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A METHOD FOR GENERATING DIFFERENTIALS FOR SOLVING THE WHITHAM EQUATIONS

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Abstract

We solve the Cauchy problem for the Whitham equations for smooth initial data, building analytic differentials on open hyperelliptic Riemann surface.

Key words: Whitham equations, non analytic differentials.

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1 Introduction

The Whitham equations is a collection of quasilinear hyperbolic systems of the form [1],[2],[3]

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1, u_2, \ldots, u_{2g+1})\frac{\partial u_i}{\partial x} = 0, \quad x, t, u_i \in \mathbb{R}, \quad i = 1, ..., 2g+1, \quad g = 0, 1, 2, \ldots, \tag{1.1}$$

with the ordering $u_1 > u_2 > \cdots > u_{2g+1}$. For a given g the system (1.1) is called g-phase Whitham equations. The zero-phase Whitham equation has the form

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} = 0\,. \tag{1.2}$$

This equation is also called Burgers equation.

For $g > 0$ the speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, 2, \ldots, 2g+1$, depend on complete hyperelliptic integrals of genus g. Because of this the g-phase system (1.1) will be also called the genus g-Whitham system.

The initial value problem of the Whitham equations for a monotone smooth initial data $x = f(u)$ consists of the following:

1) for $t \geq 0$ the (x, t) plane is split into a number of domains D_g, where $g = 0, 1, \ldots$. In each domain D_g we look for a solution $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the g-phase Whitham equations (1.1). For any $t \geq 0$ the functions $u_1(x,t) > u_2(x,t), \cdots > u_{2g+1}(x,t)$ can be plotted on the (x, u) plane as branches of a multivalued function. The solutions of the Whitham equations for different g must be glued together in order to produce a C^1-smooth curve in the (x, u) plane evolving smoothly with t.

2) At the time $t = 0$ we have only the D_0 domain for any x. The correspondent zero-phase solution $u(x,t)$ of equation (1.2) must satisfy the initial data $x = f(u(x, 0))$.

We will say that a solution of the initial value problem globally exists and it has genus at most g_0 if it is defined for any $t > 0$ and the domain D_g are empty for $g > g_0$.

The equations (1.1) were found by Whitham [1] in the single phase case $g = 1$ and more generally by Flaschka, Forest and McLaughlin [2] in the multiphase case. The Whitham equations were found also in [3] when studying the zero dispersion limit of the Korteweg de Vries equation. The hyperbolic nature of the equations was found by Levermore [4].

The investigation of the initial value problem of the Whitham equations was initiated by Gurevich and Pitaevskii [5]. In the case $g \leq 1$ they solved the initial value problem of system (1.1) for step-like initial data and studied numerically the case of cubic initial data. The Cauchy problem for the Whitham equations has been extensively studied by Tian in the case $g \leq 1$ for generic monotone decreasing initial data.

Dubrovin and Novikov [6] found the geometric Hamiltonian structure of the equations (1.1). Based on this structure, Tsarev [7] showed that equations (1.1) can be integrated by a generalization of the method of characteristics. This result was put into an algebro-geometric setting by Krichever

[8]. In this frame he gave an algebro-geometric construction of particular self-similar solutions of the Whitham equations.
Tsarev's result enabled F.R. Tian [9] to further transform the Cauchy problem for the Whitham equations (1.1) into a Cauchy problem for a linear over-determined system of Euler-Poisson-Darboux type.

In this paper we consider the Cauchy problem for the Whitham equations in the algebro-geometric setting. For monotone analytic initial data, the solution of the Whitham equations is build in terms of some meromorphic differentials defined on hyperelliptic Riemann surfaces [8],[9]. In this paper we consider the Cauchy problem for the Whitham equations for monotone analytic initial data with a small smooth perturbation. Following a Krichever's idea, we build the solution of the Whitham equations for the smooth part of the initial data, in terms of a non analytic differential defined on some hyperelliptic Riemann surface. This differential has a prescribed jump on some contour of the Riemann surface and it is constructed solving a boundary value problem on the surface.

