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Explicit Computations for the Intersection Numbers on Grassmannians, and on the Space of Holomorphic Maps from CP^1 into $G_r(C^n)$

Noureddine Chair¹

*Faculty of Art and Sciences, Physics Department,
Al al-Bayt University Mafrag, Jordan*

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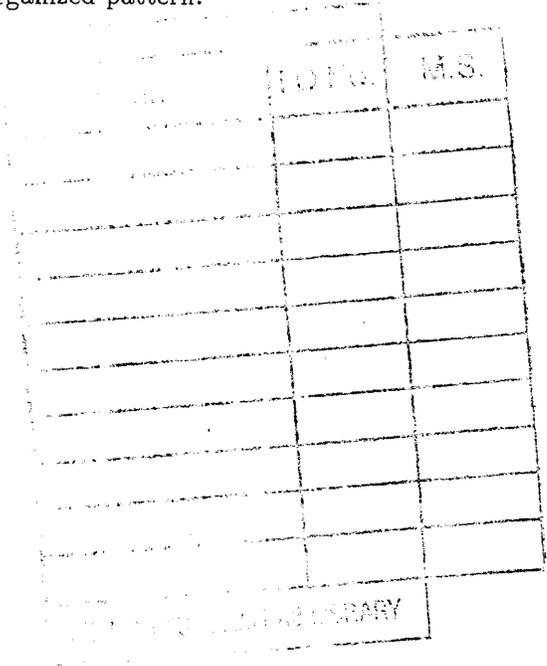
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Abstract

We derive some explicit expressions for correlators on Grassmannian $G_r(C^n)$ as well as on the moduli space of holomorphic maps, of a fixed degree d , from sphere into the Grassmannian. Correlators obtained on the Grassmannian are a first step generalization of the Schubert formula for the self-intersection. The intersection numbers on the moduli space for $r = 2, 3$ are given explicitly by two closed formulas, when $r = 2$ the intersection numbers, are found to generate the alternate Fibonacci numbers, the Pell numbers and in general a random walk of a particle on a line with absorbing barriers. For $r = 3$ the intersection numbers form a well organized pattern.



¹e-mail:n.chair@rocketmail.com



1 Introduction

The classical Schubert calculus computes intersection numbers on Grassmannians $G_r(C^n)$ of complex r -planes in C^n , by using the Giambelli and Pieri formula [1, 2, 3]. It is due primarily to Schubert more than a hundred years ago, to obtain the number,

$$\int_{G_r(C^n)} x_1^{r(n-r)} = \frac{1!2!3! \cdots (r-2)!(r-1)!(r(n-r))!}{(n-r)!(n-r+1)! \cdots (n-1)!},$$

known as the degree of the Grassmannian or the self-intersection of the first Chern class x_1 , of the r -plane bundle Q on $G_r(C^n)$. Geometrically speaking this number corresponds to the number of $(r-1)$ -planes in CP^{n-1} meeting $r(n-r)$ general $(n-r-1)$ -planes, in particular for $r=2$ there are two lines meeting 4-given lines in CP^3 .

Our goals in this paper are two-fold, first we would like to extend the above formula to other correlators that are products of Chern classes x_i , $r \leq i \leq r$. In this direction, we use the pairing residue formula that computes the correlators in topological Landau-Ginzburg Theories [4] and the explicit formula for the potential $W(x_i)$ that generate the cohomology ring [5], to do some explicit computations on the Grassmannian. The different intersection numbers obtained, show a certain pattern amongst themselves and is formulated in a proposition which in turn lead to the closed formula for $\int_{G_r(C^n)} x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}}$.

The second goal is to carry out similar computations on the space of holomorphic maps of a fixed degree d , from a Riemann surface of genus zero (CP^1) into the Grassmannian $G_r(C^n)$. Formally both computations use the same formula [4, 6], the difference between the two cases is that the potential in the second case is a deformed potential $\widetilde{W}(x_i)$ and is connected to the previous potential by $\widetilde{W}(x_i) = W(x) + (-1)^r q x_1$ [7]. This potential reproduces the quantum cohomology ring of the Grassmannian [8]. The concept of deformation of the cohomology ring was first observed in [9], in connection with the CP^1 model. On the space of holomorphic maps from CP^1 into $G_r(C^n)$, we have two closed formulas for $r=2$ and 3 for any n . When $r=2$, the intersection numbers generate well known numbers like the Fibonacci numbers and the Pell numbers for $n=5,6$ respectively, and when $n \geq 7$, the intersection numbers generates a random walk of a particle on a line with absorbing barriers [10, 11]. Our closed formula for the intersection numbers on the space of holomorphic maps into $G_3(C^n)$, when restricted to constant maps, gives all the intersection numbers on $G_3(C^n)$. Some intersection numbers on this space were computed when $n=6$ and for degree one and two. We find that these numbers organize themselves in an ordered pattern, it seems that the intersection numbers on this moduli space is a zoo of interesting numbers. This fact is already presented on the Grassmannian $G_r(C^n)$; if we set $N = n - r$ in the Schubert formula, we obtain the generating functions for the r -dimensional Catalan numbers [10]. When $r=2$, we obtain the ordinary Catalan numbers $(2N)!/(N+1)N!$.

This paper is organized as follows: in section 2 after a brief account of the cohomology ring of the Grassmannian, the pairing residue formula which computes correlators in $N=2$

topological Landau theories, and fixing our notations, we compute some intersection number on the Grassmannian $G_r(C^n)$ from which we obtain closed formula for the correlators. Section 3 and 4 will be devoted to computations of correlators on the space of holomorphic maps from CP^1 into $G_r(C^n)$, in which we find connections between intersection numbers, Fibonacci numbers, Pell numbers and the random walk. Our conclusions are given in section 5.

