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Self-similar asymptotic solutions of the Whitham equations

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Abstract

We prove a conjecture of Novikov, namely that the solution of the Whitham equations with monotone polynomial initial data has a universal one-phase self-similar asymptotics.

Key words: Whitham equations, self-similar solutions.

1 Introduction

The Whitham equations is a collection of quasilinear hyperbolic systems of the form [1],[2],[3]

$$\frac{\partial u_i}{\partial t}+\lambda_i(u_1,u_2,\ldots,u_{2g+1})\frac{\partial u_i}{\partial x}=0, \quad x,t,u_i\in\mathbb{R}, \quad i=1,...,2g+1, \quad g=0,1,2,\ldots, \tag{1}$$

with the ordering $u_1>u_2>\ldots>u_{2g+1}$. For a given g the system (1) is called g-phase Whitham equations. The zero-phase Whitham equation has the form

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}=0\,. \tag{2}$$

This equation is also called Burgers equation.

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For $g > 0$ the speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, 2, \ldots, 2g+1$, depend on complete hyperelliptic integrals of genus g. Because of this the g-phase system (1) will be also called the genus g-Whitham system.

The initial value problem of the Whitham equations for a monotone smooth initial data $x = f(u)$ consists of the following:

1) for $t \geq 0$ the (x, t) plane is split into a number of domains D_g, where $g = 0, 1, \ldots$. In each domain D_g we look for a solution $u_1(x,t) > u_2(x,t) > \ldots > u_{2g+1}(x,t)$ of the g-phase Whitham equations (1). For any $t \geq 0$ the functions $u_1(x,t) > u_2(x,t), \ldots > u_{2g+1}(x,t)$ can be plotted on the (x, u) plane as branches of a multivalued function. The solutions of the Whitham equations for different g must be glued together in order to produce a C^1-smooth curve in the (x, u) plane evolving smoothly with t.

2) At the time $t = 0$ we have only the D_0 domain for any x. The correspondent zero-phase solution $u(x,t)$ of equation (2) must satisfy the initial data $x = f(u(x, 0))$.

We will say that a solution of the initial value problem globally exists and it has genus at most g_0 if it is defined for any $t > 0$ and the domain D_g are empty for $g > g_0$.

The equations (1) were found by Whitham [1] in the single phase case $g = 1$ and more generally by Flaschka, Forest and McLaughlin [2] in the multiphase case. The hyperbolic nature of the equations was found by Levermore [4]. The investigation of the initial value problem of the Whitham equations was initiated by Gurevich and Pitaevskii [5]. In the case $g \leq 1$ they solved the initial value problem of system (1) for step-like initial data and studied numerically the case of cubic initial data. The Whitham equations were found also in [3],[6] when studying the zero dispersion limit of the Korteweg de Vries equation. Dubrovin and Novikov [7] found the geometric Hamiltonian structure of the equations (1). Based on this structure, Tsarev [8] showed that equations (1) can be integrated by a generalization of the method of characteristics. This result was put into an algebro-geometric setting by Krichever [9]. In this frame he gave an algebro-geometric construction of particular self-similar solutions of the Whitham equations.

Tsarev's result enabled F.R. Tian [10] to further transform the Cauchy problem for the Whitham equations (1) into a Cauchy problem for a linear over-determined system of Euler-Poisson-Darboux type. Based on this result Tian [11] was also able to find sufficient conditions for the global existence of the solution of the Cauchy problem for the Whitham

equations (see Theorem 2.1 below). He proved that the genus of his solution cannot be bigger then one. In [12] the author has shown that for polynomial initial data the solution of the Whitham equations has a bounded number of interacting oscillatory phases.

In this paper we continue our investigation of the initial value problem of the Whitham equations for monotone decreasing polynomial initial data $x = f(u)$ of degree $2N+1$, N positive integer. We show that for $t \to +\infty$ the solution of the Whitham equations for such initial data is asymptotically close to the self-similar solution of the Whitham equations with the initial data $x = -u^{2N+1}$.

