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Non-collision periodic orbits with zero total angular momentum for the Newtonian Three-Body Problem

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Abstract

We study the existence of a periodic orbit without collisions, with zero total angular momentum, in the planar Three-Body Problem by means of Lagrangian reduction, variational methods and local analysis of the flow.

In a preceeding paper [7] we described a class of compact set of trajectories where to find minima for the Action-functional of the Three-Body Problem. In this paper we find critical points perturbatively in the masses, these critical points are in slightly bigger compact sets. We study a system composed of three bodies: two of them with small *different* masses compared with the mass of the third body. For the unperturbed problem we construct explicit solutions, which are regular critical points for the unperturbed Action-functional. These critical points can be *continued* for sufficiently small values of the masses. We prove the existence of periodic orbits without collisions. Moreover for the full Three-Body Problem on the manifold $J = 0$ we verify the existence of critical points at "infinity" composed of a Kepler-problem and an "escaping" body.

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1 Introduction

In this paper we deal with the problem of the existence of periodic solutions in the Newtonian Three-Body Problem. In particular we consider the problem under the constrain that the total angular momentum J be zero.

For the Newtonian N-body problem in three dimensions, K.Sundman in a very old paper [1] proved that $J = 0$ is a necessary condition for total collision solutions. This fact implies that collision orbits must live in the submanifold of the phase space identified by the condition $J = 0$. Recently in [2] it is proved that in the N-body problem, under proper assumptions, collision orbits are not minima for the Action functional; (the assumptions are known to hold for $N \leq 4$) so that non-collision periodic solutions exist.

These facts led us to study the problem of the existence of periodic solutions and collision orbits by means of an Action functional reduced on the manifold of zero total angular momentum. Also in this case collision solutions are not minima and regular critical points, but now there is a lack of compactness and we cannot conclude immediately about the existence of regular critical points. In a preceeding paper [7] we studied the geometry of the reduced configuration manifold. Moreover by a weak Poincaré inequality, we showed that the lack of compactness is concentrated in a suitable neighborhood of collinear orbits. In this paper we prove the existence of non-collision periodic orbits on $J = 0$ in the case in which the system is composed of one large mass and two small *different* masses: the unperturbed system is essentially a small "Solar system" with two non-interacting small *different* "planets". For this system we exhibit a solution $\zeta_0(t)$ which is a C^1 critical point for the unperturbed Action-functional, and it lies in one of the compact sets that we found using a result proved in [7]. By means of *continuation* argument of critical points for the perturbed functional (see ([5]), we prove the existence of a solution for sufficiently small values of the masses. In the last section we study the behaviour of the Action functional on unbounded trajectories. Some of these trajectories describe the escape of one of the bodies.

2 Lagrangian reduction

We study a system composed of three point-like bodies whose masses are m_1, m_2, m_3, interacting with a newtonian potential. We will study the manifold of trajectories with zero total angular momentum, the system lies in a plane since every mechanical system composed of three material points interacting thorough a potential depending on the relative distances, with zero total angular momentum, has only planar motions. There is no loss of generality assuming that the system has the configuration space

$$Q \doteq \{\vec{q}_i \in \mathbf{R}^2; i = 1, 2, 3\} \simeq \mathbf{R}^6$$

In Q there are configurations on which the dynamics of our system is not defined. Those are the *Collision* configurations:

Double collisions

$$\{\vec{q}_1 = \vec{q}_2\} \quad \{\vec{q}_3 = \vec{q}_2\} \quad \{\vec{q}_1 = \vec{q}_3\}$$

Triple (total) collision

$$\{\vec{q}_1 = \vec{q}_2 = \vec{q}_3\}$$

The dynamics of the system is described by the Lagrangian:

$$L : TQ \to \mathbf{R}$$

$$L = \sum_{i=1}^{3} \frac{m_i}{2} \|\dot{\vec{q}}_i\|^2 + \sum_{i,j,i\neq j}^{3} \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} \tag{1}$$

where we denote with $< .,. >$ $\|.\|$ respectively the scalar product and the norm in $\mathbf{R}^2$.

In order to simplify the notations we put:

$$\vec{q}_i = (q_i^1, q_i^2) \doteq q_i$$

The Angular momentum J of the system is conserved. It is a scalar since we consider only rotation in the plane of the motion. We describe the motion for fixed values of J (in particular $J = 0$) with a reduced Lagrangian obtained by the procedure described by Routh.

Definition 1.1

A Mechanical System is a triple $(M, < .,. >_M, V)$ where:

$(M, < .,. >_M)$ is a Riemmanian Manifold of dimension n, and $V : M \to \mathbf{R}$ a function (the potential) which is smooth excluding only a finite number of compact sets.

For this system we can construct a Lagrangian:

$$\begin{aligned} L : TM &\longrightarrow \mathbf{R} \\ (m, v_m) &\longrightarrow L(m, v_m) \end{aligned} \tag{2}$$

where:

$$L(m, v_m) \doteq \frac{1}{2} < v_m, v_m >_M + (V \circ \tau_M)(m, v_m)$$

$$\begin{aligned} \tau_M : TM &\longrightarrow M \\ (m, v_m) &\longrightarrow m \end{aligned} \tag{3}$$

Assume that on M acts freely a Lie group G:

$$\begin{aligned} \Phi : M \times G &\longrightarrow M \\ q, m &\longrightarrow \Phi_g(m) \end{aligned} \tag{4}$$

with lifted action on the tangent bundle TM:

$$\begin{aligned} \Phi^T : TM \times G &\longrightarrow TM \\ (m, v_m), g &\longrightarrow (\Phi_g(m), T\Phi_g(v_m)) \end{aligned} \tag{5}$$

We assume that G is a symmetry for the system, so that:

$$L \circ \Phi_g^T(m, v_m) = L(m, v_m)$$

Then Noether theorem implies the existence of integrals of the motion.

The Three Body Problem can be easily described in this context observing that we can put:

$$M \doteq Q, \quad < .,. >_Q \doteq \sum_{i=1}^{3} m_i < .,. >$$

and the whole symmetry group is given by the diagonal action on Q of:

$$G_0 \doteq T_{\mathbf{R}^2} \times O(2)$$

but for the Routh procedure we are interested in:

$$G \doteq T_{\mathbf{R}^2} \times SO(2)$$

where we denote with $T_{\mathbf{R}^2}$ the translations group in the plane. G is an Abelian Lie group.

Remark 2.1

In the sequel, the presence of the planar reflections S_2 in the whole symmetry group G_0 of our system will play a crucial role in the study of the reduced configuration space (see ([7]). S_2 symmetry will survive the Lagrangian reduction.

If G is an Abelian symmetry group, one can find a complete set of *cyclic coordinates*, there exists a system of coordinates on M such that the action of G on M can be written additively:

$$m \in M \qquad m = (x_1.....x_r, \theta_{r+1}.....\theta_n) = (\vec{x}, \vec{\theta})$$

$$\Phi_g(m) = \Phi_g(\vec{x}, \vec{\theta}) = (\vec{x}, \vec{\theta} + \vec{\lambda})$$

with $\vec{\lambda} \in \mathbf{R}^a \times (S^1)^b \quad a + b = n - r - 1$ (in our case $n = 6, a = 2, b = 1$).