2 Intersection Numbers on a Grassmannian

In this section we shall first recall briefly the definition of the cohomology ring of the Grassmannian $G_r(C^n)$ in the Landau-Ginzburg formulation [12, 9, 13] and the pairing residue formula of the N=2 topological Landau-Ginzburg model that computes the correlators [4]. We then use the specialized form of “the pairing residue formula” and the explicit expression for the Landau-Ginzburg potential in terms of the generators $x_i (1 \leq i \leq r)$ of the cohomology ring of the Grassmannian [5] to compute some intersection numbers. The intersection numbers computed are exactly those obtained using the Schubert Calculus [2, 3]. These computations show that the two-point function $\langle x_1^{\alpha_1} x_2^{\alpha_2} \rangle$ on $G_2(C^{n+1})$ is equal to the two point function $\langle x_1^{\alpha_1} x_2^{\alpha_2-1} \rangle$ on $G_2(C^n)$. In general the r-point functions on $G_r(C^{n+1})$ and $G_r(C^n)$ are related in the same way. This fact will be proved in the proposition below and as a consequence we obtain an explicit expression for the two point functions on $G_r(C^n)$ involving the Chern classes x_1 and x_r .

The cohomology ring of the complex Grassmannian manifold, denoted by $H^*(G_r(C^n))$ is a truncated polynomial ring in several variables [14] given by

$$H^*(G_r(C^n)) \cong C[x_1, \dots, x_r, y_1, \dots, y_{n-r}] / I, \quad (1)$$

where $x_i = c_i(Q)$ (for $1 \leq i \leq r$) are the Chern classes of the quotient bundle Q of rank r , i.e., $x_i \in H^{2i}(G_r(C^n))$ and $y_j = c_j(S)$ (for $1 \leq j \leq n-r$) are the Chern classes of the universal bundle S of rank $n-r$. The ideal I in $C[x_1, \dots, x_r, y_1, \dots, y_{n-r}]$ is given by

$$(1 + x_1 + x_2 + \dots + x_r)(1 + y_1 + y_2 + \dots + y_{n-r}) = 1, \quad (2)$$

which is the consequence of the tautological sequence on $G_r(C^n)$

$$0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0,$$

where $V = G_r(C^n) \times C^n$. By using equation (2), one may rewrite $H^*(G_r(C^n))$ as

$$H^*(G_r(C^n)) \cong C[x_1, \dots, x_r] / y_j, \quad (3)$$

where y_j are expressed in terms of x_i , and $y_j = 0$ for $n - r + 1 \leq j \leq n$, and $x_0 = y_0 = 1$. The classes y_j can be written inductively as a function of x_1, \dots, x_r via

$$y_j = -x_1 y_{j-1} - \dots - x_{j-1} y_1 - x_j, \quad \text{for } j = 1, \dots, n - r. \quad (4)$$

In the Landau-Ginzburg formulation, the potential that generates the cohomology ring of the grassmannian [12, 13, 9], is given by

$$W_{n+1}(x_1, \dots, x_r) = \sum_{i=1}^r \frac{q_i^{n+1}}{n+1}, \quad (5)$$

where, x_i and q_i are related by

$$x_i = \sum_{1 \leq l_1 < l_2 < \dots < l_i \leq r} q_{l_1} q_{l_2} \dots q_{l_i}. \quad (6)$$

The cohomology ring of the grassmannian is then given by

$$\frac{\partial W_{n+1}}{\partial x_i} = (-1)^n y_{n+1-i}, \quad \text{for } 1 \leq i \leq r, \quad (7)$$

implying that $d_i W_{n+1} = 0$, for $i = 1, \dots, r$. In terms of the x_i 's, [5] the explicit formulas for the y_j 's and the cohomology potential $W(x_1, \dots, x_r)$ are,

$$y_j = (-1)^j \sum_{k_1=0}^{\lfloor \frac{j}{2} \rfloor} \dots \sum_{k_{r-1}=0}^{\lfloor \frac{j}{r} \rfloor} \frac{(-1)^{k_1+2k_2+\dots+(r-1)k_{r-1}}}{k_1! \dots k_{r-1}!} \times \frac{(j - \sum_{l=1}^{r-1} l k_l)!}{(j - \sum_{l=2}^r l k_{l-1})!} x_1^{j-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}}. \quad (8)$$

$$W_{n+1}(x_1, \dots, x_r) = \sum_{k_1=0}^{\lfloor \frac{n+1}{2} \rfloor} \dots \sum_{k_{r-1}=0}^{\lfloor \frac{n+1}{r} \rfloor} \frac{(-1)^{k_1+2k_2+\dots+(r-1)k_{r-1}}}{k_1! \dots k_{r-1}!} \times \frac{(n - \sum_{j=1}^{r-1} j k_j)!}{(n+1 - \sum_{j=2}^r j k_{j-1})!} x_1^{n+1-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}}. \quad (9)$$

