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Bang-Bang Property for Bolza Problems in Two-Dimensions

Graziano Crasta and Benedetto Piccoli

S.I.S.S.A. – Via Beirut 4, Trieste 34014, ITALY

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Abstract

Consider the following Bolza problem:

$$\dot{x} = F(x) + u\, G(x), \qquad |u| \leq 1,\ x \in \Omega \subset I\!R^2$$

$$\min \int h(x,u)$$

$$x(0) = x_0, \quad x(1) = x_1$$

We show that, under suitable assumptions on F, G, h, all optimal trajectories are bang–bang.

The proof relies on a geometrical approach that works for every smooth two–dimensional manifold.

As a corollary we obtain existence results for non–convex optimization problems.

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Via Beirut 2-4 - 34013 Trieste - ITALY

1. Introduction.

The minimum time problem for control systems with control appearing linearly on a two dimensional manifold were studied by Sussmann in [Su1,2,3]. In particular in [Su1] there is a deep analysis of the structure of time optimal trajectories and conditions to assure that every time optimal trajectory is bang–bang are given. In the same paper it is shown that the bang–bang property is not generic even for the minimum time problem.

In this paper we consider the more general minimization problem:

$$\dot{x} = F(x) + u\, G(x), \qquad x \in I\!R^2,\ |u| \leq 1$$

$$(P) \qquad x(0) = x_0, \quad x(1) = x_1$$

$$\min \int_0^1 h(x(t), u(t))\, dt$$

where F, G are C^2 vector fields, $x_0, x_1 \in I\!R^2$ and $h(\cdot, +1), h(\cdot, -1) \in C^2(I\!R^2, I\!R)$, and we give conditions to guarantee that any optimal trajectory is bang-bang.

The proof of our result is based on the idea of "bang-bang" variations. More precisely, we use these variations as substitutes for those based on Liapunov's Convexity Theorem as is done in problems of the Calculus of Variations [CC] or of Linear Control Theory [R].

We consider an optimal trajectory γ of our system. If γ is not bang-bang in a neighborhood of a point x we construct a new trajectory that is near the previous one, is bang-bang in a neighborhood of x and achieves a lower cost. To assure that this new trajectory satisfies the initial condition we use the fact that, under suitable assumptions, all time optimal trajectory are bang-bang.

Problem (P) has a particular dynamics with respect to the usual problems of calculus of variations but on the other hand the assumption on h are not too strict.

The tools used in this paper are geometric, so it is easy to show that the statements hold also for a smooth two dimensional manifold.

As corollaries we prove existence results for nonconvex optimization problem.

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2. Basic Definitions.

Throughout this paper we use the symbol $I\!R^n$ to denote the n-dimensional Euclidean space.

With $m(A)$ we denote the Lebesgue measure of the set A. We say that a point t is a Lebesgue point for the set $A \subset I\!R$ if

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon}\, m(A \cap [t-\varepsilon, t+\varepsilon]) = 1.$$

It is well known [Ru] that if A is Lebesgue–measurable, then almost every point of A is a Lebesgue point.

A curve in $\mathbb{R}^2$ is a continuous map $\gamma : I \to \mathbb{R}^2$, where $I = [a, b] \subset \mathbb{R}^2$ is an interval. We use the symbol $\restriction$ to denote the restriction, e.g. $\gamma \restriction [a_0, b_0]$.

A vector field X on $\mathbb{R}^2$ is an $\mathbb{R}^2-$valued function. Every vector field X can be written in the form:

$$X = \alpha\partial_x + \beta\partial_y \tag{2.1}$$

where ∂_x, ∂_y are the vector fields with components $(1,0),(0,1)$, respectively, and $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}$. If X admits the representation (2.1) we denote by $J(X)$ the matrix:

$$J(X) := \begin{pmatrix} \partial_x\alpha & \partial_y\alpha \\ \partial_x\beta & \partial_y\beta \end{pmatrix}.$$

The Lie-bracket of two vector fields X, Y is the vector fields defined by:

$$[X, Y] = J(X) \cdot Y - J(Y) \cdot X.$$

A control u is a measurable function $u : [a, b] \to \mathbb{R}$ and a trajectory for u is an absolutely continuous curve γ such that:

$$\dot{\gamma}(t) = F(\gamma(t)) + u(t)G(\gamma(t))$$

for almost all t.