2 Riemann surfaces and Abelian differentials: notations and definitions

Let

$$\Gamma_g := \left[P = (\lambda, \mu) : \mu^2 = \prod_{j=1}^{2g+1} (r - u_j) \right], \quad u_1 < u_2 < \cdots < u_{2g+1}, \tag{2.3}$$

be the hyperelliptic Riemann surface of genus $g \geq 0$. We shall use standard representation of Γ_g as a two-sheets covering of CP^1 with the cuts along the intervals

$$[u_{2k}, u_{2k+1}], \quad k = 0, \ldots, g, \quad u_0 = -\infty .$$

We choose the basis $\{\alpha_j, \beta_j\}_{j=1}^g$ of the group $H_1(\Gamma_g)$ so that β_j lies fully on the upper sheet and encircles clockwise the interval $[u_{2j}, u_{2j+1}]$, $j = 1, \ldots, g$, while α_j emerges on the upper sheet at the point u_{2j-1}, passes to the point u_{2j} and return to the initial point u_{2j-1} through the lower sheet.
The one-forms that are analytic on the closed Riemann surface Γ_g, except for a finite number of points, are called Abelian differentials.

We define on Γ_g the following differentials [10]:

1) The canonical basis of holomorphic one-forms or Abelian differentials of the first kind $\phi_1, \phi_2 \ldots \phi_g$:

$$\phi_k(r) = \frac{r^{g-1}\gamma_1^k + r^{g-2}\gamma_2^k + \cdots + \gamma_g^k}{\mu(r)} dr, \quad k = 1, \ldots, g. \tag{2.4}$$

The constants γ_j^k are uniquely determined by the normalization conditions

$$\int_{\alpha_j} \phi_k = \delta_{jk}, \quad i, j = 1, \ldots g. \tag{2.5}$$

2) The set σ_k, $k \geq 0$, of abelian differentials of the second kind with a pole of order $2k+2$ at infinity, with asymptotic behaviour

$$\sigma_k(r) = \frac{1}{2}\left[r^{k-\frac{1}{2}} + O(r^{-\frac{3}{2}})\right] dr\,, \quad \text{for large } r \tag{2.6}$$

and normalized by the condition

$$\int_{\alpha_j} \sigma_k = 0, \quad j = 1, \ldots g\,. \tag{2.7}$$

We use the notation

$$\sigma_0(r) = dp(r)\,, \qquad 12\sigma_1(r) = dq(r)\,. \tag{2.8}$$

In literature the differentials $dp(r)$ and $dq(r)$ are called quasi-momentum and quasi-energy respectively [6]. We have that

$$dp = \frac{P_g(r,\vec{u})}{2\mu} dr\,, \quad P_g(r,\vec{u}) = r^g + \alpha_{g-1} r^{g-1} + \cdots + \alpha_0\,, \tag{2.9}$$

$$dq = \frac{Q_g(r,\vec{u})}{\mu} dr\,, \quad Q_g(r,\vec{u}) = 12r^{g+1} - 6\left(\sum_{i=1}^{2g+1} u_k\right) r^g + \beta_{g-1} r^{g-1} + \cdots + \beta_0\,, \tag{2.10}$$

where $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ and the coefficients $\alpha_i = \alpha_i(\vec{u})$, $\beta_i = \beta_i(\vec{u})$, , $i = 0, 1, \ldots, g-1$, are uniquely determined by (2.7).

3) The abelian differential of the third kind $\omega_{q_1 q_2}(r)$ with first order poles at the points $r = q_1, q_2$ with residues ± 1 respectively. Its periods are normalized by the relation

$$\int_{\alpha_j} \omega_{q_1 q_2}(r) = 0, \quad j = 1, \ldots, g\,. \tag{2.11}$$

The Riemann bilinear period relation is an important tool in the study of differentials. Let df and dg be two closed differentials on the closed Riemann surface Γ_g, having a finite number of singularities on Γ_g. Cutting the surface Γ_g along the cycles of the canonical homological basis $\{\alpha_j, \beta_j\}_{j=1}^g$, we obtain the 4g-sided polygon $\hat{\Gamma}_g$ with the sides $\alpha_1, \beta_1, -\alpha_1, -\beta_1, \ldots, \beta_g$. If all the residues of df and dg are equal to zero, the integrals f and g are single-value on $\hat{\Gamma}_g$. If the differentials df or dg have nonzero residues, then the integrals f or g have corresponding logarithmic singularities. In order to provide their single-valuedness, it is necessary to cut the polygon $\hat{\Gamma}_g$ along some curves connecting the singular points of the integral f and g. Denote the cut by s. Now consider the differential fdg. We have the relation

$$\int\limits_{\partial\hat{\Gamma}_g + (s^+ - s^-)} fdg = 2\pi i \sum_{\Gamma_g} \operatorname{Res} fdg\,. \tag{2.12}$$

