2i} = 0,\ i=1,\ n\ . \tag{3.6b}$$

With these rules, the map $\mathscr{A}$ will reduce to the Courant-Snyder transformation if the system is decoupled and in 4-dimension it will reduce to the Edwards-Teng parametrization. Equation (3.6) fixes $\mathscr{A}$ completely and therefore the fractional part of the phase advance between two points is fixed. It should be noted that a possible permutation of the sub-spaces of R leaves some ambiguity. This must be resolved once and for all at the beginning by assigning each tune μ_i to a definite block of R. For example, if the system is mildly coupled, one can rely on the approximate formula $\cos \mu_i \simeq \frac{1}{2} (M_{2i-1\ 2i-1} + M_{2i\ 2i})$ to order the subspaces of R and select $\mathscr{A}$ unambiguously.

In the last part of this section, we derive an expression for $\vec{\phi}$, A, and M using the conditions of Eq. (3.6) and the assumption that M is generated by some quadratic Hamiltonian. This violates our prejudice of avoiding the Hamiltonian and using mappings only. We provide this only to facilitate the connection to previous treatments.

We start with a Hamiltonian H given by

$$H(\vec{z};\tau) = \frac{1}{2} \sum_{ij} S_{ij}(\tau)\, z_i z_j . \tag{3.7}$$

Let us assume that $S(\tau)$ has a periodicity of 1

$$S(\tau+1) = S(\tau) . \tag{3.8}$$

Denoting by $\mathscr{N}_{\tau_0\tau}$ the transformation connecting the ray at τ_0 to the ray at τ it can be proven that $\mathscr{N}$ obeys the equation[5]

$$\frac{d}{d\tau} \mathscr{N}_{\tau_0\tau} = \mathscr{N}_{\tau_0\tau} :-H(\vec{z}(\tau_0);\, \tau)$$

$$\mathscr{N}_{\tau_0\tau_0} = \mathscr{I} = \text{Identity mapping.} \tag{3.9}$$

Integrating Eq. (3.9) from $\tau = \tau_0$ to $\tau = \tau_0 + 1$ will produce the one-turn map $\mathscr{M}_{\tau_0}$. Let us assume that we extracted the map $\mathscr{A}_{\tau_0}$ associated to the prescriptions of Eq. (3.6).

We now move infinitesimally from τ_0 by an amount $\Delta\tau$. Using Eq. (1.10), we must have

$$\mathscr{B}^{-1}_{\Delta\vec{\phi}} \quad \mathscr{A}_{\tau_0} \mathscr{N}_{\tau_0\tau_0+\Delta\tau} = \mathscr{A}_{\tau_0+\Delta\tau} \tag{3.10a}$$

$$\mathscr{B}^{-1}_{\Delta\vec{\varphi}} = \exp(:\Delta\vec{\varphi}\cdot\vec{J}:) . \tag{3.10b}$$

Using line (3.9), we obtain $\mathscr{N}_{\tau_0\tau_0} + \Delta\tau$ to first order in $\Delta\tau$

$$\mathscr{N}_{\tau_0\tau_0+\Delta\tau} = \mathscr{I} + \Delta\tau :-H(\vec{z}_0;\tau_0); + \ldots \quad . \tag{3.11}$$

Similarly, we expand $B^{-1}_{\Delta\vec{\phi}}$ to first order in $\Delta\vec{\phi}$.

$$B^{-1}_{\Delta\vec{\varphi}} = \mathscr{I} + :\Delta\vec{\varphi}\cdot\vec{J}: + \ldots \quad . \tag{3.12}$$

Substitution of Eq. (3.11) and (3.12) in (3.10a) gives the result

$$\mathscr{A}_{\tau_0+\Delta\tau} = \mathscr{A}_{\tau_0} + \mathscr{A}_{\tau_0} :-H:\Delta\tau + :\Delta\vec{\varphi}\cdot\vec{J}: \mathscr{A}_{\tau_0} + \ldots \quad . \tag{3.13}$$