Given a trajectory $\gamma : [a, b] \to \mathbb{R}^2$, of a control u, and a cost function h we denote $T(\gamma)$ the time along γ, i.e. $b - a$, and by $\Gamma_h(\gamma)$ the cost of γ :

$$\Gamma_h(\gamma) := \int_a^b h(\gamma(t), u(t))\ dt.$$

We say that γ is time optimal if $T(\gamma) \leq T(\gamma')$ for every trajectory γ' that steers $\gamma(a)$ to $\gamma(b)$. A *bang-bang* trajectory is a trajectory corresponding to a control u such that $|u(t)| = 1$ for almost all t.

If $\gamma_1 : [a, b] \to \mathbb{R}^2, \gamma_2 : [b, c] \to \mathbb{R}^2$ are trajectories such that $\gamma_1(b) = \gamma_2(b)$ then $\gamma_2 * \gamma_1$ is the trajectory:

$$(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(t), & t \in [a, b] \\ \gamma_2(t), & t \in [b, c]. \end{cases}$$

An *admissible pair* is a pair (γ, u) such that $u : [0, 1] \to [-1, +1]$ is a control and γ is a trajectory of u such that:

$$\gamma(0) = x_0 \qquad \gamma(1) = x_1.$$

We say that (γ, u) is *optimal* (or simply that γ is optimal) if $\Gamma_h(\gamma) \leq \Gamma_h(\gamma')$ for all admissible pair (γ', u').

Given the problem (P) we define the function:

$$\Delta_A := \det(F, G) \tag{2.3a}$$

$$\Delta_B := \det(G, [F, G]) \tag{2.3b}$$

$$\Delta_C := \det(F, [F, G]) \tag{2.3c}$$

where "det" stands for determinant. Writing F, G, $[F, G]$ as column vectors we can form a 2×2 matrix having two of them as columns. The functions (2.3a,b) has been introduced in [Su1]. For simplicity we use the notations:

$$X := F - G, \qquad Y := F + G$$

Let us define:

$$m(x) := \frac{h(x, +1) - h(x, -1)}{2} \qquad q(x) := \frac{h(x, +1) + h(x, -1)}{2} \tag{2.4a}$$

$$l(x, u) := m(x)u + q(x) \tag{2.4b}$$

i.e. $l(x, u)$ is the linear interpolation between $h(x, +1)$ and $h(x, -1)$.

The assumptions on (P) will be:

$$(H.1) \qquad \Delta_A(x)\ \Delta_B(x) \neq 0 \qquad \forall x \in I\!R^2$$

$$(H.2) \qquad h(x, u) \geq l(x, u) \qquad \forall x \in I\!R^2,\ \forall u \in [-1, 1].$$

In [Su1] was proved that under the assumption (H.1) all time optimal trajectory are bang-bang.

Consider the two trajectories $\gamma_1 : [a, b] \to I\!R^2$, $\gamma_2 : [b, c] \to I\!R^2$ such that $\gamma_1(b) = \gamma_2(b)$ and $\gamma_2^{-1} * \gamma_1$ is a simple closed curve oriented counterclockwise, where γ_2^{-1} is γ_2 run backwards. From (H.1) it follows that F, G are independent at each point, and then we can define the two 1-differential forms ω, ω_l in the following way:

$$\langle \omega(x), F(x) \rangle = 1 \qquad \langle \omega(x), G(x) \rangle = 0 \tag{2.5a}$$

$$\langle \omega_l(x), F(x) \rangle = q(x) \qquad \langle \omega_l(x), G(x) \rangle = m(x). \tag{2.5b}$$

Following [Su1] we have, by Stokes' theorem:

$$T(\gamma_1) - T(\gamma_2) = \int_{\gamma_2^{-1} * \gamma_1} \omega = \int_{\mathcal{R}} d\omega \tag{2.6a}$$

$$\Gamma_l(\gamma_1) - \Gamma_l(\gamma_2) = \int_{\gamma_2^{-1} * \gamma_1} \omega_l = \int_{\mathcal{R}} d\omega \tag{2.6a}$$

where $\mathcal{R}$ is the region enclosed by $\gamma_2^{-1} * \gamma_1$. Being F, G independent we can write:

$$[F, G](x) = f(x)F(x) + g(x)G(x)$$

then from (2.3) one has:

$$\Delta_B = \det(G, fF + gG) \quad \Rightarrow \quad f = -\frac{\Delta_B}{\Delta_A} \tag{2.7a}$$

$$\Delta_C = \det(F, fF + gG) \quad \Rightarrow \quad g = \frac{\Delta_C}{\Delta_A}. \tag{2.7b}$$