Remark 2.2

The additivity of the action Φ and the invariance condition $L \circ \Phi_g^T(m, v_m) = L(m, v_m)$ imply that

$$L(m, v_m) = L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}})$$

In the case of *cyclic coordinates* the Euler-Lagraian equations have the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \quad i = 1...r$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = 0 \quad i = r + 1...n$$

Therefore the Integrals of motion, we have the following coordinate representation:

$$J_i = \frac{\partial L}{\partial \dot{\theta}_i}$$

Due to the Abelian character of G these are independent integrals of the motion of the system. Now we can recall the theorem which describe the reduction "a la Routh":

Theorem A

Let $(M, <.,.>_M, V, G)$ be a Mechanical System with Symmetry group G which is an Abelian Lie group which acts freely and transitively. Assume that

$$\det\left(\frac{\partial^2 L}{\partial\dot{\theta}_i \partial\dot{\theta}_j}\right) \neq 0 \tag{6}$$

Define:

$$R_\nu(\vec{x}, \dot{\vec{x}}) \doteq L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}) - \sum_i \nu_i \dot{\theta}_i \tag{7}$$

where $\dot{\theta}_i$ are computed inverting:

$$\nu_i = J_i(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}})$$

Then R_ν is a Lagrangian which describe the motion constrained to

$$\{(m, v_m) \in TM \ \ / \ \ J(m, v_m) = \nu\}$$

with configuration manifold

$$\tilde{M} \doteq M/G$$

(the reduced configuration space).

For interesting geometrical descriptions of this theorem one can see [3], [4].

Remark 2.3

If the Lagrangian L is a positive function then also R_0 is positive. In fact one has:

$$R_0(\vec{x}, \dot{\vec{x}}) \doteq L(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}(\vec{x}, \dot{\vec{x}}))$$

where $\dot{\vec{\theta}}(\vec{x}, \dot{\vec{x}})$ is the unique solution of

$$J_i(\vec{x}, \dot{\vec{x}}, \dot{\vec{\theta}}) = 0 \qquad i = r+1 \ldots n$$

Let us now apply the previous Theorem to the Three Body Problem.

$$M \doteq Q$$

where:

$$Q \doteq \{\vec{q}_i \in \mathbf{R}^2; i = 1, 2, 3\} \simeq \mathbf{R}^6$$

the metric:

$$< .,. >_Q \doteq \sum_{i=1}^{3} m_i < .,. >$$

where $< .,. >$ is the standard scalar product in $\mathbf{R^2}$; the potential:

$$V(q) \doteq \sum_{i,j,i\neq j}^{3} \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|}$$

and the symmetry group turns out to be:

$$G \doteq T_{\mathbf{R}^2} \times SO(2)$$

Now we find the *cyclic coordinates* for our system. This will be done in some steps.

First step:

$$\mathcal{T}_1 : Q \to Q$$

$$\begin{cases} x_1 = q_2 - q_1 \\ x_2 = q_3 - q_1 \\ x_3 = (m_1 q_1 + m_2 q_2 + m_3 q_3)/(m_1 + m_2 + m_3) \end{cases} \tag{8}$$

by means of $\mathcal{T}_1$ after some algebraic manipulations (1) becomes:

$$L = \frac{\mu}{2}\|\dot{x}_3\|^2 + \frac{1}{2}\sum_{i=1}^{3} \tilde{M}_{ij} < \dot{x}_i, \dot{x}_j > + \frac{m_1 m_2}{\|x_1\|} + \frac{m_1 m_3}{\|x_2\|} + \frac{m_2 m_3}{\|x_2 - x_1\|} \tag{9}$$

where $\|x\| =< x, x >$ and

$$\tilde{M} \doteq \begin{pmatrix} M_{11} & -M_{12} \\ -M_{12} & M_{22} \end{pmatrix} \tag{10}$$

$$M_{11} \doteq \frac{m_2(m_1 + m_3)}{\mu} \qquad M_{22} \doteq \frac{m_3(m_1 + m_2)}{\mu} \qquad M_{12} \doteq \frac{m_2 m_3}{\mu} \qquad \mu \doteq \sum_{i=1}^{3} m_i \tag{11}$$

Now in the second step, we consider the following transformation:

$$\begin{aligned} \mathcal{T}_2 : Q &\longrightarrow Q \\ (x_1, x_2, x_3) &\longrightarrow (\rho_1, \theta_1, \rho_2, \theta_2, x_3) \end{aligned} \tag{12}$$

where:

$$\begin{cases} x_i^1 = \rho_i \cos\theta_i \\ x_i^2 = \rho_i \sin\theta_i \end{cases} \tag{13}$$

By means of $\mathcal{T}_2$ one finds:

$$Q \sim \mathbf{R}^2 \times \mathbf{R}^2_+ \times [0, 2\pi] \times [0, 2\pi]$$

in this coordinates one find that the potential V becomes:

$$V(\rho_1, \rho_2, \theta_1 - \theta_2) = \frac{m_1 m_2}{\rho_1} + \frac{m_1 m_3}{\rho_2} + \frac{m_2 m_3}{[\rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2) + \rho_1^2]^{1/2}} \tag{14}$$

hence the Lagrangian (9) becomes:

$$\begin{aligned} L = \tfrac{\mu}{2}\|\dot{x}_3\|^2 + \tfrac{1}{2}\textstyle\sum_{i=0}^2 \tilde{M}_{ii}(\dot{\rho}_i^2 + {\rho_i}^2\dot{\theta}_i^2) + \\ -M_{12}(\dot{\rho}_1\dot{\rho}_2 + \rho_1\rho_2\dot{\theta}_1\dot{\theta}_2)\cos(\theta_1 - \theta_2) + \\ -M_{12}(\dot{\rho}_1\dot{\theta}_2\rho_2 - \dot{\theta}_1\rho_1\dot{\rho}_2)\sin(\theta_1 - \theta_2) + V(\rho_1, \rho_2, \theta_1 - \theta_2) \end{aligned} \tag{15}$$

we can rewrite it as:

$$L = \frac{\mu}{2}\|\dot{x}_3\|^2 + \frac{1}{2}\sum_{i,j=1}^{2} \bar{M}_{ij}\dot{\rho}_i\dot{\rho}_j + \frac{1}{2}\sum_{i,j=1}^{2} I_{ij}\dot{\theta}_i\dot{\theta}_j - M_{12}(\dot{\rho}_1\dot{\theta}_2\rho_2 - \dot{\theta}_1\rho_1\dot{\rho}_2)\sin(\theta_1 - \theta_2) + V(\rho_1, \rho_2, \theta_1 - \theta_2) \tag{16}$$

where:

$$\bar{M} \doteq \begin{pmatrix} M_{11} & -M_{12}\cos(\theta_1 - \theta_2) \\ -M_{12}\cos(\theta_1 - \theta_2) & M_{22} \end{pmatrix} \tag{17}$$

$$I \doteq \begin{pmatrix} M_{11}\rho_1^2 & -M_{12}\rho_1\rho_2\cos(\theta_1 - \theta_2) \\ -M_{12}\rho_1\rho_2\cos(\theta_1 - \theta_2) & M_{22}\rho_2^2 \end{pmatrix} \tag{18}$$

