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Supermanifolds as Sub–supermanifolds

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Abstract

We show that every supermanifold is a sub–supermanifold of a higher odd dimensional supermanifold and supermanifolds are categorically equivalent to graded manifolds as its immediate consequence.

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1. Introduction

Supersymmetry provides a natural framework for unified description of fermions and bosons and it has been widely used for the construction of field theory and elementary particle models. In the context of supersymmetry, Salam and Strathdee[1] have introduced the concepts of superspace which is parametrized by eight even and odd coordinates ($x^1,\ldots,x^4,\theta^1,\ldots,\theta^4$). But if we recall the importance of global topology in quantum field theory relating to monopoles, instantons and anomalies etc., it is natural to construct globally well–defined supermanifolds. Especially, in the superstring theory, it is indispensable to investigate the global aspects of supermanifolds, that is super Riemannn surfaces, in order to understand the super moduli space.

As rigorous mathematical models for superspace, several supermanifolds have been defined in the literature. However, we may classify them into two main categories – differential geometric and algebro geometric(or sheaf–theoretic) one. While the differential geometric models built by Rogers[4] and DeWitt[5] have a huge higher real dimensions in order to put on an equal footing the even and odd coordinates, the sheaf–theoretic models given by Berezin, Leites[6] and Kostant[2] have ordinary manifolds such as spacetime or Riemann surfaces respectively for supergravity and superstring theory as their body in the beginning together with algebraic structures given by sheaves. In this case, the even and odd coordinates are treated to be equal under the algebraic structures. Even though the two approaches look to be very different, it is known that they are categorically equivalent[3,7,8]. In this paper we shall investigate the relationships between various supermanifolds and the equivalence between the categories of the differential geometric and the sheaf–theoretic models from a different view–point.

Let us recall the definition of a graded(or super) algebra. A graded vector space is a vector space V which is a direct sum of its two subspaces V_0, V_1 called respectively even and odd part. Non–zero elements of even and odd part are said to be homogeneous. If

$x\ (\neq 0) \in V_i$, i is called the degree of x and denoted by $\langle x \rangle$. The tensor product $V \otimes W$ of two graded vector spaces V and W is again a graded vector space. A graded commutative algebra B is an associative algebra with identity which is graded as a vector space and satisfies $1 \in B_0$ and, for homogeneous elements x and y,

$$\langle xy \rangle = \langle x \rangle + \langle y \rangle \pmod 2$$

and

$$xy = (-1)^{\langle x \rangle \langle y \rangle} yx.$$

As an important instance, a Grassmann algebra Λ over a vector space V is a graded commutative algebra with

$$\Lambda_0 = \sum_k \Lambda^k V \quad (k \text{ even})$$

$$\Lambda_1 = \sum_k \Lambda^k V \quad (k \text{ odd}).$$

The number of generators for the Grassmann algebra is the dimension of the vector space V. For two graded commutative algebras A and B, $A \otimes B$ is a graded commutative algebra with the multiplication

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\langle a' \rangle \langle b \rangle} (aa') \otimes (bb').$$

2. Supermanifolds and Graded Manifolds

Let K be a Grassmann algebra with L generators with K_0, K_1 even and odd part. The superEuclidean space $K^{m,n}$ of even dimension m and odd dimension n is defined to be $(K_0)^m \times (K_1)^n$. If we write $(x,\theta) \equiv (x^1,\ldots, x^m;\theta^1,\ldots,\theta^n)$ for a coordinate in an open subset O of the superEuclidean space, the even coordinate x has a real part x_0 under the natural map