Here $s^{\pm}$ are the sides of s and $\partial\hat{\Gamma}_g$ is the boundary of $\hat{\Gamma}_g$. We define

$$L(f,g) = \int_{\partial\hat{\Gamma}_g} f dg = \sum_{j=1}^{g} \left[\int_{\alpha_j} dg \int_{\beta_j} df - \int_{\alpha_j} df \int_{\beta_j} dg \right] . \tag{2.13}$$

Then we obtain the relation

$$L(f,g) + \int_{s^+} \Delta f dg = 2\pi i \sum_{\Gamma_g} \operatorname{Res} f dg , \tag{2.14}$$

where Δf is the difference of values of f on both sides of the cut s. This formula is known as the first Riemann bilinear period relation.
Assuming $df = \omega_{qq_0}$ and $dg = \phi_k$ in (2.14) we obtain

$$\int_{\beta_k} \omega_{qq_0} = 2\pi i \int_{q_0}^{q} \phi_k , \quad k = 1, \ldots, g . \tag{2.15}$$

Assuming $df = \omega_{qq_0}(\lambda)$ and $dg = \omega_{pp_0}$ in (2.14) we obtain

$$\int_{p_0}^{p} \omega_{qq_0} = \int_{q_0}^{q} \omega_{pp_0} . \tag{2.16}$$

Differentiating with respect to p and q the above expression we obtain the identity

$$d_q[\omega_{qq_0}(p)] = d_p[\omega_{pp_0}(q)] \tag{2.17}$$

From the expression (2.16) it follows that $\omega_{qq_0}(\lambda)$ is a many-value analytic function of the variable q with simple pole at $q = \lambda$. The many-value character of $\omega_{qq_0}(\lambda)$ as a function of q can be described by the equations

$$\int_{\alpha_k} d_q[\omega_{qq_0}(r)] = 0 , \quad \int_{\beta_k} d_q[\omega_{qq_0}(r)] = 2\pi i \phi_k(r), \quad k = 1, \ldots, g , \tag{2.18}$$

where d_q denotes differentiation with respect to q.

3 Preliminaries on the theory of the Whitham equations

The speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$ of the g-phase Whitham equations (1.1) are given by the ratio [1],[2]:

$$\lambda_i(\vec{u}) = \frac{Q_g(u_i, \vec{u})}{P_g(u_i, \vec{u})} = \left. \frac{dq}{dp} \right|_{r=u_i} , \quad i = 1, 2, \ldots, 2g+1 , \tag{3.19}$$

where $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ and dp and dq have been defined in (2.8). In the case $g = 0$

$$dp(r) = \frac{dr}{2\sqrt{r-u}} , \quad dq(r) = \frac{12r - 6u}{2\sqrt{r-u}} dr ,$$

so that one obtains the zero-phase Whitham equation (1.2).
We consider monotone decreasing analytic initial data of the form

$$x = f(u) = c_0 + c_1 u + \cdots + c_k u^k + \dots \,, \tag{3.20}$$

where we assume that only a finite number of c_k is different from zero.

The solution of the g-phase Whitham equations for the above initial data can be written in the algebro-geometric form [9, 8]

$$(xdp - tdq + ds)\,|_{r=u_i} = 0\,, \quad i = 1, 2 \dots, 2g+1\,, \tag{3.21}$$

where dp, dq have been defined in (2.8) and ds is the differential

$$ds = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k-1)!!} c_k \sigma_k\,. \tag{3.22}$$

The differential σ_k has been defined in (2.6).

In the case $g = 0$ (3.21) becomes (with the notation $u_1 = u$)

$$\frac{1}{2}\left[\frac{x - t(12r - 6u) - f(u)}{\sqrt{r-u}}dr - \int_u^r \frac{f'(\theta)d\theta}{\sqrt{r-\theta}}dr\right]\Bigg|_{r=u} = 0,$$

which is equivalent to the equation

$$x = 6t\,u + f(u)\,, \tag{3.23}$$

that solves (1.2) according to the method of characteristic. For higher genus, equations (3.21) are equivalent to the generalization of the characteristic equation (3.23) [7],[9]

$$x = \lambda_i(\vec{u})\,t + w_i(\vec{u}) \quad i = 1, \dots, 2g+1\,, \tag{3.24}$$

where $\lambda_i(\vec{u})$ have been defined in (3.19) and

$$w_i(\vec{u}) = \frac{ds}{dp}\bigg|_{r=u_i}\,, \quad i = 1, \dots 2g+1. \tag{3.25}$$