2 Preliminaries on the theory of the Whitham equations

The speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, of the Whitham equations (1) are constructed in the following way. On the Riemann surface

$$S_g := \left\{\mu^2 = (r-u_1)(r-u_2)\ldots(r-u_{2g+1})\right\} \ , \quad u_1 > u_2 > \ldots > u_{2g+1} \, , \tag{3}$$

we define the two abelian differentials of the second kind with poles at infinity of second and fourth order respectively [13]

$$dp = \frac{P_g(r,\vec{u})}{2\mu} dr \ , \quad P_g(r,\vec{u}) = r^g + \alpha_{g-1} r^{g-1} + \ldots + \alpha_0 \, , \tag{4}$$

$$dq = \frac{Q_g(r,\vec{u})}{2\mu} dr \, , \quad Q_g(r,\vec{u}) = 12 r^{g+1} - 6\left(\sum_{i=1}^{2g+1} u_k\right) r^g + \beta_{g-1} r^{g-1} + \ldots + \beta_0 \, , \tag{5}$$

where $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$. The coefficients $\alpha_i = \alpha_i(\vec{u})$ and $\beta_i = \beta_i(\vec{u})$, $i = 0, 1, \ldots, g-1$, are uniquely determined by the normalization conditions

$$\int_{u_{2k+1}}^{u_{2k}} dp = 0 \, , \qquad \int_{u_{2k+1}}^{u_{2k}} dq = 0 \, , \quad k = 1, 2, \ldots, g \, . \tag{6}$$

In the literature the differential dp is called quasi-momentum and the differential dq quasi-energy [7].

The speeds $\lambda_i(\vec{u})$ of the g-phase Whitham equations (1) are given by the ratio [1],[2]:

$$\lambda_i(\vec{u}) = \frac{Q_g(u_i, \vec{u})}{P_g(u_i, \vec{u})} = \left.\frac{dq}{dp}\right|_{r=u_i} , \quad i = 1, 2, \ldots, 2g+1 . \tag{7}$$

In the case $g = 0$, (4) and (5) become $P_0(r) = 1$ and $Q_0(r) = 12r - 6u$ respectively so that one obtains the zero-phase Whitham equation (2). For the monotone decreasing initial data $x = f(u)$, the solution of the zero-phase equation (2) is obtained by the method of characteristics and it is given by the expression

$$x = 6t\,u + f(u) . \tag{8}$$

This solution is globally well defined only for $0 \leq t < t_c$ where $t_c = \frac{1}{6} min_{u \in \mathbb{R}}[-f'(u)]$ is the time of gradient catastrophe of the solution. The breaking is caused by an inflection point in the initial data. Without loss of generality we may assume the breaking point to be at the origin of the x, u, t plane. It immediately follows that $f(0) = f'(0) = f''(0) = 0$. For $t \geq t_c = 0$ the solution of the Whitham equations is obtained gluing together C^1-smoothly solutions of different genera as shown in Fig. 1 in the case $g \leq 1$. The functions

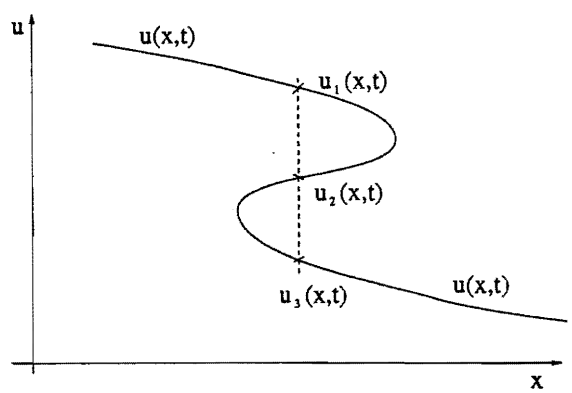

Figure 1: On the picture the evolution of $u_1(x,t) > u_2(x,t) > u_3(x,t)$ is ruled by the one-phase Whitham equations, while $u(x,t)$ satisfies the Burgers equation.

$u_1(x,t), u_2(x,t), u_3(x,t)$ plotted on Fig. 1 match the Burgers solution at the boundaries of the multi-valued region, namely:

a) trailing edge

$$\begin{cases} u_1 &= \text{solution of the Burgers equation outside the multi-valued region} \\ u_2 &= u_3\,, \end{cases} \tag{9}$$

b) leading edge

$$\begin{cases} u_1 &= u_2 \\ u_3 &= \text{solution of the Burgers equation outside the multi-valued region}\,. \end{cases} \tag{10}$$

The Cauchy problem for the Whitham equations has been extensively studied by Tian in the case $g \leq 1$. The following theorem gives conditions for the existence of a global solution of the Cauchy problem for the Whitham equations.

Theorem 2.1 *[11] Suppose that the monotone decreasing initial data $x = f(u)$ has only one inflection point at the origin of the x, u plane and satisfies the condition $f'''(u) < 0$ for all $u \neq 0$. Then the solution of the Whitham equations exists for all $t \geq 0$. The solution is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) < x^+(t)$ are two real functions satisfying the condition $x_-(0) = x_+(0) = 0$; it is of genus zero outside this interval. The solution satisfies the boundary conditions (9) and (10).*

The following result of [10] will be important for our considerations.