The self-intersection numbers $\langle x_1^{r(n-r)} \rangle$, and other correlation functions on the Grassmannian $G_r(C^n)$ of products of monomials in the cohomology classes x_i ($1 \leq i \leq r$) such that the total power of this product is the dimension of $G_r(C)$, i.e., $r(n-r)$, may be computed using the

residue pairing formula [4]. This formula computes the correlators in the topological Landau-Ginzburg theories, which for genus zero, reads

$$\begin{aligned} \left\langle \prod_{i=1}^N F_i(x_j) \right\rangle &= (-1)^{N(N-1)/2} \sum_{dW=0} \frac{\prod_{i=1}^N F_i(x_j)}{H} \\ &= (-1)^{N(N-1)/2} \frac{1}{(2\pi i)^N} \oint \dots \oint \frac{dx_1 \dots dx_N \prod_{i=1}^N F_i(x_j)}{\partial_1 W \dots \partial_N W}, \end{aligned} \quad (10)$$

where $F_i(x_j)$ are polynomials in the superfields x_i , $H = \det(\partial_i \partial_j W)$ is the hessian and the summation on the right hand side in the first expansion is over the critical points of W . In this section, the maps from sphere into $G_r(C^n)$ are considered constant, i.e., the moduli space of instantons is nothing but the Grassmannian $G_r(C^n)$ itself, and the correlators are the intersections of the cycles over $G_r(C^n)$. Therefore the residue pairing formula reads²

$$\left\langle x_1^{n+1-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}} \right\rangle = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r} \oint \dots \oint \frac{x_1^{n+1-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}}}{\partial_1 W \dots \partial_r W} \quad (11)$$

The closed form for these correlators is, in general, not known, except for the self-intersection $\langle x_1^{r(n-r)} \rangle$, which was given by the Schubert calculus [1, 2, 3] (also called the degree of the Grassmannian) and has the following expression

$$\langle x_1^{r(n-r)} \rangle = (r(n-r))! \prod_{\ell=0}^{r-1} \frac{\ell!}{(n-r+\ell)!}. \quad (12)$$

In particular for $G_2(C^4)$ the self-intersection $\langle x_1^4 \rangle$, is 2 which is the number of lines meeting 4 given lines in CP^1 , and in general the right hand side of the above equation gives the number of $(r-1)$ -planes meeting $N = r(n-r)$ given $(n-(r+1))$ -planes in general position in CP^{n-1} . The simplest non-trivial Grassmannian for which the residue pairing formula can be used is $G_2(C^4)$. Here the potential that generates the cohomology ring (intersection ring) $H^*(G_2(C^4))$, is $W(x_1, x_2) = \frac{1}{5}x_1^5 - x_1^3x_2 + x_1x_2^2$ and the possible correlators are $\langle x_1^{4-2k}x_2^k \rangle$ where $0 \leq k \leq 2$. Applying the residue pairing formula we have,

$$\langle x_1^{4-2k}x_2^k \rangle = -\frac{1}{(2\pi i)^2} \oint \oint \frac{x_1^{4-2k}x_2^k dx_1 dx_2}{(x_1^4 - 3x_1^2x_2 + x_2^2)(-x_1^3 + 2x_1x_2)}. \quad (13)$$

Explicit computation for $k = 0, 1, 2$ gives $(\partial_2 w = 0$ for $x_2 = \frac{1}{2}x_1^2)$ $\langle x_1^4 \rangle = 2$, $\langle x_1^2x_2 \rangle = 1$ and $\langle x_2^2 \rangle = 1$, which agree with the Schubert calculus [3]. In the same way, we have computed the

²This is a natural parametrization for the powers of x_i 's since the total power sums up to $r(n-r)$, otherwise the correlators vanish.

correlators $I_k^n := \langle x_1^{2(n-2)-2k} x_2^k \rangle$ for $n = 5, 6, 7$, on $G_2(C^n)$ and $I_{k_1 k_2}^n := \langle x_1^{3(n-3)-2k_1-3k_2} x_2^{k_1} x_3^{k_2} \rangle$ on $G_3(C^n)$ for $n = 5, 6, 7$ and the results obtained are indicated in tables 1 and 2.

We have checked our computations using the property, $Res_W(H) = \mu$, where μ is the criticality index of W [4], *i.e.*, the dimension of chiral ring $R = \frac{C[x_i]}{dW_i}$. The above computations on $G_2(C^n)$ and $G_3(C^n)$ indicate that we should have $\langle x_1^{\alpha_1} x_2^{\alpha_2} \rangle_{G_2(C^{n+1})} = \langle x_1^{\alpha_1} x_2^{\alpha_2-1} \rangle_{G_2(C^n)}$, $\langle x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \rangle_{G_3(C^{n+1})} = \langle x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3-1} \rangle_{G_3(C^n)}$ and, in general,

$$\langle x_1^{\alpha_1} x_2^{\alpha_2} \dots x_r^{\alpha_r} \rangle_{G_r(C^{n+1})} = \langle x_1^{\alpha_1} x_2^{\alpha_2} \dots x_r^{\alpha_r-1} \rangle_{G_r(C^n)}$$

with $\sum_{i=1}^{r-1} \alpha_i = r(n-r)$. This is indeed the case as we shall show in the proposition below; but first we need the following lemma

lemma 1 *Given an inclusion $i:Z \hookrightarrow X$ (non singular subvariety) with $\dim_C X - \dim_C Z = r = n - m$ and suppose there exists a complex vector bundle E on X such that $E|_Z = N_{Z,X}$ (normal bundle of Z in X) and $\alpha \in H^{2m}(X) = H^{2n-2r}(X)$ then $i^*(\alpha) = \alpha X_r(E)$.*