We perform now the last transformation:

$$\mathcal{T}_3 : Q \to Q$$

$$\begin{cases} \varphi = & \theta_1 - \theta_2 \\ \psi = & \theta_1 + \theta_2 \end{cases} \tag{19}$$

So we obtain:

$$L = \frac{\mu}{2}\|\dot{x}_3\|^2 + \frac{1}{2}\sum_{i,j=1}^{2} \bar{M}_{ij}\dot{\rho}_i\dot{\rho}_j + \frac{1}{2}(I_\psi\dot{\psi}^2 + I_\varphi\dot{\varphi}^2 + 2I_{\psi\varphi}\dot{\psi}\dot{\varphi}) - A_\psi\dot{\psi} + A_\varphi\dot{\varphi} + V(\rho_1, \rho_2, \varphi) \tag{20}$$

where:

$$\begin{cases} I_\psi = & \frac{1}{4}(I_{11} + I_{12} + I_{22}) \\ I_\varphi = & \frac{1}{4}(I_{11} - I_{12} + I_{22}) \\ I_{\psi\varphi} = & \frac{1}{4}(I_{11} - I_{22}) \\ A_\psi = & \frac{M_{12}}{2}(\dot{\rho}_1\rho_2 - \dot{\rho}_2\dot{\rho}_1)\sin\varphi \\ A_\varphi = & \frac{M_{12}}{2}(\dot{\rho}_1\rho_2 + \dot{\rho}_2\dot{\rho}_1)\sin\varphi \end{cases} \tag{21}$$

hence the cyclic variables are: ψ, x_3. The Lagrangian (20) is invariant under the action of $G = T_{\mathbf{R}^2} \times SO(2)$:

$$\begin{aligned} \Phi : Q \times G &\longrightarrow Q \\ (\rho_1, \rho_2, x_3, \varphi, \psi) \times (l, \alpha) &\longrightarrow (\rho_1, \rho_2, x_3 + l, \varphi, \psi + \alpha) \end{aligned}$$

the associated integrals of the motion are:

Total Linear Momentum

$$P \doteq \sum_{i=1}^{3} m_i \dot{q}_i = \mu \dot{x}_3 = (\mu \dot{x}_3^1, \mu \dot{x}_3^2)$$

Total Angular Momentum

$$J \doteq \sum_{i=1}^{3} m_i q_i \wedge \dot{q}_i = \mu x_3 \wedge \dot{x}_3 + I_\psi \dot{\psi} + I_{\psi\varphi} \dot{\varphi} - A_\psi$$

Let us apply Theorem **A** to construct the reduced Lagrangian on the submanifold:

$$\{P = p^*, J = j^*\} \subset TQ$$

then we will put $p^* = 0, j^* = 0$. Hence we have:

$$R_{p^*,j^*}(\rho_1, \rho_2, \dot{\rho}_1, \dot{\rho}_2, \varphi, \dot{\varphi}) \doteq L(\rho_1, \rho_2, \dot{\rho}_1, \dot{\rho}_2, \varphi, \dot{\varphi}, \dot{\psi}, \dot{x}_3) - J\dot{\psi} - P \cdot \dot{x}_3 \tag{22}$$

where $\dot{\psi}$ and $\dot{x}_3$ are computed inverting the expressions of $P = p^*$ and $J = j^*$, so one gets:

$$\begin{aligned} R_{p^*,j^*} = \tfrac{1}{2} p^{*2} + \tfrac{1}{2} \textstyle\sum_{i=1,2} \bar{M}_{ij} \dot{\rho}_i^2 + V(\rho_1, \rho_2, \varphi) + \\ + \tfrac{\dot{\varphi}^2}{2} [4 \det(I) + 3 I_{12}^2] / I_\psi + [((j^* - x_3 \wedge p^*) + A_\psi) I_{\psi\varphi} / I_\psi + A_\varphi] \dot{\varphi} + \\ - (j^* - x_3 \wedge p^*) A_\psi / I_\psi + (j^* - x_3 \wedge p^*)^2 / 2 I_\psi - A_\psi^2 / I_\psi \end{aligned} \tag{23}$$

now we restrict (23) on $\{p^* = 0, j^* = 0\} \subset TQ$ getting:

$$L^0 \doteq R_{0,0} = \frac{1}{2} \sum_{i=1,2} \bar{M}_{ij} \dot{\rho}_i^2 + V(\rho_1, \rho_2, \varphi) + \frac{\dot{\varphi}^2}{2} [4 \det(I) + 3 I_{12}^2] / I_\psi + [A_\psi I_{\psi\varphi} / I_\psi + A_\varphi] \dot{\varphi} - A_\psi^2 / I_\psi \tag{24}$$

moreover we have to notice that by means of Theorem **A** we have that the Lagrangian (24) is constrained on

$$\{P = 0, J = 0\}$$

and describes the dynamics on the configuration manifold

$$\tilde{Q} = Q/G \sim \mathbf{R}_+{}^2 \times [0, 2\pi]$$

We denote each point of $\tilde{Q}$ with ζ and we give it coordinates $(\rho_1, \rho_2, \varphi) \in \mathbf{R}_+^2 \times [0, 2\pi]$.

Remark 2.4

From Remark 2.3 one conclude that the reduced Lagrangian L^0 is a positive function.

3 The Perturbation setting

Let us choose now:

$$m_1 \doteq m \quad m_2 \doteq m a_1 \epsilon \quad m_3 \doteq m a_2 \epsilon \tag{25}$$

where ϵ is a small parameter $(0 < \epsilon < 1)$, $a_1, a_2 \in (0, 1)$ m will be chosen later on. From (25) the entries of the "mass" matrix $\bar{M}$ are:

$$M_{11} = m\epsilon a_1 \frac{1 + a_2\epsilon}{1 + (a_1 + a_2)\epsilon}$$

$$M_{22} = m\epsilon a_2 \frac{1 + a_2\epsilon}{1 + (a_1 + a_2)\epsilon}$$

$$M_{12} = m\epsilon^2 \frac{a_1 a_2}{1 + (a_1 + a_2)\epsilon}$$

The potential $V(\rho_1, \rho_2, \varphi)$ becomes:

$$V = \frac{m^2 a_1 \epsilon}{\rho_1} + \frac{m^2 a_2 \epsilon}{\rho_2} + \frac{m^2 a_1 a_2 \epsilon^2}{[\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi]^{1/2}} \tag{26}$$

We want to write the Lagrangian in the following form:

$$L^0 = \epsilon \left\{ \sum_{i=1,2} \left[\frac{m a_i}{2} \dot{\rho}_i{}^2 + \frac{m^2 a_i}{\rho_i} \right] + \frac{m a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{2 \sum_{i=1,2} a_i \rho_i^2} + V_\epsilon \right\} \tag{27}$$

where V_ϵ is of order $O(\epsilon)$. This term include the contributions due to the gravitational interaction between the the smaller bodies and also due to the kinetic energy.