We have:

$$\langle d\omega, F \wedge G\rangle = F\langle \omega, G\rangle - G\langle \omega, F\rangle - \langle \omega, [F, G]\rangle$$

$$d\omega = \rho\, dx \wedge dy \qquad \rho = \langle d\omega, \partial_x \wedge \partial_y\rangle = \frac{1}{\Delta_A}\langle d\omega, F \wedge G\rangle$$

and the same for ω_l. With straightforward calculations from (2.4,5,6,7) we obtain:

$$T(\gamma_1) - T(\gamma_2) = \int_{\mathcal{R}} \varphi\, dx \wedge dy \tag{2.8a}$$

$$\Gamma_l(\gamma_1) - \Gamma_l(\gamma_2) = \int_{\mathcal{R}} \varphi_l\, dx \wedge dy \tag{2.8b}$$

where:

$$\varphi := \frac{\Delta_B}{\Delta_A^2} \tag{2.9}$$

$$\varphi_l := \frac{\nabla m \cdot F - \nabla q \cdot G}{\Delta_A} + \frac{\dot{q}\, \Delta_B - m\, \Delta_c}{\Delta_A^2}. \tag{2.10}$$

Given $f_1, f_2 : I\!R^2 \to I\!R$ we define a vector field by:

$$[f_1 | f_2](x) := f_1(x)\, \nabla f_2(x) - f_2(x)\, \nabla f_1(x). \tag{2.11}$$

Define:

$$S_{AB} := \operatorname{sgn}(\Delta_A\, \Delta_B) \tag{2.12}$$

$$\mathcal{K}(x) := \{v \in I\!R^2 : S_{AB}\, v \cdot X(x) < 0 \text{ or } S_{AB}\, v \cdot Y(x) > 0\} \tag{2.13}$$

(note that from (H.1) S_{AB} is independent of x). The last assumption for (P) is:

$$(H.3) \qquad D(x) := [\varphi | \varphi_l](x) \in \mathcal{K}(x) \qquad \forall x \in I\!R^2.$$

3. The Main Result.

In this section we will prove the main result; the key role is played by the following lemma.

Lemma 3.1. Let (γ, u) be an admissible pair of

$$\dot{x} = F(x) + u\, G(x) \tag{3.1}$$

and let

$$E := \{t \in [0,1] \ : \ |u(t)| < 1\}$$

Suppose that there exists a Lebesgue point τ for E of differentiability for γ. If (H.1)–(H.2) hold and $D(\overline{x}) \in \mathcal{K}(\overline{x})$, $\overline{x} := \gamma(\tau)$, then there exist $\varepsilon > 0$, $\sigma > 0$ and an admissible pair $(\hat{\gamma}, \hat{u})$ such that:

$$\hat{\gamma}(t) = \gamma(t) \qquad \text{for } t \notin [\tau - \varepsilon, \tau + \sigma]$$

$$|\hat{u}(t)| = 1 \qquad \text{for } t \in [\tau - \varepsilon, \tau + \sigma]$$

$$\int_0^1 h(\gamma(t), u(t))\, dt \ > \ \int_0^1 h(\hat{\gamma}(t), \hat{u}(t))\, dt$$

Proof. Now we will construct a pair of admissible trajectories $\gamma^{\pm}$, that are bang–bang in a neighborhood of τ, and correspond to controls $u^{\pm}$ that change sign two times in this neighborhood. This is possible thanks to (H.1); by (H.2–3) we will show that one of these trajectories achieves a better performance with respect to γ.