For simplicity consider the case $G_2(C^n) \hookrightarrow G_2(C^{n+1})$, and let ${}^{n+1}Q$ and nQ denote the quotient subbundles on $G_2(C^{n+1})$ and $G_2(C^n)$ respectively, both of rank 2. Then the induced pullback gives $i^*({}^{n+1}Q) = {}^nQ$, furthermore ${}^{n+1}Q|_{G_2(C^n)} = N_{G_2(C^n), G_2(C^{n+1})}$. The above remarks on the intersection numbers on $G_2(C^n)$, $G_2(C^{n+1})$ computed by the residue pairing formula are equivalent to the following proposition:

proposition 1 *The correlators on $G_2(C^{n+1})$ and $G_2(C^n)$ are identical in the following sense*

$$\langle x_1({}^{n+1}Q)^{2n-4-2k} x_2({}^{n+1}Q)^{k+1} \rangle = \langle x_1({}^nQ)^{2n-4-2k} x_2({}^nQ)^k \rangle \quad (14)$$

Proof: setting $x_1({}^nQ)^{2n-4-2k} = U$, $x_1({}^{n+1}Q)^{2n-4-2k} = U'$, $x_2({}^nQ)^k = V^k$ and $x_2({}^{n+1}Q)^k = V'^k$, applying the above lemma; $i^(\alpha) = \alpha x_r({}^{n+1}Q)$ with $r = 2$ then $i^*(U'V'^k) = (U'V'^k)V' = U'V'^{k+1}$ and by using the homomorphism of the pullback, the lefthand side is $U.V$, hence the proof of the proposition.*

The above proposition can be generalized to correlators on $G_r(C^n)$, namely, we will have the following

$$\begin{aligned} \langle x_1({}^{n+1}Q)^{r(n-r)-2k_1-3k_2-\dots-rk_{r-1}} x_2({}^{n+1}Q)^{k_1} \dots x_r({}^{n+1}Q)^{k_{r-1}} \rangle = \\ \langle x_1({}^nQ)^{r(n-r)-2k_1-3k_2-\dots-rk_{r-1}} x_2({}^nQ)^{k_1} \dots x_r({}^nQ)^{k_{r-1}-1} \rangle. \end{aligned} \quad (15)$$

As a consequence of the proposition, we have a closed formula for the two point functions on $G_r(C^n)$, containing x_1 and x_r given by

$$\langle x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}} \rangle = (r(n - k_{r-1} - r))! \prod_{\ell=0}^{r-1} \frac{\ell!}{(n - k_{r-1} - \ell + 1)!}, \quad (16)$$

which is obtained from the self-intersection formula equation (12) simply by the shift $n \rightarrow n - k_{r-1}$. In particular, on $G_2(C^n)$, we have the following closed formula

$$\langle x_1^{2n-4-2k} x_2^k \rangle = \frac{(2(n-2-k))!}{(n-k-2)!(n-k-1)!}, \quad (17)$$

This particular case, that we denote by I_k^n , was also obtained using topological Kazama-Suzuki models based on complex Grassmannian [15]. If we set $k_{r-1} = n - r$ in equation (16) one obtains $\langle x_r^{n-r} \rangle = \int_{G_r(C^n)} x_r^{n-r} = 1$ as was shown in [9]. The closed formula given by equation (16) is consistent with the proposition above, since the formula is automatically invariant under the shifts $n \rightarrow n + 1$ and $k_r \rightarrow k_r + 1$, which in turn gives the two point function on $G_r(C^{n+1})$.

3 Intersection numbers on the space of holomorphic maps to a Grassmannian

Here and in the next section we will give two explicit formulas for the intersection numbers on the space of holomorphic maps of degree d from CP^1 into $G_r(C^n)$ for $r = 2, 3$. This space of maps is denoted by $Hol_d(CP^1 \rightarrow G_r(C^n))$, the space of instantons of degree d . The intersection numbers on $Hol_d(CP^1 \rightarrow G_r(C^n))$ will be computed using the deformed potential $\widetilde{W}_{n+1}(x_1, \dots, x_r) = W_{n+1}(x_1, \dots, x_r) + (-1)^r q x_1$, that reproduces the quantum cohomology $H_q^*(G_r(C^n), C) = \frac{C[x_1, \dots, x_r, q]}{(\partial \widetilde{W}_{n+1} / \partial x_1, \dots, \partial \widetilde{W}_{n+1} / \partial x_r)}$ [9]. This means that we will use formally the same formula for the intersections on the Grassmannian carried out in the last section, however, the objects inserted in the correlators are the pullbacks of the cohomology classes (Chern classes) to the parametrizing space of holomorphic maps of degree d . These will be denoted again by x_i ($1 \leq i \leq r$) such that the total power of the product of these classes is the dimension of $Hol_d(CP^1 \rightarrow G_r(C^n))$, which is, $r(n-r) + nd$ [16]. We will see that the intersection numbers on $Hol_d(CP^1 \rightarrow G_2(C^n))$ generates alternating Fibonacci numbers for $n = 5$, the Pell numbers for $n = 6$, and for $n \geq 7$ the intersection numbers generates a random walk of a particle on a line with absorbing barriers [10, 11]. The self-intersection formula for $\langle x_1^{2(n-2)+nd} \rangle$, which is a special case of our two-point function given below on $Hol_d(CP^1 \rightarrow G_2(C^n))$ agrees with that computed in [6] for $n = 5$. We have also checked the geometrical meaning of the quantum correction [17] associated with the topological σ -model on CP^1 with values in the Grassmannian $G_r(C^n)$, in which computing correlators on $Hol_d(CP^1 \rightarrow G_r(C^n))$ is equivalent to doing computations on $Hol_{d-1}(CP^1 \rightarrow G_r(C^n))$ provided we set $x_r y_{n-r} = 1$.