The perturbation V_ϵ in the Lagrangian L^0 can be so defined:

$$\begin{aligned} V_\epsilon = {} & \tfrac{1}{2} \textstyle\sum_{i=1,2} \frac{\bar{M}_{ii}}{\epsilon} \dot{\rho}_i^2 - \sum_{i=1,2} \frac{m a_i}{2} \dot{\rho}_i{}^2 + \\ & + \left[\frac{[4\det(I) + 3I_{12}^2]}{I_\psi \epsilon} - \frac{m a_1 a_2 (\rho_1 \rho_2)^2}{\sum_{i=1,2} a_i \rho_i^2} \right] \frac{\dot{\varphi}^2}{2} + \frac{m^2 a_1 a_2 \epsilon}{[\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi]^{1/2}} + \\ & + m\epsilon/(1 + (a_1 + a_2)\epsilon)[((I_{\psi\varphi}/I_\psi)(\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1) + \\ & + (\dot{\rho}_1\rho_2 + \dot{\rho}_2\rho_1))\dot{\varphi}\sin\varphi - (\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1)^2 (M_{12}/I_\psi)\sin^2\varphi] \end{aligned} \tag{28}$$

We give the expression of the first term in ϵ of V_ϵ:

$$V_\epsilon = m\epsilon \left\{ -\sum_i (a_i \dot{r}_i)^2 - \frac{(a_1 a_2 \rho_1 \rho_2)^2}{[\sum_{i=1,2} a_i \rho_i^2]^2} \frac{\dot{\varphi}^2}{2} + \right.$$
$$+ \frac{m^2 a_1 a_2 \epsilon}{[\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi]^{1/2}} +$$
$$\left. + \frac{a_1\rho_1^2 - a_2\rho_2^2}{\sum_i a_i r_i^2} (\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1) + (\dot{\rho}_1\rho_2 + \dot{\rho}_2\rho_1))\dot{\varphi}\sin\varphi \right\} + o(\epsilon) \tag{29}$$

In the preceding formulation of the perturbation term we have written explicitly the perturbation parameter ϵ only in the relevant place where it appear.

From the form of the Lagrangian one can easily deduce the Collisions set K_c:

$$K_c \doteq \{(\rho_1, \rho_2, \varphi) \in \mathbf{R}^2 \times [0, 2\pi] \ / \ (0, 0, \varphi) \vee (\rho, \rho, 0) \vee (0, \rho_2, \varphi) \vee (\rho_1, 0, \varphi)\} \tag{30}$$

The first condition represents the *triple collision* and the other three conditions represent the *double collisions.*

$V_\epsilon(\rho_1, \rho_2, \dot{\rho}_1, \dot{\rho}_2, \varphi, \dot{\varphi})$ is a smooth function out on $T(\tilde{Q} \backslash K_c)$. Using (28) one can easily verify that:

$$\lim_{\epsilon \to 0} V_\epsilon = 0 \quad \forall \zeta \in \tilde{Q} \backslash K_c$$

Moreover all the derivatives of V_ϵ go to zero when $\epsilon \to 0$.

We can now define the perturbation problem. Chosen $T > 0$, we consider as Action functional for Three Body Problem reduced on $\tilde{Q}$:

$$\mathcal{A}_T[\zeta, \epsilon] \doteq \int_0^T dt (L^0(\zeta(t) + V_\epsilon(\zeta(t))) \tag{31}$$

which can be written as:

$$\mathcal{A}_T[\zeta, \epsilon] = \epsilon \int_0^T dt \left\{ \sum_{i=1,2} \left[\frac{a_i}{2} \dot{\rho}_i^{\,2} + \frac{a_i}{\rho_i} \right] + \frac{a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{2 \sum_{i=1,2} a_i \rho_i^2} + V_\epsilon \right\} \tag{32}$$

Without loss of generality we have chosen $m = 1$. The Critical points of $\mathcal{A}_T[., \epsilon]/\epsilon$ and $\mathcal{A}_T[., \epsilon]$ are the same for $\epsilon > 0$; moreover $\mathcal{A}_T[., \epsilon]/\epsilon$ has a limit when ϵ goes to 0. Therefore perturbation theory can then be applied to $\mathcal{A}_T[., \epsilon]/\epsilon$ to find the critical points of $\mathcal{A}_T[., \epsilon]$.

4 Geometry of the Reduced Phase Space and the Action functional

We now introduce some results proved in [7] in order to describe the geometry of the reduced Configuration space. In [7] we found a class of compact sets of orbits which are good candidates to contain critical points of the full problem. Moreover it was proved that the Collision solutions are not minima for the Action functional reduced on $J = 0$ and on such trajectories the functional is not C^1. We now describe the geometry of the reduced Configuration space, then in the next paragraph we will construct a solution ζ_0 for the unperturbed problem, then we we show that this critical point is regular and it can be *extended* to ζ_ϵ for $\epsilon > 0$ for the perturbed problem. Then we will show that both the unperturbed critical point ζ_0 and the continued ζ_ϵ live in the class of compact sets. By means of the properties of collision solutions turns out that the perturbed critical points, is not a collision solution.

In the quoted paper we proved that if one uses the relative distances $\rho_i = \|q_j - q_k\|$ (*cyclic permutation of* $i, j, k = 1, 2, 3$) as coordinates for the Configuration Space, the reduction is described by the map:

$$\begin{aligned} p : Q &\longrightarrow C \\ (q_1, q_2, q_3) &\longrightarrow (\rho_1, \rho_2, \rho_3) \end{aligned} \tag{33}$$

where :

$$C \doteq \left\{ \vec{\rho} = (\rho_1 \ \rho_2 \ \rho_3) \in \mathbf{R}^3_+ \ / \ \rho_i + \rho_j \geq \rho_k \ \textit{and cyclic permutations} \right\}$$

and

$$\partial C \doteq \left\{ \vec{\rho} = (\rho_1 \ \rho_2 \ \rho_3) \in \mathbf{R}^3_+ \ / \ \rho_i + \rho_j = \rho_k \ \textit{and cyclic permutations} \right\}$$

Taking account of reflection symmetry the reduction turns out to be a *double covering.* Hence in order to study global aspects of the motion one needs a new Configuration manifold which takes account of the nature of the double covering.

Our "new" reduced Configuration manifold is so defined:

$$\begin{aligned} Q &\longrightarrow \tilde{Q} \\ (q_1, q_2, q_3) &\longrightarrow (\rho_1, \rho_2, \rho_3, z) \doteq \zeta \end{aligned} \tag{34}$$

where:

$$\tilde{Q} \doteq C \times \{+1\} \cup \partial C \times \{0\} \cup C \times \{-1\}$$

Let $(\rho_1, \rho_2, \rho_3, z) \doteq (\vec{\rho}, z)$. In the definitions two copies of C are glued along the boundary ∂C. The z coordinate takes account of the reflexion and parameterized the sheets of the covering ($z = +1, -1$, $z = 0$ parameterizes the boundary ∂C).