Fix $0 < \varepsilon < \min(\tau, 1 - \tau)$. If ε is sufficiently small, there exist $\sigma_1^{\pm} \geq \tau - \varepsilon$ and $\tau^{\pm} \geq \sigma_1^{\pm}$, depending on ε, two one–parameter families of pairs $(\gamma_\mu^{\pm}, u_\mu^{\pm})$, $\mu \in [\tau^{\pm}, 1]$, $\sigma_2^{\pm} = \sigma_2^{\pm}(\varepsilon, \mu) \in [\mu, 1]$, $\sigma_3^{\pm} = \sigma_3^{\pm}(\varepsilon, \mu) \in [\tau, 1]$ such that:

$$\gamma_\mu^{\pm}(\tau - \varepsilon) = \gamma(\tau - \varepsilon)$$

$$\gamma_\mu^{\pm}(\tau^{\pm}) = \gamma(\tau)$$

$$\gamma_\mu^{\pm}(\sigma_2^{\pm}) = \gamma(\sigma_3^{\pm})$$

$$u_\mu^{\pm}(s) = \pm 1, \qquad \text{for } s \in [\tau - \varepsilon, \sigma_1^{\pm}) \cup (\mu, \sigma_2^{\pm}]$$

$$u_\mu^{\pm}(s) = \mp 1, \qquad \text{for } s \in [\sigma_1^{\pm}, \mu]$$

Let $A^{\pm}$, $B_\mu^{\pm}$ be the regions enclosed respectively by $(\gamma_\mu^{\pm} \restriction [\tau - \varepsilon, \tau^{\pm}])^{-1} * \gamma \restriction [\tau - \varepsilon, \tau]$ and $(\gamma_\mu^{\pm} \restriction [\tau^{\pm}, \sigma_2^{\pm}])^{-1} * \gamma \restriction [\tau, \sigma_3^{\pm}]$. Define the functions $\psi^{\pm}(\mu) := \pm \mathrm{sgn}\,(\Delta_A \cdot \Delta_B)[T(\gamma) - T(\gamma_\mu^{\pm})]$. By (2.8) we have:

$$\psi^{\pm}(\mu) = \mathrm{sgn}\,(\Delta_B) \cdot \left[\int_{A^{\pm}} \varphi\, dx - \int_{B_\mu^{\pm}} \varphi\, dx \right] \tag{3.2}$$

Infact if $\Delta_A > 0$ we have $\det(X, Y) > 0$, and the regions A^+ and B^- are enclosed by curves oriented counterclockwise, while A^- and B^+ are enclosed by curves oriented clockwise; the opposite happens if $\Delta_A < 0$.

Moreover $\psi^{\pm}(\tau^{\pm}) > 0$, $\psi^{\pm}$ is decreasing, and for ε sufficiently small there exists $\mu^{\pm} \in [\tau^{\pm}, 1]$ such that $\psi^{\pm}(\mu^{\pm}) = 0$. Define the admissible pairs $(\gamma^{\pm}, u^{\pm})$ as follows:

$$\gamma^{\pm}(t) := \begin{cases} \gamma^{\pm}_{\mu^{\pm}}, & \text{if } t \in I^{\pm} := [\tau - \varepsilon, \sigma_3^{\pm}(\mu^{\pm})] \\ \gamma(t), & \text{if } t \in [0,1] \backslash I^{\pm} \end{cases}$$

$$u^{\pm}(t) := \begin{cases} u^{\pm}_{\mu^{\pm}}, & \text{if } t \in I^{\pm} \\ u(t), & \text{if } t \in I^{\pm} \end{cases}$$

Define $B^{\pm} := B^{\pm}_{\mu^{\pm}}$; writing $\varphi(x) = \varphi(\overline{x}) + \nabla\varphi(\overline{x}) \cdot (x - \overline{x}) + o(|x - \overline{x}|)$ we obtain:

$$0 = \int_{A^{\pm}} \varphi \, dx - \int_{B^{\pm}} \varphi \, dx = \left[m(A^{\pm}) - m(B^{\pm})\right] \cdot \varphi(\overline{x}) + \nabla\varphi(\overline{x}) \cdot d^{\pm} + o(\varepsilon^3) \tag{3.3}$$

where $d^{\pm} = d^{\pm}(\varepsilon) := m(A^{\pm})b(A^{\pm}) - m(B^{\pm})b(B^{\pm})$ and $b(\Omega) := \frac{1}{m(\Omega)} \int_{\Omega} (x - \overline{x}) \, dx$ denotes the baricentre of Ω relative to $\overline{x}$. Expanding φ_l we obtain in the same way:

$$\delta^{\pm} := \Gamma_l(\gamma) - \Gamma_l(\gamma^{\pm}) = \pm \text{sgn}\,(\Delta_A) \cdot \left\{ \left[m(A^{\pm}) - m(B^{\pm})\right] \varphi_l(\overline{x}) + \nabla\varphi_l(\overline{x}) \cdot d^{\pm} \right\} + o(\varepsilon^3) \tag{3.4}$$