In the following, we first write the correlators on $Hol_d(CP^1 \rightarrow G_r(C^n))$ in terms of the Chern roots q_i , [9, 18, 19] then we will compute explicitly the formula for the intersection

numbers for $r = 2, 3$. The computations on $Hol_d(CP^1 \rightarrow G_3(C^n))$ are lengthy we will only give the final formula. The correlators on $Hol_d(CP^1 \rightarrow G_r(C^n))$ are given by

$$\langle x_1^{r(n-r)+nd-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}} \rangle = (-1)^{\frac{r(r-1)}{2}} \sum_{d\tilde{W}_{n+1}=0} \frac{x_1^{r(k-r)+nd-2k_1-\dots-rk_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}}}{h} \quad (18)$$

where the summation is over a finite number of critical points of $\tilde{W}_{n+1}(x_1, \dots, x_r)$ and $h = \det(\partial_i \partial_j \tilde{W})$. In terms of the Chern roots q_i , the potential is given by

$$\tilde{W}_{n+1}(q_i) = \sum_{i=1}^r \frac{q_i^{n+1}}{n+1} + (-1)^r q_i. \quad (19)$$

The hessian in terms of the q_i 's on the critical points [9] is

$$\det \left[\frac{\partial \tilde{W}_{n+1}}{\partial q_i \partial q_j} \right]_{d\tilde{w}_{n+1}=0} = \det \left[\frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j} \right] \Delta^2, \quad (20)$$

where $\Delta = \prod_{j < k} (q_j - q_k)$ is the Vandermonde determinant which is the Jacobian for the change of variables from q_i to x_j . Therefore, the hessian in terms of the Chern roots is given by

$$h(q_1, \dots, q_r) = \det \left[\frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j} \right] = \frac{n^r (q_1, \dots, q_r)^{n-1}}{\Delta^2}. \quad (21)$$

Since the Vandermonde determinant vanishes for $q_i = q_j$, the summation over the critical points given by equation (18) involves only distinct roots $q_i (1 \leq i \leq r)$ of the polynomial of degree n , of the form $d\tilde{W}_{n+1} = x^n + (-1)^r$ and hence the product of the roots satisfy the identity $(q_1 \dots q_r)^n = 1$. By using the facts $q_i^n = -1, i = 1, 2, x_1 = q_1 + q_2, x_2 = q_1 q_2$ for $r = 2$ and making the change of variables $q_i = \omega \xi_i$ with $\omega^n = -1, \xi_i^n = 1$, the two-point functions on $Hol_d(CP^1 \rightarrow G_2(C^n))$, that we denote by $I_k^{n,d}$, in terms of the new variables ξ_i are

$$\begin{aligned} \langle x_1^{2(n-2)+nd-2k} x_2^k \rangle &= - \sum_{d\tilde{W}=0} \frac{x_1^{2(n-2)+nd-2k} x_2^k}{h} \\ &= \frac{(-1)^{d+1}}{2n^2} \sum_{\xi_i^n=1, \xi_1 \neq \xi_2} [(\xi_1 + \xi_2)^2 - 4\xi_1 \xi_2] (\xi_1 + \xi_2)^{2(n-2)+nd-2k} (\xi_1 \xi_2)^{k+1}, \end{aligned} \quad (22)$$

where a factor $1/2$ was inserted in order to avoid overcounting, since the x_i 's are symmetric in the q_i 's. The restriction $\xi_1 \neq \xi_2$ can be lifted provided we subtract from the sum terms

with $\xi_1 = \xi_2$. In our case these terms do not contribute, therefore, we obtain

$$\begin{aligned} \langle x_1^{2(n-2)+nd-2k} x_2^k \rangle &= \frac{(-1)^{d+1}}{2} \frac{1}{n^2} \sum_{\xi_i^n=1} [(\xi_1 + \xi_2)^2 - 4\xi_1\xi_2] \times \\ &\quad (\xi_1 + \xi_2)^{2(n-2)+nd-2k} (\xi_1\xi_2)^{k+1} \end{aligned} \quad (23)$$

If we set $z = \xi_1\xi_2^{-1}$ in equation (23), then the above summation will be over a single n -th root of unity z , *i.e.*,

$$\begin{aligned} \langle x_1^{2(n-2)+nd-2k} x_2^k \rangle &= \frac{(-1)^{d+1}}{2} \frac{1}{n^2} \sum_{z^n=1} [(1+z)^2 - 4z] (1+z)^{2(n-2)+nd-2k} (z)^{k+1} \\ &= \frac{(-1)^{d+1}}{2} \frac{1}{n} \sum_{z^n=1} \left[\sum_{\ell \geq 0} \binom{2n-2+nd-2k}{\ell} z^{\ell+k+1} - 4 \sum_{\ell' \geq 0} \binom{2n-4+nd-2k}{\ell'} z^{\ell'+k+2} \right]. \end{aligned} \quad (24)$$

The summations of $z^{\ell+k+1}$, $z^{\ell'+k+2}$ over the n -th roots of unity are non-vanishing only if ³ $\ell+k+1 = nq$, $\ell'+k+2 = nq'$. Finally, explicit computation yields:

$$\langle x_1^{2(n-2)+nd-2k} x_2^k \rangle = \frac{(-1)^{d+1}}{2} \sum_{q \in \{1, 2, \dots\}} \left[\binom{2n-2+nd-2k}{qn-(k+1)} - 4 \binom{2n-4+nd-2k}{qn-(k+2)} \right]. \quad (25)$$