Remark 4.1

We denote with the same symbol $\tilde{Q}$ the new reduced Configuration manifold since, as we now show, there is a homeomorphism (which is a diffeomorphism out of collinear trajectories) between the two systems of coordinates.

$$\{(\rho_1, \rho_2, \varphi)\} \qquad \{(\rho_1, \rho_2, \rho_3, z)\}$$

In the first system of coordinates φ is the angle between the two "planets", measured by the oriented line on which mass m_1 and mass m_2 stay. Hence the angle φ is oriented. We have to stress that the choice of the oriented line is arbitrary since we are considering lines up to rotations. Let us consider the map:

$$\begin{aligned} \mathcal{T} : \tilde{Q} &\longrightarrow \tilde{Q} \\ (\rho_1, \rho_2, \varphi) &\longrightarrow (\rho_1, \rho_2, \rho_3, z) \end{aligned} \tag{35}$$

$$\begin{cases} \rho_i & = \rho_i \quad i = 1, 2 \\ \rho_3 & = (\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi)^{1/2} \\ z & = \begin{cases} +1 & \text{if } \varphi \in (0, \pi) \\ 0 & \text{if } \varphi = 0, \pi \\ -1 & \text{if } \varphi \in (\pi, 2\pi) \end{cases} \end{cases}$$

This transformation is invertible, and fails to be a diffeomorphism only at $z = 0 \sim \varphi = 0, \pi$ (Collinear Configurations). The inverse is:

$$\begin{cases} \rho_i & = \rho_i \quad i = 1, 2 \\ \varphi & = \arccos[(\rho_1^2 + \rho_2^2 - \rho_3^2)/(2\rho_1\rho_2)] \end{cases}$$

$$z = 1 \Rightarrow \varphi \in (0, \pi)$$

$$z = 0 \ , \ \rho_3 = \rho_2 + \rho_1 \ , \ \rho_1 = \rho_3 + \rho_2 \Rightarrow \varphi = 0$$

$$z = 0 \ , \ \rho_2 = \rho_3 + \rho_1 \Rightarrow \varphi = \pi$$

$$z = -1 \Rightarrow \varphi \in (\pi, 2\pi)$$

In $\tilde{Q}$ equipped with the "mutual distances" coordinates we can introduce an operation σ:

$$\begin{aligned} \sigma : \tilde{Q} &\longrightarrow \tilde{Q} \\ (\vec{\rho}, z) &\longrightarrow (\vec{\rho}, -z) \end{aligned} \tag{36}$$

$\sigma^2 = id_{\tilde{Q}}$, this operation represents the reflexion w.r.t. a generic line in the plane. In the coordinates $\{(\rho_1, \rho_2, \varphi)\}$ the action of σ becomes:

$$\begin{aligned} \sigma : \tilde{Q} &\longrightarrow \tilde{Q} \\ (\rho_1, \rho_2, \varphi) &\longrightarrow (\rho_1, \rho_2, 2\pi - \varphi) \end{aligned} \tag{37}$$

On $\tilde{Q}$ we defined the following function spaces:

$$\tilde{\Lambda}_T^a = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ / \ \zeta(t + T/2) = \sigma\zeta(t)\} \tag{38}$$

$$\tilde{\Lambda}_T^s = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ / \ \zeta(t + T/2) = \zeta(t)\} \tag{39}$$

so the space of T-periodic, H^1-functions with values in $\tilde{Q}$

$$B_{\tilde{Q}}^1 \doteq H^1([0,T], \tilde{Q}) \doteq \left\{(\vec{\rho}, z) \in \tilde{Q} \ / \ \rho_i \in H^1([0,T], \mathbf{R}_+) \ \ i = 1,2,3 \ \ z(t) \in \{+1, 0, -1\}\right\}$$

can be decomposed into:

$$B_{\tilde{Q}}^1 = B_{\tilde{Q}}^1 \cap \tilde{\Lambda}_T^s \oplus B_{\tilde{Q}}^1 \cap \tilde{\Lambda}_T^a \tag{40}$$

In the coordinates $\{(\rho_1, \rho_2, \varphi)\}$ we can construct $\tilde{\Lambda}_T^a$ and $\tilde{\Lambda}_T^a$ by means of $\mathcal{T}$.

In [7] it is proved that the Reduced Action functional for the full Three Body Problem can be defined on $H_{\tilde{Q}}^1$ and written as:

$$\bar{\mathcal{A}}_T[\zeta] = \int_0^T dt \left\{ \sum_{ij}^{3} B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j + \sum_{i,j,k} \frac{m_i m_j}{\rho_k} \right\}$$

where $\vec{\rho} \doteq (\rho_1, \rho_2, \varphi)$ and B is a matrix valued-function:

$$B : \tilde{Q} \to M(3 \times 3, \mathbf{R})$$

the entries are smooth functions out of Collision configurations, and moreover the following results hold:

Proposition 4.1

The matrix $B(\vec{\rho})$ is symmetric and such that there exist $b_1, b_2 \in \mathbf{R}_+$ such that for all vectors $\vec{V} \in \mathbf{R}^3$:

$$b_1\|\vec{V}\|^2 \leq (\vec{V}, B(\vec{\rho})\vec{V}) \leq b_2\|\vec{V}\|^2 \tag{41}$$

where with $(.,.)$ we denote the natural scalar product in $\mathbf{R}^3$.

(see [7])

In [7] we proved a sort of "weak Poincarè inequality":

Proposition 4.2

Let be $\zeta(t) = (\vec{\rho}(t), z(t)) \in \tilde{\Lambda}_T^a$. The following inequality holds:

$$\int_0^T dt \sum_{ij}^{3} B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j = \int_0^T dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) \geq \frac{4b_1}{T} \sup_{s\in[0,T]} \min_{i,j,k}{}' \left\{(\rho_i(s) + \rho_j(s) - \rho_k(s))^2\right\} \tag{42}$$

This inequality allows to show that there are C^0-compact sets. Let us now define the following subsets (depending on k_2):

$$M_{k_2} \doteq \left\{\zeta \in \tilde{\Lambda}_T^a \cap B_{\tilde{Q}}^1 \ \ such \ \ that: \ \sup_{s\in[0,T]} g(\vec{\rho}(s)) \geq k_2\right\}$$

with $k_2 > 0$ and where:

$$g(\vec{\rho}(s)) \doteq \frac{1}{\|\vec{\rho}(s)\|} \min_{i,j,k}{}' \left\{(\rho_i(s) + \rho_j(s) - \rho_k(s))^2\right\}$$

$$\|\vec{\rho}(s)\| = \sqrt{\sum_{i=1}^{3}(\rho_i)^2}$$

We found [7] that for any k_1 , $k_2 > 0$, the following sets

$$M_{k_1,k_2} \doteq M_{k_2} \cap \left\{\zeta \in \tilde{\Lambda}_T^a \cap B_{\tilde{Q}}^1 \ / \ \bar{\mathcal{A}}_T[\zeta(t)] \leq k_1\right\}$$

are $L^2([0,T], \tilde{Q})$-compact subsets of the sublevels of the Action functional. In the remark below we show that the compactness is in C^0 topology.