To impose the condition (3.66), we operate on q_i using $\mathscr{A}_{\tau_0+\Delta\tau}$. For example, let us use $q_1 = z_1$. Assuming that repeated indices are summed and that A_{ij} is the matrix representation of $\mathscr{A}_{\tau_0}$, we get the equality

$$\mathscr{A}_{\tau_0+\Delta\tau}\, z_1 = A_{1i}z_i + \Delta\tau\, S_{2i}A_{ik}z_k + \Delta\varphi_k\cdot[J_k, z_i]\, A_{1i}. \tag{3.14}$$

We assumed that A_{12} is zero. This will be satisfied at $\tau_0+\Delta\tau$, if the term in Eq. (3.14) proportional to z_2 vanishes. This condition applied to $\mathscr{A}_{\tau_0+\Delta\tau}$ forces a relation between $\Delta\varphi$ and $\Delta\tau$

$$\Delta\tau\ S_{2i}\, A_{i2} - \Delta\varphi_1\, A_{11} = 0 \tag{3.15a}$$

$$\frac{d\varphi_1(\tau)}{d\tau} = \frac{S_{2i}\, A_{i2}}{A_{11}} \ . \tag{3.15b}$$

The other components of $\vec{\Phi}$ must obey the equation

$$\frac{d\varphi_k}{d\tau} = \sum_i \frac{S_{2k,i}\, A_{i,\,2k}}{A_{2k-1,\,2k-1}} \ . \tag{3.16}$$

Notice that in the case of strong violation of mid-plane symmetry the denominator of Eq. (3.16) can vanish. In that case, our definition of the phase advance may or may not be adequate, depending on the nature of the singularity. In any case, condition (3.6a) is probably maintainable for most systems.

From Eq. (3.10a), we derive an expression for $dA/d\tau$.

$$\frac{dA}{d\tau} = JS\ A + A\Phi' \ , \tag{3.17a}$$

with

$$\Phi'_{2i-1,2i} = -\frac{d\varphi_i}{d\tau}\ , \quad \Phi'_{2i,2i-1} = \frac{d\varphi_i}{d\tau} \ . \tag{3.17b}$$

and $A(\tau_0) = A_0$ as the initial condition. (3.17c)

Collecting all the equations, we get a hierarchy of equations which describes the linear system

$$\frac{dN}{d\tau} = JSN, \quad N_{\tau_0\tau_0} = \text{Identity} \tag{3.18a}$$

$$\frac{d\varphi_k}{d\tau} = \sum_i \frac{S_{2k,i}A_{i,2k}}{A_{2k-1,2k-1}}, \quad \varphi_k(\tau_0) = 0. \tag{3.18b}$$

$$\frac{dA}{d\tau} = JSA + A\Phi', \quad A_{\tau_0} = A_0 \tag{3.18c}$$

Equation (3.18a) is the matrix form of Eq. (3.9). As a simple exercise, we show now how this produces the correct two-dimensional results. A comparison with the four-dimensional result of Edwards and Teng is found in Appendix I.

In two-dimensions, we have the well-known results

$$M_0 = \begin{pmatrix} \cos\mu + \alpha_0\sin\mu & \beta_0\sin\mu \\ -\gamma_0\sin\mu & \cos\mu - \alpha_0\sin\mu \end{pmatrix} \tag{3.19a}$$

$$A(\tau) = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}, \tag{3.19b}$$

Substituting (3.19b) into (3.18b), we get for the phase

$$\frac{d\varphi}{d\tau} = \frac{S_{22}\ A_{12}}{A_{11}} = \frac{S_{22}}{\beta} \tag{3.20a}$$

or,

$$\varphi(\tau) = \int_{\tau_0}^{\tau} \frac{S_{22}}{\beta(\tau)}\, d\tau \quad . \tag{3.20b}$$

Using Eq. (3.19b) and (3.17a), we can derive two equations

$$\frac{d\sqrt{\beta}}{ds} = \frac{-\alpha}{\sqrt{\beta}} \text{ , (equation for } A_{11}) \tag{3.21a}$$

$$d[\frac{-\alpha}{\sqrt{\beta}}] = - K \sqrt{\beta} + \frac{1}{\beta^{3/2}} \text{ , (equation for } A_{12}) \text{ .} \tag{3.21b}$$

From (3.21a), we get

$$\frac{d\beta}{ds} = -2 \alpha \text{ .} \tag{3.22}$$

And, finally, similarly for α, we obtain using (3.22) and (3.21b) and

$$H = \frac{p^2 + K q^2}{2}$$

$$\frac{d\alpha}{ds} = K\beta + \frac{\alpha^2-1}{\beta} = K\beta - \gamma \text{ .} \tag{3.23}$$

Equations (3.22) and (3.23) agree with the usual result derivable from the expression for dN/ds and the Courant-Snyder invariant.