By (3.3) and multiplying (3.4) by $\varphi(\overline{x})$ one has:

$$\varphi(\overline{x})\delta^{\pm} = \pm \text{sgn}\,(\Delta_A) \, [\varphi|\varphi_l](\overline{x}) \cdot d^{\pm} + o(\varepsilon^3)$$

where $[\cdot|\cdot]$ is defined in (2.11). Define the cone:

$$\mathcal{C}(v, w) := \{\lambda v + \mu w \mid \lambda \geq 0, \ \mu \geq 0\}$$

For ε sufficiently small, we have that $d^{\pm} \in \mathcal{C}(-X(\overline{x}), -Y(\overline{x}))$; moreover sgn $[\det(d^+, d^-)] =$ sgn (Δ_A), and then

$$\{v \in I\!R^2 \mid S_{AB} \, v \cdot d^{\pm} \leq 0\} \cap \mathcal{K}(\overline{x}) = \emptyset$$

Being sgn $(\Delta_B) =$ sgn (φ), it follows that either $\delta^+ > 0$ or $\delta^- > 0$. Suppose, for example, that $\delta^+ > 0$. Then, from (H.2) and (2.4), one has:

$$\int_{\gamma} h(x,u) \geq \int_{\gamma} l(x,u) > \int_{\gamma^+} l(x,u) = \int_{\gamma^+} h(x,u)$$

Q.E.D.

From this lemma one obtains immediately the following:

Theorem 3.2. If (H) holds, then all optimal trajectories of (P) are bang–bang.

Proof. Let (γ, u) be an optimal pair for (P), and suppose that $m(E) > 0$. Almost every $\tau \in E$ is a Lebesgue point for E and a point of differentiability for γ; if τ is such a point, all the assumptions of Lemma 3.1 are satisfied, and then γ is not optimal.

Q.E.D.

4. Examples and Applications.

In this section we give two examples of minimization problems without the bang–bang property and some applications of the main result.

Example 1. Consider the following problem in $\Omega := \{(x_1, x_2) \in I\!R^2 \mid x_2 < 0\}$:

$$(P_1) \qquad \begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -x_2^2 - x_2 + u \end{cases}$$

$$\min \int_0^1 u$$

$$x(0) = \left(0, -\frac{1}{2}\right), \quad x(1) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$$

i.e. problem (P) with

$$F(x) = \begin{pmatrix} x_2 \\ -x_2^2 - x_2 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h(x,u) = u$$

There is a unique optimal pair (γ, u),

$$\gamma(t) = \left(-\frac{t}{2}, -\frac{1}{2}\right), \quad u(t) = -\frac{1}{4}, \qquad t \in [0,1]$$

with a cost $\Gamma(\gamma) = -\frac{1}{4}$. By a straightforward calculation we have that $\Delta_A = x_2$, $\Delta_B = 1$, $\Delta_C = x_2^2$, $S_{AB} = -1$, $m(x) = 1$, $q(x) = 0$.

Conditions (H.1) and (H.2) are satisfied in Ω. We now check that (H.3) does not hold. Recalling (2.9, 10) it is easy to verify that:

$$\varphi = \frac{1}{x_2^2}, \quad \varphi_l = -1, \quad D = \begin{pmatrix} 0 \\ -\frac{2}{x_2^3} \end{pmatrix}$$

$$X = \begin{pmatrix} x_2 \\ -x_2^2 - x^2 - 1 \end{pmatrix}, \quad Y = \begin{pmatrix} x_2 \\ -x_2^2 - x_2 + 1 \end{pmatrix}$$

$$S_{AB}\, D \cdot X = -2\, \frac{x_2^2 + x_2 + 1}{x_2^3}, \qquad S_{AB}\, D \cdot Y = -2\, \frac{x_2^2 + x_2 - 1}{x_2^3}$$

Being $S_{AB}\, D(\gamma(t)) \cdot X(\gamma(t)) > 0$ and $S_{AB}\, D(\gamma(t)) \cdot Y(\gamma(t)) < 0$ for every $t \in [0,1]$, we have that $D(\gamma(t)) \notin \mathcal{K}(\gamma(t))$ for every $t \in [0,1]$.