The same result can be concluded for the set:

$$\tilde{M}_{k_1,k_2} \doteq \left\{\zeta \in B_{\tilde{Q}}^1 \ / \ \exists t^* \in [0,T] \ / \ \sigma(\zeta(t^*)) = \zeta(t^* + T/2), \ \ \bar{\mathcal{A}}_T[\zeta] \leq k_1, \ \sup_{t\in[0,T]} g[\zeta(t)] \geq k_2\right\}$$

We have to stress that the proof of the compactness of M_{k_1,k_2} depends only on the existence at least of one time of antisymmetricity. This result will be very important in the next sections.

Remark 4.2

Let us consider the following decomposition for the set $\tilde{M}_{k_1,k_2}$:

$$\tilde{M}_{k_1,k_2} = B_{k_2} \cup B^c_{k_2}$$

where

$$B_{k_2} \doteq \left\{ \zeta \in \tilde{M}_{k_1,k_2} \ / \ \sup_{s\in[0,T]} \|\vec{\rho}(s)\| \leq 3\sqrt{Tk_2/b_1} \right\}$$

The set B_{k_2} is trivially C^0-compact. On the other hand, for its complement, we can consider the facts:

(i) by hipothesis $\sup_s g[\zeta(s)] \geq k_1$

(ii) the following inequality holds:

$$\sup_s \|\vec{\rho}(s)\| - \inf_s \|\vec{\rho}(s)\| \leq \int_0^T dt \left| \frac{d}{dt} \|\vec{\rho}(t)\| \right|$$

We prove the compacness for $B^c_{k_2}$.

Using polar coordinates $r \doteq \|\vec{\rho}\|, \vec{\Sigma} \doteq \vec{\rho}/\|\vec{\rho}\|$, it turns out that

$$k_2 \geq \bar{\mathcal{A}}_T[\zeta] \geq a \int_0^T dt \dot{r}^2 + b \int_0^T dt r^2 + c \int_0^T dt (\dot{\vec{\Sigma}}, \dot{\vec{\Sigma}})$$

With $a, b, c > 0$.

Now taking a sequence $\{r_n(t), \vec{\Sigma}_n(t) \in S^2\} \in B^c_{k_2}$ with $r_n(t) = \tilde{r}_n(t) + 1 \int_0^T \frac{dt}{T} r_n(t)$ one finds that there exist $C_1, C_2, C_3 > 0$ such that:

$$\int_0^T dt r_n \leq C_1 \ , \ \int_0^T dt \dot{\tilde{r}}_n^2 \leq C_2 \ , \ \int_0^T dt \|\dot{\vec{\Sigma}}\|^2 \leq C_3$$

then, the first term, the mean of $r_n(t)$, gives a bounded sequence in $\mathbf{R}$. Using standard Sobolev inequality, the second term give that $\sup_s r_n(s)$is bounded. Since $\vec{\Sigma}(t) \in S^2 \ \forall t \in [0,T]$ then $\sup_s \|\vec{\Sigma}(s)\|$ is bounded. hence we can extract a subsequence converging in the C^0 topology. Therefore $B^c_{k_2}$ is C^0-compact. @

In [2],[7] it is proved that collision solution are not minima for the Action functional of the full Three Body Problem. Indeed the following Proposition holds:

Proposition 4.3

In $\vec{\rho}$ coordinates the collision solutions are described, near the collision instant t_c, by:

$$\rho_i(t) = (c_i + \gamma_i(t))(t - t_c)^{\frac{2}{3}} \quad i = 1, 2, 3$$

with c_i functions of the limit Central Configuration ([2]) and

$$\|\gamma_i(t)\| = O(t^d) \quad 0 < d < 1 \quad i = 1, 2, 3$$

There exist continuous functions $(g_1(t), g_2(t), g_3(t)) \doteq \vec{g}(t)$ and δ small enough such that:

$$\bar{\mathcal{A}}_T[\vec{\rho}_c(t) + \delta\vec{g}(t)] - \bar{\mathcal{A}}_T[\vec{\rho}_c(t)] \leq C_1\delta^{\beta} - C_2\delta^{1/2}$$

with $C_1, C_2 > 0$ and $\beta > 1/2$.

Hence Collision solutions are neither minima nor regular critical points for the the Action functional of the Three-Body Problem reduced on $J = 0$; thus regular critical points are not collision orbits.

5 Critical manifold and its perturbation

In this section we construct an explicit solution for the unperturbed problem. This solution turns out to be a smooth critical point for the Action-functional therefore we can apply the perturbation theory to the full problem.

The full Action functional is denoted by :

$$\mathcal{A}_T[\zeta, \epsilon] = \mathcal{A}_T^0[\zeta] + \mathcal{A}_T^{\epsilon}[\zeta] \tag{43}$$

where:

$$\mathcal{A}_T^{\epsilon}[\zeta] \doteq \int_0^T dt V_{\epsilon}$$

with:

$$\lim_{\epsilon \to 0} \mathcal{A}_T^{\epsilon}[\zeta] = 0$$

uniformly out of the collisions, but if ζ is a collision solution the limit holds.

The Lagrangian at $\epsilon = 0$, $\tilde{L}_0$, is given by:

$$\tilde{L}_0 = \sum_{i=1,2} \left[\frac{a_i}{2} \dot{\rho}_i^2 + \frac{a_i}{\rho_i} \right] + \frac{a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{2 \sum_{i=1,2} a_i \rho_i^2} \tag{44}$$

whose Action functional is given by:

$$\mathcal{A}_T^0[\zeta] = \int_0^T dt \tilde{L}_0(\zeta(t)) \tag{45}$$

The Lagrangian (44) describes a system formed by non-interacting masses a_1 a_2, each of which is attracted to the other by a Keplerian force, under the constrain that the total angular momentum is zero.

We define $\mathcal{A}_T^0[.]$ on the Hilbert space:

$$H^1_{\tilde{Q}} = \left\{(\rho_1(t), \rho_2(t), \varphi(t)) \ / \ \varphi \in H^1([0,T],[0,2\pi]) \ \ \rho_i \in H^1([0,T],\mathbf{R}_+) \ i = 1,2\right\}$$

with scalar product:

$$(\zeta^1, \zeta^2)_{H^1_{\tilde{Q}}} \doteq (\varphi^1, \varphi^2)_{H^1([0,T],[0,2\pi])} + \sum_{i=1,2} (\rho_i^1, \rho_i^2)_{H^1([0,T],\mathbf{R}_+)}$$

where $(.,.)_{H^1}$ is the standard H^1-scalar product.