Theorem 2.2 *[10] Let us assume that the solution of the Cauchy problem for the Whitham equations with monotone decreasing initial data $x = f(u)$ exists for any $x, t \geq 0$. Suppose that the function $f(u)$ defined on the whole real axis satisfies the conditions*

$$\begin{aligned} &\lim_{u\to-\infty} f''(u) = +\infty\,, \qquad f''(u) < 0 \quad \text{for } u \to +\infty\,, \\ &f'''(u) < 0 \quad \text{for} \quad u > u_+ \quad \text{and} \quad u < u_- \end{aligned} \tag{11}$$

where $u_+ \geq u_-$ are some real numbers. Then there is a time $T \geq 0$ such that for all $t > T$ the solution of the Whitham equations is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) < x^+(t)$ are two real functions of t. It is of genus zero outside this interval and satisfies the boundary conditions (9) and (10).

The solution of the one-phase equations which satisfies the boundary conditions (9) and (10) can be written in the form [8] [14]

$$x = \lambda_i t + w_i\,, \quad i = 1, 2, 3\,, \tag{12}$$

where the $\lambda_i = \lambda(u_1, u_2, u_3)$ have been defined in (7) and the w_i's are given by the expression [14]

$$w_i = (\frac{1}{2}\lambda_i - u_1 - u_2 - u_3)\frac{\partial q}{\partial u_i} + q\,, \quad i = 1, 2, 3\,. \tag{13}$$

The function $q = q(u_1, u_2, u_3)$ is given by the expression [11]

$$q(u_1, u_2, u_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f(\frac{1+\mu}{2}\frac{1+\nu}{2}u_1 + \frac{1+\mu}{2}\frac{1-\nu}{2}u_2 + \frac{1-\mu}{2}u_3)}{\sqrt{(1-\mu)(1-\nu^2)}}d\mu d\nu\,. \tag{14}$$

For analytic initial data $f(u) = c_0 + c_1 u + \ldots + c_k u^k + \ldots$ the function $q(u_1, u_2, u_3)$ can be written in the form [10]

$$q(u_1, u_2, u_3) = \sum_{j=0}^{\infty}\frac{c_k}{Z_k}\Gamma_k(u_1, u_2, u_3) \tag{15}$$

where $Z_k = \dfrac{(2k-1)!!}{2^k k!}$ and the Γ_k's are the coefficients of the expansion for $\xi \to \infty$ of

$$\frac{1}{\sqrt{(\xi - u_1)(\xi - u_2)(\xi - u_3)}} = \xi^{-\frac{3}{2}}(\Gamma_0 + \Gamma_1\xi + \ldots + \Gamma_k\xi^k + \ldots)\,. \tag{16}$$

Lemma 2.3 *If the monotone initial data $f(u) \in C^1(\mathbb{R})$, then the one phase solution (12)-(14) is attached C^1-smoothly to the zero-phase solution (8).*

When the initial data is of the form $f(u) = -u^k$, $k = 3, 5, 7, \ldots$, one obtains the distinguished self-similar solutions [15],[9]. Their genus is at most equal to one. These solutions have the form $u_i(x,t) = t^{\frac{1}{k-1}}U_i(t^{-\frac{k}{k-1}}x)$, $i = 1, 2, 3$. Indeed, the $\lambda_i(u_1, u_2, u_3)$'s are homogeneous functions of u_1, u_2 and u_3 of degree one while the $w_i(u_1, u_2, u_3)$'s are homogeneous of degree k. Introducing the new variables

$$X = t^{-\frac{k}{k-1}}x\,, \quad u_i(x,t) = t^{\frac{1}{k-1}}U_i(t^{-\frac{k}{k-1}}x)\,, \quad u(x,t) = t^{\frac{1}{k-1}}U(t^{-\frac{k}{k-1}}x)\,, \tag{17}$$

the system (12) becomes time free. Indeed

$$X = \lambda_i(U_1, U_2, U_3) + w_i(U_1, U_2, U_3)\,, \quad i = 1, 2, 3 \tag{18}$$

and the characteristic equation (8) becomes

$$X = 6U - U^k\,. \tag{19}$$

The w_i's in (18) are given by the expression

$$w_i(\vec{U}) = -\left[\frac{1}{2}\lambda_i(\vec{U}) - U_1 - U_2 - U_3\right]\frac{\partial}{\partial U_i}\frac{\Gamma_k(\vec{U})}{Z_k} - \frac{\Gamma_k(\vec{U})}{Z_k}\,, \tag{20}$$

where $\vec{U} = (U_1, U_2, U_3)$ and Γ_k has been defined in (16).
The following result can be derived from Theorem 2.1 and Lemma 2.3.