If we set $k = 0$ in the above formula then we obtain the explicit formula for the self-intersection on $Hol_d(CP^1 \rightarrow G_2(C^n))$, on the other hand, setting $d = 0$ gives the two point functions on $G_2(C^n)$ obtained in the previous section, equation (17). Using conformal field theory an expression for the self-intersection on $Hol_d(\Sigma_g \rightarrow G_2(C^5))$ was obtained [6] where Σ_g is a Riemann surface of genus g . When the genus $g = 0$, this formula can be written as follows:

$$\begin{aligned} F(0, d) &= \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5}+1}{2} \right)^{5(d+1)} + (-1)^d \left(\frac{\sqrt{5}-1}{2} \right)^{5(d+1)} \right] \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5}+1}{2} \right)^{5(d+1)} - \left(\frac{1-\sqrt{5}}{2} \right)^{5(d+1)} \right], \end{aligned} \quad (26)$$

which is the well known Binet's formula for the Fibonacci numbers $F_{5(d+1)}$, [20]. We have checked for many values of d that this formula agrees with ours, and therefore we should have the following mathematical identity on $Hol_d(CP^1 \rightarrow G_2(C^5))$:

$$\langle x_1^{6+5d} \rangle = \frac{(-1)^{d+1}}{2} \sum_{q \in \{1, 2, \dots\}} \left[\binom{8+5d}{5q-1} - 4 \binom{6+5d}{5q-2} \right] = F_{5(d+1)}. \quad (27)$$

³we have used the identity, $\sum_{z^n=1} z^r = n$, if $r \equiv 0 \pmod{n}$, and vanishing otherwise.

By an explicit computation for the two point functions on $Hol_d(CP^1 \rightarrow G_2(C^5))$ see table 3 for various k and for fixed d , one can see that we should have the identity

$$\langle x_1^{6+5d-2k} x_2^k \rangle = F_{5(d+1)-2k}. \quad (28)$$

The intersection numbers given by $F_{5(d+1)-2k}$ corresponds to the alternate Fibonacci numbers for $0 \leq k \leq [5(d+1)/2]$ with d fixed, and for $k = [5(d+1)/2]$ the intersection numbers are equal to one or zero depending on whether the degree of the holomorphic maps d is even or odd. This seems to hold for every n . For $n = 4$, the intersection numbers $Hol_d(CP^1 \rightarrow G_2(C^4))$ are powers of 2, as one can see from the table 3. For $n = 6$, we obtain two possible sequences of Pell numbers [10], when d is odd the general term is $\frac{3^m-1}{2}$, and the other sequence given by $\frac{3^m+1}{2}$ for even degree. In general, for $n \geq 7$, the intersection numbers generate a random walk with absorbing barriers [10, 5]. This is a one dimensional random walk, in which the particle starts at point 1 and arrives eventually at the point n , the particle may never visit 0, i.e. the points 0 and n are absorbing barriers this happens when the degree is odd. When the degree d is even, the intersection numbers generate a random walk on a line for a particle that starts at point $n-1$ see table 4.

4 The Correlators on $Hol_d(CP^1 \rightarrow G_3(C^n))$

In computing all the correlators on $Hol_d(CP^1 \rightarrow G_3(C^n))$ that we denote by $I_{k_1, k_2}^{n, d}$ we follow the same technique as for the two-point functions computed in section 3. Using equations (18), (21) and after some algebra, the correlators can be written as

$$\begin{aligned} \langle x_1^{3(n-3)+nd-2k_1-3k_2} x_2^{k_1} x_3^{k_2} \rangle &= -\frac{1}{n^3} \sum_{q^n=1, i=1,2,3} (q_1 - q_2)^2 (q_2 - q_3)^2 (q_1 - q_3)^2 \times \\ &\quad (q_1 + q_2 + q_3)^{3(n-3)+nd-2k_1-3k_2} (q_1 q_2 + q_1 q_3 + q_2 q_3)^{k_1} (q_1 q_2 q_3)^{k_2+1} \\ &= -\frac{1}{6} \sum_{p, q \in \{1, 2, \dots\}} \sum_{s, t=0}^2 (-1)^{s+t} \binom{2}{s} \binom{2}{t} \sum_{\ell'=0}^{k_1} \sum_{\ell''=0}^{\ell'} \binom{k_1}{\ell'} \binom{\ell'}{\ell''} \times \\ &\quad \frac{x!}{(x-y)!} \left[\frac{1}{(z)!(w)!} + \frac{1}{(z+2)!} - \frac{2}{(z+1)(w-1)!} \right], \end{aligned} \quad (29)$$

where

$$\begin{aligned} w &= qn - (\ell' + k_2 + t + 1), \\ x &= 3(n-3) + nd - 2k_1 - 3k_2, \\ y &= (p+q)n - (\ell' + \ell'' + k_1 + 2k_2 + 4 + s + t), \\ z &= pn - (k_1 + k_2 + s + 3 - \ell''). \end{aligned} \quad (30)$$

If we set $d = 0$, $k_1 = k_2 = 0$ in the above formula, then we obtain the number

$$I_{0,0}^{n,0} = \frac{2(3(n-3))!}{(n-3)!(n-2)(n-1)!},$$

which is the self-intersection formula for $\langle x_1^{3(n-3)} \rangle$ on $G_3(C^n)$ given by equation (12). We also have checked that the above formula for $d = 0$ gives the intersections numbers on $G_3(C^5)$ and $G_3(C^6)$ and therefore setting $d = 0$ in equation (29) we obtain the formula for the intersection numbers I_{k_1, k_2}^n , on $G_3(C^n)$.