The Euler-Lagrange equations of the system are:

for $i = 1,2; j \neq i$

$$a_i \frac{d^2\rho_i}{dt^2} = -\frac{a_i}{\rho_i^2} + \frac{a_1 a_2 \rho_i (\rho_j \dot{\varphi})^2}{\sum_{l=1,2} a_l \rho_l^2} - \frac{a_1 a_2 (\rho_1 \rho_2 \dot{\varphi})^2}{(\sum_{i=l,2} a_l \rho_l^2)^2} a_i \rho_i$$

and

$$\frac{a_1 a_2 (\rho_1 \rho_2)^2 \dot{\varphi}}{\sum_{l=1,2} a_l \rho_l^2} = c$$

with $c \in \mathbf{R}$. These equations can be written as:

$$\begin{cases} a_i \frac{d^2\rho_i}{dt^2} = & -\frac{a_i}{\rho_i^2} + \frac{c}{a_i \rho_i^3} \quad i = 1,2 \\ \dot{\varphi} = & \frac{c \sum_k a_k \rho_k^2}{a_1 a_2 (\rho_1 \rho_2)^2} \end{cases} \tag{46}$$

The system (46) admits circular solution $\zeta_0 : [0,T] \to \tilde{Q}$ given by:

$$\begin{cases} \rho_i^0(t) = & c^2/a_i^2 \quad i = 1,2 \\ \varphi^0(t) = & \varphi^*(0) + \omega^* t \ ; \ \ \omega^* = (a_1^3 + a_2^3)/(c^3) \end{cases} \tag{47}$$

The solutions are parameterized by c and $\varphi(0)$. We have now to specify c and $\varphi(0)$ such that ζ_0 be circular periodic orbit, antisymmetric in the sense defined in the preceding section , i.e. reflected with respect to the line which connects two of the three masses.

In order to seek for these condition one has to take account of the fact that φ is the relative between the two bodies. φ is measured by this *oriented line*, therefore $\varphi_0(t) \in [0, 2\pi]$ has a a minimal period $T_0 = T/2$ where with T we denote the period of the trajectories in $\tilde{\Lambda}^a_T$.

Now $\zeta_0(t)$ belongs to set of trajectories defined by the following relations

$$\begin{cases} \rho_i(t + T_0/2) - \rho_i(t) = & 0 \quad i = 1, 2 \quad \forall t \in [0, T_0] \\ \varphi(t + T_0/2) - \varphi(t) = & -\pi \quad \forall t \in [0, T_0] \end{cases} \tag{48}$$

In the next section we will prove that ζ_0 belongs to $\tilde{\Lambda}^a_{T_0}$ and the same for the *continued* solution ζ_ϵ.

Substituting the solution $\zeta_0(t)$ in (48) for c one finds:

$$c = \left[\frac{T_0(a_1^3 + a_2^3)}{2\pi}\right]^{1/3} \tag{49}$$

Remark 5.1

Using the Schwartz Inequality the second condition of (48) implies:

$$\|\dot{\varphi}\|^2_{L^2} \geq \frac{(\pi)^2}{T_0} \tag{50}$$

With this condition we have a circular T_0-periodic antisymmetric solutions for the unperturbed problem.

We described solution which lives in $\tilde{Q}$, its image in Q, reduced the action of the translations, is given by the couple (φ, ψ). Using the condition that the total angular momentum is zero and that we are considering the case $\epsilon = 0$, one obtains:

$$\begin{cases} \rho_1^0 = (T_0/2\pi)^{2/3}(1 + a^3)^{2/3} \\ \rho_2^0 = (T_0/2\pi)^{2/3}(1 + a^3)^{2/3}\frac{1}{a^2} \\ \varphi^0(t) = \frac{2\pi}{T_0}t + \varphi(0) \\ \psi^0(t) = -\frac{2\pi}{T_0}t\frac{a_1^3 - a_2^3}{a_1^3 + a_2^3} + \psi(0) \quad \psi(0) \in [0, 2\pi] \end{cases} \tag{51}$$

From the expression of $\dot{\psi}$ one gets the condition for the periodicity of the motion in Q:

$$\left(\frac{a_2}{a_1}\right)^3 \in \mathbf{Q} \tag{52}$$

When this ratio is not a rational number then $\varphi^0(t), \psi^0(t)$ fill a two dimensional torus $[0, 2\pi] \times [0, 2\pi] \doteq \mathbf{T}^2$ in Q. From (52) one can write the trajectory in $\mathbf{T}^2$ corresponding

to ζ_0:

$$\frac{a_1^3 - a_2^3}{a_1^3 + a_2^3}\psi^0(t) + \varphi^0(t) = \frac{a_1^3 - a_2^3}{a_1^3 + a_2^3}\psi(0) + \varphi(0) \bmod 2\pi \tag{53}$$

If (52) holds the ζ is a family of periodic orbits on $\mathbf{T}^2$ parameterized by the r.h.s. of (53) which we denote with $\Delta(a_1, a_2, \varphi(0), \psi(0))$. Fixed $T_0 > 0$, one has that $\frac{a_1^3 - a_2^3}{a_1^3 + a_2^3} < 1$. Let be $r \doteq (a_1^3 - a_2^3)/(a_1^3 + a^3) \in \mathbf{Q}$, hence $\exists p, q \in \mathbf{N} \ / \ r = p/q$, then using (51) the minimal period for the motion in $[0, 2\pi] \times [0, 2\pi]$ is qT_0.

Now we show that the circular antisymmetric solution $\zeta_0(t)$ living in $\tilde{Q}$ is not a minimum for the Action functional (45). This trajectory lies in a manifold which is non-degenerated in in a sense which will be defined below. This fact will allow us to prove that when the perturbation is turned on the critical point $\zeta_0(t)$ is *continued* in the small parameter.

Proposition 5.1

Let

$$\zeta_0(t) = (\rho_1^0, \rho_2^0, \varphi^0(t)) \tag{54}$$

$$\begin{cases} \rho_1^0 = (T_0/2\pi)^{2/3}(1 + a^3)^{2/3} \\ \rho_2^0 = (T_0/2\pi)^{2/3}(1 + a^3)^{2/3}\frac{1}{a^2} \\ \varphi^0(t) = \frac{2\pi}{T_0}t + \varphi(0) \quad \varphi(0) \in [0, 2\pi] \end{cases} \tag{55}$$

be the circular solution of the unperturbed problem. For all $a_2/a_1 \in \mathbf{R}_+$ ζ is a Critical point for for the Action functional:

$$\mathcal{A}_{T_0}^0[\zeta] = \int_0^{T_0} dt \tilde{L}_0(\zeta(t))$$

Moreover the circular solution ζ_0 belongs to the non-degenerate manifold $\Sigma \doteq \cup_{\varphi(0)}\zeta(t; \varphi(0))$.

Proof

We start with a definition:

Definition 4.1

Given a C^2 functional $F : H \to \mathbf{R}$ on H Hilbert space, $\Sigma \subset H$ compact and connected. Σ is called **non-degenerate** critical manifold for F iff:

(i) Σ has no boundary

(ii) $DF(x) = 0$ for all $x \in \Sigma$

(iii) $\ker D^2F(x) = T_x\Sigma$ for all $x \in \Sigma$

Obviously $\zeta_0 \in \Sigma$. Since $\Sigma \sim [0, 2\pi]$ *mod* 2π (i) is verified, then we already know that condition (ii) hold, we are left to verify (iii).