We need to consider what happens to the equations (3.24) when one of the u_l coalesces with either u_{l-1} or u_{l+1}. From [9] it can be checked that the abelian differentials of the second kind $\sigma_k = \sigma_k(r, \vec{u}, g)$ defined on Γ_g satisfy the relation

$$\sigma_k(r, \vec{u}, g)|_{u_l = u_{l+1} = u} = \sigma_k(r, \vec{u}^*, g-1), \quad l = 1, \dots 2g\,, \quad k \geq 0\,, \tag{3.26}$$

where $\vec{u}^* = (u_1, \dots, u_{l-1}, u_{l+2}, \dots, u_{2g+1})$.

From (3.26) the speeds $\lambda_i(\vec{u})$ satisfy the following equalities

$$\begin{aligned} \lambda_i(u_1, \dots, u_{l-1}, u, u, u_{l+2}, \dots, u_{2g+1}) &= \lambda_i(u_1, \dots, u_{l-1}, u_{l+2}, \dots, u_{2g+1}) \\ &= \frac{Q_{g-1}(u_i, \vec{u}^*)}{P_{g-1}(u_i, \vec{u}^*)} \quad i \neq l, i = 1, \dots, 2g+1\,, \end{aligned} \tag{3.27}$$

$$\lambda_l(u_1,\dots,u_{l-1},u,u,u_{l+2},\dots,u_{2g+1}) = \lambda_{l+1}(u_1,\dots,u_{l-1},u,u,u_{l+2},\dots,u_{2g+1})$$
$$= \frac{Q_{g-1}(u,\vec{u}^*)}{P_{g-1}(u,\vec{u}^*)} \quad l = 1,\dots,2g\,. \tag{3.28}$$

Analogous relation can be obtained for the $w_i(\vec{u})$, $i = 1,\dots,2g+1$ and this shows that the g-phase solution of the Whitham equations can be attached continuously to $g-1$-phase solution. The solution of the Whitham equations $u_1 > u_2 > \cdots > u_{2g+1}$ is implicitly defined as a function of x and t by the equations (3.21) or (3.24). The solution is uniquely defined only for those x and t such that the functions $u_i(x,t)$ are real and the partial derivatives $\partial_x u_i(x,t)$, $i = 1,\dots,2g+1$, are not vanishing. We have the following important theorem of Tsarev.

Theorem 3.1 *[7] The solution $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the equation (3.24) satisfies the g-phase Whitham system if and only if $w_i(u_1,u_2,\dots,u_{2g+1})$ solves the linear overdetermined system*

$$\frac{\partial w_i}{\partial u_j} = \frac{w_i - w_j}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j}\,, \quad i,j = 1,2\dots,2g+1,\ \ i \neq j\,. \tag{3.29}$$

The linear overdetermined system (3.29) was transform by Tian [9] into the linear overdetermined system of Euler-Poisson-Darboux type

$$\begin{aligned} &2(u_i - u_j)\frac{\partial^2 q}{\partial u_i \partial u_j} = \frac{\partial q}{\partial u_i} - \frac{\partial q}{\partial u_i} \quad i,j = 1,2,\dots,2g+1 \\ &q(u,u,\dots,u) = F(u)\,. \end{aligned} \tag{3.30}$$

Tian then showed that the solution of the Whitham equations (1.1) can be obtained using sytem (3.30) with a suitable initial data $F(u)$.
In this work we obtain the solution of the Cauchy problem for the Whitham equations for smooth initial data building a non analytic differential on the hyperelliptic Riemann surface Γ_g.

4 Construction of the solution of the Whitham equations for smooth initial data

We consider monotone decreasing initial data of the form

$$x = f(u) + f_1(u) \tag{4.31}$$

where $f(u)$ is analytic and given by the expression (3.20) and $f_1(u)$ is a smooth function rapidly decreasing at infinity. In order to build the solution of the Whitham equations for such initial data, we follow an idea of Krichever. In [8] he suggests to build, for the smooth part of the initial data, a differential Ω with a given jump on a contour $\mathcal{L}$ of the Riemann surface Γ_g. The jump must not depend on the moduli of the surface.