Let us check the implication of the the geometrical meaning of the quantum correction [17] using our formula, equation (29). As was mentioned in the beginning of this section, the quantum correction implies that correlators on $Hol_d(CP^1 \rightarrow G_3(C^n))$ are identical to those on $Hol_{d-1}(CP^1 \rightarrow G_3(C^n))$ provided we set $x_3 y_{n-3} = 1$. This can be seen by considering the following simple example: suppose we want to evaluate the correlator $\langle x_1^7 x_2 x_3 y_3 \rangle_{d=1}$, where $y_3 = x_1^3 - 2x_1 x_2 + x_3$. Then using the results indicated in table 5, where $I_{1,1}^{6,1} = 171$, $I_{2,1}^{6,1} = 86$ and $I_{1,2}^{6,1} = 22$ we have $\langle x_1^7 x_2 x_3 y_3 \rangle_{d=1} = 21$ which is $\langle x_1^7 x_2 \rangle_{d=0} = I_{1,0}^6$ see table 2.

5 Conclusion

In section two, we obtained a closed formula for the two-point functions $\langle x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}} \rangle$ on $G_r(C^n)$ given by equation (16). When we set $r = 2$, two-point functions on $G_2(C^n)$ are obtained. Also, all the correlators $\langle x_1^{3(n-r)-2k_1-3k_2} x_2^{k_1} x_3^{k_2} \rangle$ on $G_3(C^n)$ are obtained by restricting our formula on the space of holomorphic maps of degree d to constant maps, i.e., $d = 0$. The closed formulas, obtained here are extensions of the Schubert formula equation (12) that computes the self-intersection $\langle x_1^{r(n-r)} \rangle$.

In section three, we obtained an explicit formula for the two-point functions on the space of holomorphic maps of degree d from CP^1 into $G_2(C^n)$. This formula generates well known numbers like the Fibonacci numbers for $n = 5$ and the Pell numbers for $n = 6$ [10]. However, when $n \geq 7$ the formula generates a random walk of a particle on line with absorbing barriers [10, 5] that starts at the point 1 and eventually reaches the point n , if the d is odd. When d is even the particle starts at the point $n - 1$ see table 4 and [11]. At the moment, we do not understand this connection. It would be nice if this can be understood from both mathematics and physics.

Also on the moduli space of holomorphic maps into $G_r(C^n)$, we have computed intersection numbers using equation (29), these numbers follow well organized patterns for given n . For $n = 5$, the intersection numbers are given by the Fibonacci numbers, this is expected since $G_3(C^5)$ and $G_2(C^5)$ are dual to each other. Setting $n = 6$, one obtains a pattern like that in tables 5 and 6 for any degree d . For $n = 7$, we have found sequences of numbers such

that the ratio of any consecutive numbers behave like that of L_n/F_n , the n^{th} Lucas number by the n^{th} Fibonacci number, which is known to be $\sqrt{5}$.

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References

- [1] W.V.D. Hodge and D. Pedoe *Methods of Algebraic Geometry* **2**, Cambridge Univ. Press, (1952).
- [2] P. Griffiths and J. Harris, *principle of Algebraic Geometry*, J. Wiley (1978).
- [3] W. Fulton *Intersection Theory*, Springer -Verlag Berlin Heidelberg (1984).
- [4] C. Vafa *Mod. Phys. Lett.* **A6**, 337 (1991).
- [5] N. Chair *Grassmannian Cohomology Rings and Fusion Rings from Algebraic Equations* *hep-th/9704138*
- [6] K. Intriligator *Mod. Phys. Lett.* **A6** 3543 (1991).
- [7] C. Vafa *Topological Mirrors and Quantum Rings, in Essays on Mirrors Manifolds, ed. S-T. Yau* (International Press, 1992).
- [8] E. Witten *Nucl. Phys.* **B340**, 281 (1990).
- [9] E. Witten, *The Verlinde Algebra and the Cohomology of the Grassmannian*, preprint IASSNS-HEP 93/41, December (1993).
- [10] N.J.A. Sloane and Simon Plouffe, *The Encyclopidia of Integer Sequences* Academic Press (1995).
- [11] C. J. Everett and P.R. Stein *Discrete Mathematics* **17** 27 (1977).
- [12] W. Lerche, C. Vafa and N. Warner, *Nuc. Phys.* **B324** 427-474 (1989).
- [13] D. Gepner, *Comm. Math. Phys.* **141** 381-411 (1991).

- [14] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer (1982).
- [15] M. Blau, F. Hussein, G. Thompson *Nucl. Phys.* **B488** 599 (1997).
- [16] S. Stromme *On Parametrized Rational Curves in Grassmannian Varieties* In space curve eds. F. Ghione et al. Springer -Verlag Lecture Notes in Mathematics (springer, 1987).
- [17] D. Franco and C. Reina *Comm. Math. Phys.* **174** 137 (1995).
- [18] A. Bertram, G. Daskalopoulos and R. Wentworth, *Gromov Invariant for Holomorphic Maps from Riemann Surfaces to a Grassmannian*, Preprint (April, 1993).
- [19] M.S. Ravi, J.Rosenthal and X. Wang, *Degree of the Generalized Plucker Embedding of a Quote Scheme and Quantum Cohomology* Preprint (1993).
- [20] T.J. Rivlin, *The Chebyshev Polynomials*, 2nd edition - J. Wiley New York (1990).

| | | | | | | | |
|---|---|----|----|---|---|---|---|
| | k | 0 | 1 | 2 | 3 | 4 | 5 |
| n | | | | | | | |
| 4 | | 2 | 1 | 1 | | | |
| 5 | | 5 | 2 | 1 | 1 | | |
| 6 | | 14 | 5 | 2 | 1 | 1 | |
| 7 | | 42 | 14 | 5 | 2 | 1 | 1 |