Aug from $K^{m,n}$ to $\mathbb{R}^m$ and hence a supersmooth function can be defined: Let Λ be a subalgebra of K with q generators. We assume here that L is big enough so that q + n is not greater than L for self–consistency. A Λ–supersmooth function[4,5,9] f is a map from O to K which is expressed as a terminating formal power series in θ such that each coefficient function, which is a function of only even coordinate x, has a smooth function of x_0 for the zeroth order term whenever it is Tayor–expanded in $x - x_0$. In a similar way as the ordinary manifold theory, we define a Λ–supermanifold of dimension (m,n) to be a smooth manifold covered by open subsets of the superEuclidean space of m even and n odd dimension such that the coordinate transformations are Λ–supersmooth. We define Λ–supersmooth maps and Λ–superdiffeomorphisms between Λ–supermanifolds in the obvious way with the following notions on the superEuclidean space: Let O and O′ be open subsets of $K^{m,n}$, $K^{m',n'}$ respectively. A map $\phi : O \longrightarrow O'$ is called Λ–supersmooth if ϕ is smooth and pulls back Λ–supersmooth functions on O′ to Λ–supersmooth functions on O. It follows then that ϕ is Λ–supersmooth if and only if the functions followed by projection onto any coordinates of O′ are Λ–supersmooth. A Λ–superdiffeomorphism is a Λ–supersmooth map with a Λ–supersmooth inverse.

For physical applications we impose one more condition. A DeWitt open set[5] is an open subset O of the superEuclidean space such that $\mathrm{Aug}^{-1}(\mathrm{Aug}\ O) = O$. This implies that a DeWitt open set has a trivial topology. Now we say that a Λ–supermanifold is a Λ–DeWitt supermanifold if it is modeled on DeWitt open sets. A Λ–DeWitt supermanifold is called a Batchelor supermanifold if $\Lambda = \mathbb{R}$.

A subset Y′ of a Λ–supermanifold Y of dimension (m,n) is said to be a sub–supermanifold of dimension (m′,n′), if Y′ is contained in the union of a set $\{(U, \phi)\}$ of charts each of which satisfies the following: for all points p in $U \cap Y'$,

$$\phi(p) = (x^1,\ldots,x^{m'},a^{m'+1},\ldots,a^n;\theta^1,\ldots,\theta^{n'},b^{n'+1},\ldots,b^n)$$

for some constants $a^{m'+1},\ldots,a^n,b^{n'+1},\ldots,b^n$.

We shall now introduce a sheaf–theoretic approach to supermanifolds. A Kostant sheaf over a smooth manifold M is a sheaf $\mathscr{A}$ of graded commutative algebras such that there is a surjective homomorphism α from $\mathscr{A}$ to the sheaf C^∞ of smooth functions on M and locally $\mathscr{A}$ is isomorphic to $C^\infty \otimes \Lambda V$ for a finite–dimensional vector space V. An m–dimensional smooth manifold M together with a Kostant sheaf $\mathscr{A}$ over it is called a Kostant graded manifold of dimension (m,n) where n is the dimension of V. Since a Kostant graded manifold deals with only bosonic component fields, we extend it to contain fermionic fields as well.

A graded commutative algebra B is said to be a graded Λ–algebra if there is a bilinear gradation–preserving map from $\Lambda \otimes B$ to B, $a \otimes b \mapsto ab$, which satisfies

$$(aa')b = a(a'b) \quad \text{and} \quad b(ab') = (-1)^{<b><a>}a(bb')$$

for $a,a' \in \Lambda$ and $b,b' \in B$. Let I_Λ be an ideal of Λ generated by the odd part Λ_1. A Λ–DeWitt sheaf over a smooth manifold M is a sheaf $\mathscr{B}$ of graded Λ–algebras such that $\mathscr{B}/I_\Lambda\mathscr{B} \cong \mathscr{K}(\mathscr{B})$ is a Kostant sheaf and locally $\mathscr{B}$ is isomorphic to $\Lambda \otimes \mathscr{K}(\mathscr{B})$. An m–dimensional smooth manifold M together with a Λ–DeWitt sheaf $\mathscr{B}$ over it is called a Λ–DeWitt graded manifold of dimension (m,n) where n is the dimension of the vector space for the Kostant sheaf $\mathscr{K}(\mathscr{B})$. For a small open subset U in M, the isomorphism between $\mathscr{B}$ and $\Lambda \otimes \mathscr{K}(\mathscr{B}) \cong \Lambda \otimes C^\infty \otimes \Lambda V$ defines a function factor to be the corresponding subalgebra of $\mathscr{B}(U)$ to $C^\infty(U)$.