Lemma 4.1 *Let $z(p)$ be a local coordinate on Γ_g and let $\psi(p) = h(z)dz$ be a one form defined on the Lyapunov*[1] *contour $\mathcal{L} \subset \Gamma_g$. Suppose that h is H-continuous (Hölder continuous) on $\mathcal{L}$, namely*

$$|h(z) - h(z')| < H|z - z'|^{\alpha}, \quad 0 < \alpha \leq 1, \;\; H > 0\,.$$

Then there exists a unique differential Ω on Γ_g which satisfies the conditions:

1) Ω is holomorphic outside $\mathcal{L}$;

2) the limiting values $\Omega^{\pm}$ on $\mathcal{L}$ are H-continuous and satisfy the relation

$$\Omega^{+} - \Omega^{-} = \psi \tag{4.32}$$

3) Ω is a normalized differential, namely

$$\int_{\alpha_j} \Omega = 0, \quad j = 1, 2, \ldots, g. \tag{4.33}$$

The assertion in the Lemma is a standard one in the theory of boundary value problems [11].

The unique differential Ω satisfying the properties 1),2) and 3) of the Lemma is given by the Cauchy integral

$$\Omega(r) = \frac{dr}{2\pi i} \int_{\mathcal{L}} A(r, z) h(z) dz \tag{4.34}$$

where $A(r, z)dr$ is a meromorphic analogues of the Cauchy kernel [11]. In general $A(r, z)dr$ is a meromorphic function of z and a differential in r. In order to satisfy the properties 1),2) and 3) of Lemma 4.1, the kernel $A(r, z)dr$ must be equal to the normalized abelian differential of the third kind $\omega_z(r)$ which has simple poles at the points $P^{\pm}(z) = (z, \pm\mu(z))$ with residue ± 1 respectively, and $\mu^2 = \prod_{i=1}^{2g+1}(r - u_i)$ define the hyperelliptic Riemann surface Γ_g. As $r \to z$, it satisfies the relation

$$A(r, z)dr = \left[\frac{1}{r - z} + O(1)\right] dr \tag{4.35}$$

which have the Cauchy and Sokhotskii formulas as consequence, namely

$$\Omega^{\mp}(s) = \pm\frac{1}{2} h(s) ds + \frac{ds}{2\pi i} \int_{\mathcal{L}} A(s, z) h(z) dz\,, \quad s \in \mathcal{L}\,. \tag{4.36}$$

Here the integral is taken in the sense of principal value. The differential $A(r, z)dr = \omega_z(r)$ is explicitly given by the expression

$$\omega_z(r) = \frac{dr}{\mu(r)} \frac{\mu(z)}{r - z} - \sum_{k=1}^{g} \phi_k(r) C_k(z)\,, \quad C_k(z) = \int_{\alpha_k} \frac{dt}{\mu(t)} \frac{\mu(z)}{t - z}\,, \tag{4.37}$$

[1] A Lyapunov contour is a contour whose tangent rotates Hölder-continuously

where $\phi_k(z)$, $k = 1, \ldots, g$, is the normalized basis of holomorphic differentials. As a function of z $\omega_z(r)$ is an abelian integral of the second kind with poles of the first order at the points $(r, \pm\mu(r))$. The periods of this integral are obtained from (2.18)

$$\int_{\alpha_j} d_z[\omega_z(r)] = 0, \quad \int_{\beta_j} d_z[\omega_z(r)] = 4\pi i \phi_k(r)\,, \quad j = 1, \ldots g\,. \tag{4.38}$$

In order to build a solution of the g-phase Whitham equations for the smooth initial data $x = f_1(u)$ rapidly decreasing at infinity, we make the following choice for defining the differential Ω.
As a contour we take $\mathcal{L} = \sum_{j=0}^{g} \tilde{\beta}_j$, where each $\tilde{\beta}_{j>0}$ is the beta-cycle whose projection on the complex r-plane coincides with the segment $[u_{2j-1}, u_{2j}]$ described twice and $\tilde{\beta}_0$ is the close cycle whose projection on the r-plane coincides with the cut $(-\infty, u_1]$ described clockwise.