Table 1: Intersection numbers I_k^n on $G_2(C^n)$ for $n = 4, 5, 6, 7$

| (k_1, k_2) | I_{k_1, k_2}^5 | (k_1, k_2) | I_{k_1, k_2}^6 | (k_1, k_2) | I_{k_1, k_2}^7 |
|--------------|------------------|--------------|------------------|--------------|------------------|
| (0,0) | 5 | (0,0) | 42 | (0,0) | 462 |
| (1,0) | 3 | (1,0) | 21 | (1,0) | 210 |
| (2,0) | 2 | (2,0) | 11 | (2,0) | 98 |
| (3,0) | 1 | (3,0) | 6 | (3,0) | 47 |
| (1,1) | 1 | (0,1) | 5 | (0,1) | 42 |
| (0,1) | 1 | (4,0) | 3 | (4,0) | 23 |
| (0,2) | 1 | (1,1) | 3 | (1,1) | 21 |
| | | (2,1) | 2 | (5,0) | 11 |
| | | (0,2) | 1 | (2,1) | 11 |
| | | (1,2) | 1 | (3,1) | 6 |
| | | (0,3) | 1 | (6,0) | 5 |
| | | | | (0,2) | 5 |
| | | | | (4,1) | 3 |
| | | | | (1,2) | 3 |
| | | | | (2,2) | 2 |
| | | | | (0,3) | 1 |
| | | | | (0,4) | 1 |
| | | | | (1,3) | 1 |
| | | | | (3,2) | 1 |

Table 2: Intersection numbers I_{k_1, k_2}^n on $G_3(C^n)$ for $n = 5, 6, 7$

| k | $I_k^{4,3}$ | $I_k^{4,4}$ | $I_k^{5,2}$ | $I_k^{5,3}$ | $I_k^{6,2}$ | $I_k^{6,3}$ |
|----|-------------|-------------|-------------|-------------|-------------|-------------|
| 0 | 128 | 512 | 610 | 6765 | 9842 | 265720 |
| 1 | 64 | 256 | 233 | 2584 | 3281 | 88573 |
| 2 | 32 | 128 | 89 | 987 | 1094 | 29524 |
| 3 | 16 | 64 | 34 | 377 | 365 | 9841 |
| 4 | 8 | 32 | 13 | 144 | 122 | 3280 |
| 5 | 4 | 16 | 5 | 55 | 41 | 1093 |
| 6 | 2 | 8 | 2 | 21 | 14 | 364 |
| 7 | 1 | 4 | 1 | 8 | 5 | 121 |
| 8 | 0 | 2 | 1 | 3 | 2 | 40 |
| 9 | | 1 | | 1 | 1 | 13 |
| 10 | | 1 | | 0 | 1 | 4 |
| 11 | | | | | | 1 |
| 12 | | | | | | 0 |

Table 3: Intersection numbers $I_k^{n,d}$ for $n = 4, d = 3, 4$; $n = 5, d = 2, 3$ and $n = 6, d = 2, 3$

| k | $I_k^{7,1}$ | $I_k^{7,2}$ | $I_k^{8,1}$ | $I_k^{9,1}$ | $I_k^{10,1}$ |
|----|-------------|-------------|-------------|-------------|--------------|
| 0 | 2380 | 147798 | 15504 | 100947 | 657800 |
| 1 | 728 | 45542 | 4488 | 28101 | 177859 |
| 2 | 221 | 14041 | 1288 | 7752 | 47562 |
| 3 | 66 | 4334 | 364 | 2108 | 12597 |
| 4 | 19 | 1341 | 100 | 560 | 3264 |
| 5 | 5 | 413 | 26 | 143 | 820 |
| 6 | 1 | 131 | 6 | 34 | 196 |
| 7 | 0 | 42 | 1 | 7 | 43 |
| 8 | | 14 | 0 | 1 | 8 |
| 9 | | 5 | | 0 | 1 |
| 10 | | 2 | | | 0 |
| 11 | | 1 | | | |
| 12 | | 0 | | | |

Table 4: Intersection numbers $I_k^{n,d}$ for $n = 7, d = 1, 2$ and $n = 8, 9, 10, d = 1$

| k_1 | k_2 | 0 | 1 | 2 | 3 | 4 |
|-------|-------|------|-----|----|---|---|
| 0 | | 2730 | 341 | 43 | 6 | 1 |
| 1 | | 1365 | 171 | 22 | 3 | 0 |
| 2 | | 683 | 86 | 11 | 1 | 0 |
| 3 | | 342 | 43 | 5 | 0 | |
| 4 | | 171 | 21 | 2 | | |
| 5 | | 85 | 10 | | | |
| 6 | | 42 | 5 | | | |
| 7 | | 21 | | | | |

Table 5: Intersection numbers $I_{k_1, k_2}^{6,1}$

| k_1 | k_2 | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|-------|-------|-------|------|-----|----|---|
| 0 | | 17476 | 21845 | 2731 | 342 | 43 | 5 |
| 1 | | 87381 | 10923 | 1366 | 171 | 21 | 2 |
| 2 | | 43691 | 5462 | 683 | 85 | 10 | 1 |
| 3 | | 21846 | 2731 | 341 | 42 | 5 | 1 |
| 4 | | 10923 | 1365 | 170 | 21 | 3 | |
| 5 | | 5461 | 682 | 85 | 11 | | |
| 6 | | 2730 | 341 | 43 | 6 | | |
| 7 | | 1365 | 171 | 22 | | | |
| 8 | | 683 | 86 | | | | |
| 9 | | 342 | 43 | | | | |
| 10 | | 171 | | | | | |

Table 6: Intersection numbers $I_{k_1, k_2}^{6,2}$