For a Kostant graded manifold $(M,\mathscr{A})$, a subset $\{r^1,\ldots,r^m\}$ of $\mathscr{A}(U)_0$ is called an even coordinate system in $U \subset M$ if the smooth function $\tilde{r}^1,\ldots,\tilde{r}^m$ obtained under the map α: $\mathscr{A} \longrightarrow C^\infty$ is an ordinary coordinate system in U. A subset $\{s^1,\ldots,s^n\}$ of $\mathscr{A}(U)_1$ is called an odd coordinate system in $U \subset M$ if an $(\mathscr{A}/I_{\mathscr{A}})(U)$–module $(I_{\mathscr{A}}/I^2_{\mathscr{A}})(U)$ has the classes $\{s^1,\ldots,s^n\}$ as a basis. We define even and odd coordinate systems for a Λ–DeWitt

graded manifold $(M,\mathscr{B})$ to be those for the Kostant graded manifold $(M,\mathscr{K}(\mathscr{B}))$.

3. Sub–supermanifolds

We shall now investigate the relationships between the various supermanifolds and graded manifolds. Let Λ be a Grassmann algebra of q generators and let $(M,\mathscr{B})$ be a Λ–DeWitt graded manifold of dimension (m,n). From the following sheaf isomorphisms

$$\mathscr{B}/I_{\mathscr{B}} \cong \frac{\mathscr{B}/I_\Lambda\mathscr{B}}{I_{\mathscr{B}}/I_\Lambda\mathscr{B}} \cong C^\infty,$$

it follows that a Λ–DeWitt graded manifold $(M,\mathscr{B})$ of dimension (m,n) is a Kostant graded manifold of dimension (m,n+q). Let us now consider the converse. If a Kostant sheaf $\mathscr{A}$ is a graded Λ–algebra, we have an exact sequence of sheaves from the above sheaf relationship:

$$0 \longrightarrow I_{\mathscr{K}(\mathscr{A})} \longrightarrow \mathscr{K}(\mathscr{A}) \longrightarrow C^\infty \longrightarrow 0$$

and hence the Kostant graded manifold $(M,\mathscr{A})$ of dimension (m,n + q) becomes a DeWitt graded manifold of dimension (m,n). From now on, we shall write $\mathscr{A}$ for $\mathscr{B}$ whenever we emphasize that the Λ–DeWitt sheaf $\mathscr{B}$ is a Kostant sheaf. Let U and U′ be two overlapping small open subsets in M and let $\{r^i,s^j,v^k\}$ and $\{r^{i'},s^{j'},v^{k'}\}$ be coordinate systems of $\mathscr{A}(U \cap U')$ obtained by restricting coordinate systems of $\mathscr{A}(U)$ and $\mathscr{A}(U')$ respectively to $\mathscr{A}(U \cap U')$ where $i = 1,\ldots,m$, $j = 1,\ldots,n$ and $k = 1,\ldots,q$. Here we note that $v^k,v^{k'}$ are odd coordinate systems. It follows then that the transformations of coordinates on the intersection of two open subsets are specified in general by the following:

$$r^{i\prime} = \Sigma f^{i}_{\rho\sigma} v^{\rho} s^{\sigma}$$
$$s^{j\prime} = \Sigma g^{j}_{\rho\sigma} v^{\rho} s^{\sigma}$$
$$v^{k\prime} = \Sigma h^{k}_{\rho\sigma} v^{\rho} s^{\sigma}$$

where ρ,σ are the multiplicative indices, i.e. if $\rho = (\rho_1,\ldots,\rho_l)$ then $v^{\rho} = v^{\rho_1}\cdots v^{\rho_l}$, and such that $v^{\rho}s^{\sigma}$ respect the even and odd parities depending respectively on the even coordinates r and odd coordinates s,v. The coefficients $f^{i}_{\rho\sigma}$, $g^{j}_{\rho\sigma}$ and $h^{k}_{\rho\sigma}$ are in an appropriate function factor of $\mathscr{A}(U \cap U')$. The additional condition that the Kostant sheaf is a graded Λ–algebra implies that v^k may be identified with the generators of the constant sheaf Λ and hence they transform identically, i.e. $h^{k}_{\rho\sigma} = \delta_{\rho k}\,\delta_{\sigma 0}$ for $k = 1,\ldots,q$. We recall that a Batchelor supermanifold, $Y(\mathscr{A})$, can be constructed by defining local coordinate transformations according to the method of Rogers[4]. The transformations in the above determine an atlas for $Y(\mathscr{A})$ such that each chart has m even coordinates x^i and n odd coordinates θ^j together with the extra q odd coordinates v^k which transforms trivially as follows:

$$\begin{aligned} x^{i\prime} &= \Sigma \tilde{f}^{i}_{\rho\sigma}(x) v^{\rho}\theta^{\sigma} \\ \theta^{j\prime} &= \Sigma \tilde{g}^{j}_{\rho\sigma}(x) v^{\rho}\theta^{\sigma} \\ v^{k\prime} &= v^{k} \end{aligned} \qquad (*)$$

where $\tilde{f}^{i}_{\rho\sigma}$, $\tilde{g}^{j}_{\rho\sigma}$ are the images of $f^{i}_{\rho\sigma}$, $g^{j}_{\rho\sigma}$ under the map a from $\mathscr{A}$ to C^{∞}. If we take a subset $\{\phi^{-1}(x, \theta, v) \mid v \equiv (v^1,\ldots,v^q) = \text{constant}\}$ of U from each chart (U,ϕ) of the atlas, the union $Y'(\mathscr{A})$ of these subsets is obviously a sub–supermanifold of $Y(\mathscr{A})$ of dimension (m,n). Let $Y(\mathscr{B})$ be the Λ–DeWitt supermanifold constructed from the Λ–DeWitt graded manifold $(M,\mathscr{B})$ according to the method of Rogers. Since coordinate transformations between overlapping open sets in $Y(\mathscr{B})$ are essentially the same as the first two equations

in the transformations (*) by construction, it follows that $Y(\mathscr{B})$ and $Y'(\mathscr{A})$ are Λ–superdiffeomorphic. Thus $Y(\mathscr{B})$ can be regarded as a sub–supermanifold of $Y(\mathscr{A})$. It is noted here that a Batchelor supermanifold is also a sub–supermanifold of a higher odd dimensional Batchelor supermanifold, which is the case when v is taken to be zero. Moreover, for the split case where the Λ–DeWitt sheaf $\mathscr{B}$ is just the tensor product of Λ and a given Kostant sheaf $\mathscr{S}$, $Y(\mathscr{A})$ is the product supermanifold $Y(\mathscr{S}) \times K^{0,q}$.

As an application of the sub–supermanifold structure, we can prove easily the equivalence between the categories of Λ–DeWitt graded and supermanifolds as follows : First of all, as has been shown in the above, the category of Λ–DeWitt graded manifolds $(M,\mathscr{B})$ of dimension (m,n) is equivalent to the category of Kostant graded manifolds $(M,\mathscr{A})$ of dimension (m,n+q) for graded Λ–algebras $\mathscr{A}$. It follows then, by Batchelor[3], the categories of the Kostant graded manifolds $(M,\mathscr{A})$ and Batchelor supermanifolds $Y(\mathscr{A})$ having $Y'(\mathscr{A})$ as sub–supermanifolds are equivalent. Since $Y(\mathscr{B})$ is superdiffeomorphic to $Y'(\mathscr{A})$ and hence is the sub–supermanifold of $Y(\mathscr{A})$ together with the obvious morphisms, the categories of Λ–DeWitt graded manifolds $(M,\mathscr{B})$ and Λ–DeWitt supermanifolds $Y(\mathscr{B})$ are equivalent.

4. Conclusions

We have clarified the various definitions of supermanifolds in the literature. Even though a DeWitt supermanifold is a generalization of a Batchelor supermanifold such that it contains fermionic fields in addition to bosonic fields, we have shown that a DeWitt supermanifold may be regarded as a sub–supermanifold of a Batchelor supermanifold. This leads us immediately to the equivalence between the categories of DeWitt graded and supermanifolds.

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