The "jump" differential is given by the real exact one form

$$dh(t) = \frac{\partial}{\partial t}\left[\int_{-\infty}^{t} \frac{f_1(y)}{\sqrt{t-y}} dy\right] dt = \left[\int_{-\infty}^{t} \frac{f_1'(y)}{\sqrt{t-y}} dy\right] dt\,, \quad t \in \mathbb{R} \tag{4.39}$$

where $f_1(y)$ is the smooth part initial data (4.31).
We define the differential $\Omega = \Omega(r, \vec{u}, g)$ by the expression

$$\begin{aligned} \Omega(r, \vec{u}, g) =& \frac{1}{4\pi i} \int_{\mathcal{L}} \omega_z(r) dh(z) \\ =& \frac{1}{4\pi i}\left[\frac{dr}{\mu(r)} \int_{\mathcal{L}} \frac{\mu(z)}{r-z} dh(z) - \sum_{k=1}^{g} \phi_k(r) \int_{\mathcal{L}} C_k(z) dh(z)\right] \end{aligned} \tag{4.40}$$

where the $C_k(z)$'s have been defined in (4.37). In the following we sometimes omit the dependence of Ω on the branch points $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ and on the genus g.
$\Omega(r)$ is holomorphic everywhere on Γ_g outside $\mathcal{L}$. We underly that the above differential is well defined even though $\omega_z(r)$ is a multivalued function of z. In fact as $z \to z + \beta_j$ we have that

$$\int_{\mathcal{L}} \omega_z(r) dh(z) \overset{z \to z+\beta_j}{\longrightarrow} \int_{\mathcal{L}} [\omega_z(r) + 4\pi i \phi_j(r)] dh(x) = \int_{\mathcal{L}} \omega_z(r) dh(z)$$

because $\int_{\mathcal{L}} dh(z) = 0$.
The following Sokhotskii formulae hold:

$$\Omega^{\mp}(\lambda) = \pm\frac{1}{4} dh(\lambda) + \frac{1}{4\pi i} \int_{\mathcal{L}} \omega_x(\lambda) dh(x)\,, \quad \lambda \in \mathcal{L}, \quad \lambda \neq u_j, \ j = 1, 2, \ldots 2g+1\,. \tag{4.41}$$

For $g = 0$ we have $\mu^2 = r - u$ and $\Omega^{\pm}(r)$ can be calculated explicitly

$$\Omega^{+}(r) = -\frac{1}{2} dh(r) + \frac{dr}{2\sqrt{r-u}}\left(f_1(u) - \sqrt{u-r} \int_r^u \frac{f_1'(s)}{\sqrt{s-r}} ds\right)\,, \quad r \in (-\infty, u) \tag{4.42}$$

$$\Omega^{-}(r) = \frac{dr}{2\sqrt{r-u}}\left(f_1(u) - \sqrt{u-r} \int_r^u \frac{f_1'(s)}{\sqrt{s-r}} ds\right)\,, \quad r \in (-\infty, u)\,, \tag{4.43}$$

$$\tag{4.44}$$

The following theorem enables one to construct the solution of the Whitham equations for the initial data (4.31).

Theorem 4.2 *Define*

$$v_i(u_1, u_2, \ldots, u_{2g+1}) = \left.\frac{\Omega(r)}{dp(r)}\right|_{r=u_i}, \quad i = 1, 2, \ldots, 2g+1. \tag{4.45}$$

Then $v_i(u_1, u_2, \ldots, u_{2g+1})$ *satisfies the overdetermined system*

$$\frac{\partial v_i}{\partial u_j} = \frac{v_i - v_j}{\lambda_i - \lambda_j}\frac{\partial v_i}{\partial u_j}, \quad i, j = 1, 2 \ldots, 2g+1, \ \ i \neq j.$$

Proof: Part 1)
The differential

$$\frac{\partial \Omega}{\partial u_j} - v_j \frac{\partial dp}{\partial u_j}$$

is identically zero. Indeed it is holomorphic: the differentials $\partial_{u_j}\Omega(\lambda)$ and $\partial_{u_j} dp$ are abelian differentials with a second order pole at $\lambda = u_j$; since the differential $(\Omega - v_j\, dp)$ has a first order zero at $r = u_j$, the differential $[\partial_{u_j}(\Omega - v_j\, dp) + dp\, \partial_{u_j} v_j]$ is regular at $r = u_j$. Furthermore by the normalization conditions (2.7) and (4.33)

$$0 = \partial_{u_j} \int_{\alpha_k} (\Omega - v_j\ dp) = \int_{\alpha_k} (\partial_{u_j}\Omega - v_j \partial_{u_j} dp), \quad k = 1, \ldots, g. \tag{4.46}$$