This concludes our discussion of the Courant-Snyder-Edwards-Teng phase advance.

4. The significance of $\mathscr{A}$ and its associated phase advance.

Consider two locations, "a" and "b," in the ring. Suppose we introduce pertubations in the form of multipole kicks at the locations. Calling the operators of these perturbations $:f(\vec{q}):$ and $:g(\vec{q}):$, we can write the perturbed map at location "a" as

$$\mathscr{M} = \exp(:f:)\ \mathscr{N}_{ab}\ \exp(:g:)\mathscr{N}_{ab}^{-1}\ \mathscr{M}_a\ . \tag{4.1}$$

Using Eq. (2.10), we can simplify (4.1).

$$\begin{aligned}\mathscr{M} &= \exp(:f:)\ \mathscr{A}_a^{-1}\ \mathscr{B}\ \mathscr{A}_b\ \exp(:g:)\ \mathscr{A}_b^{-1}\mathscr{B}^{-1}\mathscr{A}_a\ \mathscr{M}_a, \\ &= \mathscr{A}_a^{-1}\ \exp(:\ \mathscr{A}_a f:)\quad \mathscr{B}\exp(:\ \mathscr{A}_b g:)\ \mathscr{B}^{-1}\mathscr{R}\mathscr{A}_a^{-1}\end{aligned} \tag{4.2}$$

Equation (4.2) has the following interpretation. The ray is "extracted" from real space and put into Floquet space at location "a" by $\mathscr{A}_a^{-1}$. Then it experiences the perturbation f, also seen in its Floquet representation $\mathscr{A}_a f$. In the Floquet space, it travels to "b" through a simple phase advance. Then, it experiences a second perturbation g in the form $\mathscr{A}_b g$. Then, it "phase-advances" to "a" through the effect of $\mathscr{B}^{-1}\ \mathscr{R}$. Finally, it is brought back to the real space by $\mathscr{A}_a$. Notice that the ray is moved into Floquet space by $\mathscr{A}_a^{-1}$ while the operators are transformed by $\mathscr{A}_a$.

One trivial use of (4.2) is the computation of tune shifts. $\mathscr{M}$ can be rewritten as

$$\mathscr{M} = \mathscr{A}_a^{-1}\exp(:\mathscr{A}_a f:)\ \exp(:\mathscr{B}\ \ \mathscr{A}_b g:)\mathscr{R}\ \mathscr{A}_a. \tag{4.3a}$$

$$= \mathscr{A}_a^{-1}\ \mathscr{S}\ \mathscr{R}\ \mathscr{A}_a \tag{4.3b}$$

The map $\mathscr{S}$ ("T" for "trouble") provides all the information for the computation of tune shifts and distortions of the invariants. The first-order tune shifts are the parts of $\mathscr{S}$ which modify $\mathscr{R}$. The first-order distortions are the parts of $\mathscr{S}$ which modify $\mathscr{A}_a$. The higher order tune shifts require the knowledge of the lower distorsions of $\mathscr{A}_a$.

For example, suppose we are interested in a 1st order result. We proceed by rewriting $\mathscr{A}_a f$ and $\mathscr{A}_b g$ in action-angle variables;

$$\mathscr{A}_a f = [\sum_{\vec{m}\neq 0} W_{\vec{m}}(\vec{J})\, e^{i\vec{m}\cdot\vec{\phi}}] + W_0(\vec{J}), \tag{4.4a}$$

$$\mathscr{A}_b g = [\sum_{\vec{m}\neq 0} V_{\vec{m}}(\vec{J})\, e^{i\vec{m}\cdot\vec{\phi}}] + V_0(\vec{J}) \tag{4.4b}$$