Let us rewrite problem (P_1) in Mayer form, introducing the new variable x_3 such that $\dot{x_3} = u$, and consider the two–dimensional system:

$$(P_2) \qquad \begin{cases} \dot{x_2} = -x_2^2 - x_2 + u \\ \dot{x_3} = u \end{cases} ;$$

Being this system independent of x_1, we have that every optimal trajectory of (P_1) gives a time–optimal trajectory of (P_2). The time–optimal trajectory associated to γ in the (x_2, x_3)–plane is $\tilde{\gamma}(t) = \left(-\frac{1}{2}, -\frac{t}{4}\right)$. Using the same terminology of [Su1], $\tilde{\gamma}$ is a turnpike; in the same paper it was shown that, under generic conditions, these are the only cases of not bang–bang time–optimal trajectories.

Example 2. Consider now the problem in $\Omega := \{(x_1, x_2) \in I\!R^2 \mid x_2 < 0\}$:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \frac{1}{2}x_2 + u\, x_2 \end{cases}$$

$$\min \int_0^1 x_2^2 + x^2$$

$$x(0) = \left(0, -\frac{1}{2}\right), \quad x(1) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$$

i.e. problem (P) with

$$F(x) = \begin{pmatrix} x_2 \\ \frac{1}{2}x_2 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \quad h(x,u) = x_2^2 + x_2$$

In this case $\Delta_A = x_2^2$, $\Delta_B = x_2^2$, $\Delta_C = \frac{1}{2}x_2^2$, $S_{AB} = 1$, $m(x) = 0$, $q(x) = x_2^2 + x_2$, $\varphi = \frac{1}{x_2^2}$, $\varphi_l = -1$,

$$D = \begin{pmatrix} 0 \\ -\frac{2}{x_2^3} \end{pmatrix}, \quad X = \begin{pmatrix} x_2 \\ -\frac{1}{2}x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} x_2 \\ \frac{3}{2}x_2 \end{pmatrix}$$

$$S_{AB}\, D \cdot X = \frac{1}{x_2^2}, \quad S_{AB}\, D \cdot Y = -\frac{3}{x_2^2}$$

As in Example 1 we have that $D(\gamma(t)) \notin \mathcal{K}(\gamma(t))$ for every $t \in [0,1]$.

Passing to Mayer form as in the previous example, we obtain the system:

$$\begin{cases} \dot{x_2} = \frac{1}{2}x_2 + u\, x_2 \\ \dot{x_3} = x_2^2 + x_2 \end{cases}$$

We have $\Delta_B = -x_2^2(2x_2 + 1) = 0$ on the time–optimal trajectory $\tilde{\gamma}(t) = \left(-\frac{1}{2}, -\frac{t}{4}\right)$, i.e. $\tilde{\gamma}$ is a turnpike.

Application. We show an example of non–convex minimization problem, and give conditions depending only on the dynamics for the existence of an optimal solution:

$$\dot{x} = F(x) + u\, G(x), \qquad x \in \Omega, \ |u| = 1$$

$$(P') \qquad \min \int_0^1 h(u), \qquad h : \{-1, 1\} \not\to I\!R$$

$$x(0) = x_0, \quad x(1) = x_1$$

Define $m = (h^+ - h^-)/2$, $q = (h^+ + h^-)/2$, where $h^\pm := h(\pm 1)$, and consider the convexified problem ($|u| \leq 1$) with cost $\tilde{h}(u) = m\,u + q$. It is not restrictive to suppose $q = 0$ (otherwise we can add a constant to h); in this case we obtain

$$\varphi_l = -m\frac{\Delta_C}{\Delta_A}.$$

If $m = 0$, that is if $h^+ = h^-$, then every admissible trajectory is optimal; it is easy to show that there exists a bang–bang optimal trajectory following the proof of Lemma 3.1.

If $m \neq 0$ we have that $D = \frac{m}{\Delta_A^4}\tilde{D}$, with:

$$\tilde{D} := -(\nabla\Delta_A)\Delta_B\Delta_C + \Delta_A(\nabla\Delta_B)\Delta_C - \Delta_A\Delta_B(\nabla\Delta_C)$$

The following proposition is a direct application of Theorem 3.2:

Proposition. Consider the problem (P'). Suppose that $\Delta_A \neq 0$, $\Delta_B \neq 0$, $\pm\tilde{D}(x) \in \mathcal{K}(x)$ for every $x \in \Omega$. If there exists an admissible pair for the convexified problem, then (P') has a solution.

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