Hence $(\partial_{u_j}\Omega - v_j\partial_{u_j} dp)$ is a holomorphic differential having all the α-periods equal to zero, consequently it is identically zero (cfr [10]) and the following identity holds

$$v_j = \frac{\partial_{u_j}\Omega}{\partial_{u_j} dp}. \tag{4.47}$$

We write $\Omega(r)$ in the form $\Omega(r) = \dfrac{R(r, \vec{u})}{2\mu(r)} dr$ where

$$R(r, \vec{u}) = \frac{1}{2\pi i}\left[\int_{\mathcal{L}} \frac{\mu(z)}{r - z} dh(z) - \sum_{k=1}^{g}\sum_{l=1}^{g} \gamma_l^k r^l \int_{\mathcal{L}} C_k(z) dh(z)\right], \tag{4.48}$$

γ_l^k and C_k have been defined in (2.4) and (4.37) respectively. Form (4.47) and (4.48) (2.9) we obtain

$$\frac{\partial R}{\partial u_j} - v_j \frac{\partial P_g}{\partial u_j} = -\frac{1}{2}\frac{R - v_j P_g}{r - u_j}, \quad j = 1, 2, \ldots, 2g+1. \tag{4.49}$$

Part 2)

$$
\begin{aligned}
\frac{\partial v_i(\vec{u})}{\partial u_j} =& \frac{\partial}{\partial u_j}\frac{R(u_i,\vec{u})}{P_g(u_i,\vec{u})}, \quad i \neq j \\
=& \frac{\partial_{u_j} R(u_i,\vec{u}) - v_j \partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} + (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} \\
=& (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{R(u_i,\vec{u}) - v_j P_g(u_i,\vec{u})}{(u_i - u_j)P_g(u_i,\vec{u})} \\
=& (v_j - v_i)\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{v_i - v_j}{u_i - u_j}
\end{aligned}
\tag{4.50}
$$

which shows that

$$
\frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j} = -\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{1}{u_i - u_j} \tag{4.51}
$$

In particular the above argument also applied for $dq(r)$ and λ_i defined in (2.8) and (3.19) respectively, therefore we also have

$$
\frac{1}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j} = -\frac{\partial_{u_j} P_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} - \frac{1}{2}\frac{1}{u_i - u_j}
$$

which when combined with (4.51) proves the Theorem 4.2.

Finally we combine Theorem 3.1 and Theorem 4.2 to construct the transform

$$
\begin{aligned}
x =& \lambda_i(\vec{u})t + w_i(\vec{u}) + v_i(\vec{u}), \quad i = 1, \ldots, 2g+1 \\
=& \left[\frac{dq}{dp}t + \frac{ds}{dp} + \frac{\Omega}{dp}\right]\Bigg|_{r=u_i}.
\end{aligned}
\tag{4.52}
$$

where dq and ds have been defined in (2.8) and (3.22) respectively. The above system solves the g-phase Whitham equations for the initial data (4.31).

We need to consider what happens to the equations (4.52) when one of the u_l coalesces with either u_{l-1} or u_{l+1}. From [12] it can be checked that

$$
\Omega(r,\vec{u},g)|_{u_l = u_{l+1}} = \Omega(r,\vec{u}^*, g-1), \quad l = 1, \ldots 2g, \tag{4.53}
$$

where $\vec{u}^* = (u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1})$.
For $g = 0$ we have

$$
\Omega(r,u,g=0) = \frac{1}{4\pi i}\frac{dr}{\sqrt{r-u}}\int_{-\infty}^{u}\frac{\sqrt{z-u}}{r-z}dh(z) \tag{4.54}
$$

so that

$$
\frac{\Omega(r,u,g=0)}{dp(r)}\Bigg|_{r=u} = f_1(u). \tag{4.55}
$$

Hence the zero-phase solution of (4.52) coincides with the characteristic equation

$$x = 6tu + f(u) + f_1(u)\,.$$

Conclusions We have built the solution of the Cauchy problem for the Whitham equations for a monotone initial data that is the sum of analytic function and a smooth function rapidly decreasing at infinity. The work of our further investigations is to consider smooth initial data with more general boundary conditions.

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