Corollary 2.4 *For $x = -u^k$, $k = 3, 5, 7, \ldots$, the Whitham equations have a global self-similar one-phase solution $u_1 > u_2 > u_3$:*

$$u_i(x,t) = t^{\frac{1}{k-1}} U_i(t^{-\frac{k}{k-1}} x)\,, \quad i = 1, 2, 3 \tag{21}$$

within a cusp in the $x - t$ plane: $x_-(k)\, t^{\frac{k}{k-1}} < x < x_+(k)\, t^{\frac{k}{k-1}}$, where $x_-(k) < x_+(k)$ are two real constants and $t > 0$. On the boundary of the cusp the one-phase solution is attached C^1-smoothly to the solution $u(x,t)$ of the zero-phase equation.

The constants $x_-(k)$ and $x_+(k)$ are given by the following relations:

$$x_-(k) = -6\,\frac{k-1}{k}(2z_-(k) - 1)\left[\frac{6}{k}(1 + 2(k-1)z_-(k))\right]^{\frac{1}{k-1}}\,, \tag{22}$$

where $z_-(k) > 1$ is the unique real solution of $F(-k+2, 2, \frac{5}{2}; z) = 0$. Here $F(a, b, c; z)$ is the hypergeometric function. The quantity $x_+(k)$ is obtained from the expression

$$x_+(k) = 2\,\frac{k-1}{k}(2z_+(k) - 3)\left[\frac{2}{k}(3 + 2(k-1)z_+(k))\right]^{\frac{1}{k-1}}\,, \tag{23}$$

where the number $z_+(k) > 1$ is the unique real solution of the equation $F(-k+2, 2, \frac{7}{2}; z) = 0$.

Below we give the first numerical values of $x_-(k)$ and $x_+(k)$.

k	$x_-(k)$	$x_+(k)$
3	-20.784	1.721
5	-16.849	1.584
7	-16.215	1.606
9	-16.090	1.716

Remark For $k \to \infty$ the solution (17)-(20) goes to the step-like solution of [5]. The limiting solution loses the C^1-smoothness.

3 Asymptotic self-similar solutions

Let us consider the Cauchy problem for the Whitham equations for the monotone decreasing polynomial initial data

$$x = f(u) = -u^{2N+1} + f_1(u) \tag{24}$$

where $f_1(u) = -(c_0 + c_1 u + \ldots + c_{2N} u^{2N})$.
According to the result in [12] the solution of the Whitham equations for such initial data exists for all x and $t \geq 0$ and it has a bounded number of interacting oscillatory phases.

The monotone decreasing polynomial initial data satisfy the hypothesis of Theorem 2.2. Hence there exists a time T such that for all time $t > T$ the solution of the Whitham equations with polynomial initial data has genus $g \leq 1$.
We consider the solution of the Burgers equation (8) and of the one-phase equations (12) for the polynomial initial data (24) when $t > T$. We introduce the new variables given in (17) with $k = 2N + 1$. Then the solution of the Burgers equation (8) becomes

$$X = 6U - U^{2N+1} + \epsilon^{2N+1} f_1\left(\frac{U}{\epsilon}\right), \tag{25}$$

where $\epsilon = t^{-\frac{1}{2N}}$. The solution (12) of the one-phase Whitham equations becomes

$$X = \lambda_i(\vec{U}) + w_i(\vec{U}) + \epsilon^{2N+1} w_i^{\epsilon}(\vec{U}), \quad i = 1, 2, 3 \tag{26}$$

where the $w_i(\vec{U})$'s have been defined in (20) and

$$w_i^\epsilon(\vec{U}) = \left[\frac{1}{2}\lambda_i(\vec{U}) - U_1 - U_2 - U_3\right] \frac{\partial q_\epsilon(\vec{U})}{\partial U_i} + q_\epsilon(\vec{U}) \,, \tag{27}$$

$$q_\epsilon(\vec{U}) = \frac{1}{2\sqrt{2}\pi} \int_{-1}^{1}\int_{-1}^{1} \frac{f_1\left(\frac{1}{\epsilon}\left[\frac{1+\mu}{2}\frac{1+\nu}{2}U_1 + \frac{1+\mu}{2}\frac{1-\nu}{2}U_2 + \frac{1-\mu}{2}U_3\right]\right)}{\sqrt{(1-\mu)(1-\nu^2)}} d\mu d\nu \,. \tag{28}$$

Theorem 3.1 *The solution of the Whitham equations (1) with initial data (24) is asymptotically close for $t \to +\infty$ to the self-similar solution (18)-(20) with initial data $x = -u^{2N+1}$.*

The above theorem holds true whenever to the polynomial initial data is added a small smooth perturbation with compact support. It remains to prove that the solution of the Whitham equations exists for all $t > 0$. We are going to investigate this point in a subsequent publication.

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