In first order in f and g, the detuning component of $\mathscr{S}$ is just

$$D(\vec{J}) = W_0(\vec{J}) + V_0(\vec{J}) \ . \tag{4.5}$$

The first-order correction of $\mathscr{A}_a$, can be written as exp(:Δ:) and it obeys the equation

$$\exp(:\Delta:)\mathscr{S}\mathscr{R}\exp(:-\Delta:) = \exp(:D(J):)\,\mathscr{R} + \text{(second order)}$$

$$\Leftrightarrow \exp(:\Delta:)\,\mathscr{S}\exp(:-\mathscr{R}\,\Delta:) = \exp(:D(J):) + \text{(second order)}$$

Solving for Δ, we get

$$\Delta = \frac{\mathscr{I}}{(\mathscr{R} - \mathscr{I})}(\mathscr{A}_a f + \mathscr{B}\ \mathscr{A}_b g) - D(\vec{J}) \tag{4.6a}$$

$$\Delta = \sum_{\vec{m}\neq 0} (\exp(-i\vec{m}\cdot\vec{\nu}(\vec{J})) - 1)^{-1}\ (W_{\vec{m}} + \exp(-i\vec{m}\cdot\frac{d\Gamma}{d\vec{J}})\ V_{\vec{m}})e^{i\vec{m}\cdot\vec{\phi}} \ , \tag{4.6b}$$

where

$$\nu(\vec{J}) = \vec{\mu} - \frac{\partial F}{\partial \vec{J}} \ . \tag{4.6c}$$

Equation (4.6c) is given by (2.1b).

We can use Δ to rewrite the expression for $\mathscr{M}$.

$$\mathscr{M} = \mathscr{A}_a^{-1}\exp(:-\Delta:)\ \ \mathscr{S}_1\,\mathscr{R}\exp(:D:)\ \exp(:\Delta:)\ \mathscr{A}_a \tag{4.7a}$$

$$\mathscr{S}_1 = \exp(:\tfrac{1}{2}[\ \mathscr{A}_a f,\ \mathscr{B}\ \mathscr{A}_b g] + \tfrac{1}{2}[\Delta,\ \mathscr{A}_a f + \mathscr{B}\ \mathscr{A}_b g] + \ldots:) \tag{4.7b}$$

The structure of (4.7) is indentical to the structure of (4.3). The transformation $\mathscr{A}_a$ is modified and an extra detuning term has been added to $\mathscr{R}$.

$$\mathscr{R}\ \exp(:D:) = \exp(:\mu\cdot\vec{J} - F(\vec{J}) + D(\vec{J}):) \quad . \tag{4.8}$$

The new "trouble" map $\mathscr{S}_1$ is second order in f and g. Quite clearly the second-order tune shifts are given by the average over $\vec{\phi}$ of the Lie operator of $\mathscr{S}_1$.

These formulas can be generalized to K locations around the ring with K perturbing insertions f_K.

$$D(\vec{J}) = \frac{1}{(2\pi)^n} \int d\phi_1 \cdots d\phi_n \sum_{k'=1}^{K} \mathscr{A}_{K'}\ f_{K'} \tag{4.9a}$$

$$\Delta = \frac{\mathscr{J}}{(\mathscr{R} - \mathscr{J})} \left\{ \sum_{K'=1}^{K} \mathscr{B}_{K'}\ \mathscr{A}_{K'}\ f_{K'} - D(\vec{J}) \right\} \tag{4.9b}$$

$$\mathscr{S}_1 = \tfrac{1}{2} \sum_{K'<K''} [\mathscr{B}_{K'}\ \mathscr{A}_{K'}\ f_{K'}, \mathscr{B}_{B''}\ \mathscr{A}_{K''}\ f_{K''}]$$

$$+ \tfrac{1}{2}[\Delta, \sum_{k'=1}^{K} \mathscr{B}_{K'}\ \mathscr{A}_{K'} f_{K'}] \quad . \tag{4.9c}$$

$\mathscr{B}_{K'}$ = phase advance between location K' = 1 and K'.

Conclusion

We have shown how the concept of phase advance is generalizable to nonlinear systems with or without coupling. We have also shown how one introduces errors into such a system without using the Hamiltonian frame work. The four dimensional phase advance described in this paper has been incorporated in the program FASTRACK which can now compute the linear lattice functions and the associated phase advance in presence of errors and coupling.

Finally, we would like to point out that the idea of using mappings to describe the properties of the ring can be extended to spin systems. Indeed, Yokoya has shown how the tools presented in this paper, the $\mathscr{A}$-transformations and $\mathscr{B}$-phase advance transformation are extendable to systems mixing the orbital motion around the ring with the spin of the particles.[7]

Appendix I: Equivalence of our phase advance to the work of Edwards-Teng.

Consider the Hamiltonian

$$H = \tfrac{1}{2}\vec{p}^{\,2} + L(p_x y - p_y x) + \tfrac{1}{2}Fx^2 + Kxy + Gy^2. \tag{I.1}$$

This Hamiltonian was studied by Edwards and Teng. They succeeded in finding a canonical transformation which reduces the matrix M of the one turn map into block diagonal matrix of the Courant-Snyder type.

$$M = A_1 \bar{M} A_1^{-1} \tag{I.2a}$$

$$A_1 = \begin{pmatrix} I\cos\phi & D^{-1}\sin\phi \\ -D\sin\phi & I\cos\phi \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{I.2b}$$

$$\bar{M} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{I.2c}$$

The matrices I, D, B_1 and B_2 are two dimensional. Furthermore, B_1 and B_2 have the Courant-Snyder form

$$B_i = \begin{pmatrix} \cos\mu_i + \alpha_i \sin\mu_i & \beta_i \sin\mu_i \\ -\gamma_i \sin\mu_i & \cos\mu_i - \alpha_i \sin\mu_i \end{pmatrix}. \tag{I.3}$$

They give a prescription in their paper which fixes the quantity $\cos\phi$ unambigously. ($\cos\phi \geq 0$).

Now, consistent with our spirit, we diagonalize M

$$\bar{M} = A_2 \; R \; A_2^{-1} \tag{I.4a}$$

$$A_2 = \begin{pmatrix} \sqrt{\beta_1} & 0 & 0 & 0 \\ \frac{-\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_2} & 0 \\ 0 & 0 & \frac{-\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}$$

Hence, we can write M as

$$M = ARA^{-1}; \; A = A_1 A_2 \quad . \tag{I.5}$$

The quantity $A(=A_1 A_2)$ has the property described in section 3, Eq. (3.6). As a result, the phase advance must obey Eq. (3.16). Looking at the plane of B_1, we get

$$\frac{d\varphi_1}{d\tau} = \frac{S_{21}A_{12} + S_{22}A_{22} + S_{23}A_{32} + S_{24}\;A_{42}}{A_{11}}$$

$$= \frac{\frac{\cos\phi}{\sqrt{\beta_1}} + L\left(\frac{-b\sin\phi}{\sqrt{\beta_1}}\right)}{\sqrt{\beta_1}\;\cos\phi}$$

$$= \frac{(1 - Lb\tan\phi)}{\beta_1} \tag{I.6}$$

Similarly, for the second plane

$$\frac{d\varphi_2}{d\tau} = \frac{1 + Lb\tan\phi}{\beta_2} \quad . \tag{I.7}$$

To no one's surprise, the expressions for $d\vec{\varphi}/d\tau$ agree with Edwards-Teng's results.

References

1. E.D. Courant and H.S. Snyder, Annuals of Physics 3, 1 (1958).

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3. A.J. Dragt Fermilab Summer School, AIP Conference Proceedings, Number 87, (1981) p. 147.

4. A.J. Dragt, Nonlinear Lattice Functions, Proceedings the 1984 Study on the Design and Utilization of the Superconducting Supper Collider, p. 369.

5. A.J. Dragt and E. Forest, Computation of Nonlinear Behavior of Hamiltonian Systems using Lie Algebraic Methods, J. Math. Phys. 24(12), (1983).

6. FASTRACK is a modified version of the program RACETRACK of A. Wrulich. It is a kick code equipped with a 4x4 phase advance algorithm as described in this paper. The code has also the capability of generating a fifth order map (B.T. Leemann and E. Forest).

7. K. Yokoya, Calculation of the Equilibrium Polarization of Stored Electron Beams Using Lie Algebras.