We perform a rescaling in the Action functional, and we introduce the parameter:

$$a \doteq \frac{a_2}{a_1} \in \mathbf{R}_+$$

Parameter a does not appear in the Proposition, and, in fact, the circular solutions are critical points independently on the values of a, i.e. independently on the periodicity in the unreduced configuration space $\tilde{Q}$.

The reduced Lagrangian becomes:

$$\tilde{L}_0 = \frac{1}{2}\dot{\rho_1}^2 + \frac{a}{2}\dot{\rho_2}^2 + \frac{1}{\rho_1} + \frac{a}{\rho_2} + \frac{a(\rho_1\rho_2\dot{\varphi})^2}{2(\rho_1^2 + a\rho_2^2)}$$

and remember that $\zeta(t)$:

$$\begin{cases} \rho_1(t) = & c^2 \\ \rho_2(t) = & c^2/a^2 \\ \varphi(t) = & \varphi(0) + \omega^0 t \quad ; \quad \omega^0 = (1+a^3)/(c^3) \end{cases}$$

Where c is obtained by the periodicity condition on φ:

$$c = \left[\frac{T_0(1+a^3)}{2\pi}\right]^{1/3}$$

Firstly we compute the Hessian of the Lagrangian evaluated in $\zeta(t)$, we get:

$$Hess(\tilde{L}^0) \doteq \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & \vec{b} \\ 0 & \vec{b}^t & C \end{pmatrix}$$

where with 0 is a 2×2 matrix with null entries. The nonvanishing blocks are given by:

$$A_1 \doteq \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

$$A_2 \doteq \frac{4\pi^2}{T_0^2(1+a^3)^3} \begin{pmatrix} 3-a^3 & 4a^5 \\ 4a^5 & (3a^3-1)a^7 \end{pmatrix}$$

$$\vec{b} \doteq \frac{2(2\pi)^{1/3}}{T_0^{1/3}(1+a^3)^{4/3}} \begin{pmatrix} 1 \\ a^5 \end{pmatrix}$$

and

$$C \doteq \frac{(T_0^4(1+a^3))^{1/3}}{(2\pi)^{4/3}}$$

For the second variation of the Action functional and one gets:

$$< D^2\mathcal{A}^0_{T_0}(\zeta)\vec{v},\vec{v} >= \int_0^{T_0} dt \sum_{i,j} \left\{ \frac{\partial^2 \tilde{L}^0}{\partial \dot{\rho}_i^{\,2}} \dot{f}_i^{\,2} + \frac{\partial^2 \tilde{L}^0}{\partial \rho_i \partial \rho_j} f_i f_j + \frac{\partial^2 \tilde{L}^0}{\partial \rho_i \partial \dot{\varphi}} f_i \dot{g} + \frac{\partial^2 \tilde{L}^0}{\partial \dot{\varphi}^2} \dot{g} \right\}$$

where $f_i, g \in H^1_{\tilde{Q}}$ and:

$$\vec{v} \doteq \begin{pmatrix} f_1(t) \\ f_2(t) \\ \dot{g}(t) \end{pmatrix}$$

and c is the function in a already defined.

Now by means of the periodicity conditions the following conditions hold:

$$\begin{cases} f_i(t) & = f_i(t+T_0/2) \quad i=1,2 \quad \forall t \in [0,T_0] \\ \dot{g}(t) & = \dot{g}(t+T_0/2) \quad \forall t \in [0,T_0] \end{cases}$$

and thus we can represent the variation in term of their Fourier series:

$$\begin{cases} f_j(t) & = 1/2T_0 \sum_k [f^j_{2k} \exp 2ki\omega t + f^{j\,*}_{2k} \exp(-2ki\omega t)] \quad j=1,2 \\ \dot{g}(t) & = 1/T_0 \sum_k [g_{2k} ki\omega \exp 2ki\omega t - g_{2k}{}^* ki\omega \exp(-2ki\omega t)] \end{cases}$$

where $\omega = 2\pi/T_0$ and $.^*$ is the complex conjugation. Hence inserting these expressions into the second variations, using the Hessian of the Lagrangian and after some computations one obtains:

$$\begin{aligned} < D^2\mathcal{A}^0_{T_0}(\zeta)\vec{v},\vec{v} >= \textstyle\sum_{l,j} \frac{1}{2} A^2_{lj} (f^j_0 f^{l\,*}_0 + f^l_0 f^{j\,*}_0) + \\ + \textstyle\sum_k \sum_{l,j} (A^2_{jl} + 4\omega^2 k^2 A^1_{jl}) \frac{1}{2} (f^{j\,*}_{2k} f^l_{2k} + f^{l\,*}_{2k} f^j_{2k}) + \\ + \textstyle\sum_k \sum_l 2b_l i\omega (k g_{2k} f^{l\,*}_{2k} - k g_{2k}{}^* f^l_{2k}) + \\ + 4\omega^2 \textstyle\sum_k C |g_{2k}|^2 k^2 \end{aligned} \tag{56}$$

we can write:

$$< D^2\mathcal{A}^0_{T_0}(\zeta)\vec{v},\vec{v} >\geq \sum_{k=1} \vec{v}^\dagger_k H_k \vec{v}_k + \vec{v}^\dagger_0 A_2 \vec{v}_0 \tag{57}$$

where:

$$\vec{v}_k \doteq \begin{pmatrix} f^1_{2k} \\ f^2_{2k} \\ ikg_{2k} \end{pmatrix}$$

and

$$H_k \doteq \begin{pmatrix} A_2 + 4\omega^2 k^2 A_1 & \omega\vec{b} \\ \omega\vec{b}^t & 4\omega^2 C \end{pmatrix}$$

Firstly we note that when one considers *constant* variations in (57) one has:

$$< D^2\mathcal{A}^0_{T_0}(\zeta)\vec{v}, \vec{v} >= \vec{v}_0^\dagger A_2 \vec{v}_0$$

and one easily verifies that the matrix A_2 has a negative direction for all values of a, hence the *constant* directions are negative directions of the Hessian, and thus $\zeta_0(t)$ is not local minimum for the Action functional.

We have to study the kernel of $D^2\mathcal{A}^0_{T_0}[\zeta]$. By means of the expression for the second variation, computed above, one can find the equations for the kernel:

$$\sum_j \left\{ -\frac{d}{dt}\left\{ \frac{\partial^2 \tilde{L}^0}{\partial \dot{\rho_j}^2} \dot{f}_j \right\} + \frac{\partial^2 \tilde{L}^0}{\partial \rho_i \partial \rho_j} f_j \right\} + \frac{\partial^2 \tilde{L}^0}{\partial \rho_i \partial \dot{\varphi}} \dot{g} = 0$$

$$-\frac{d}{dt}\left\{ \frac{\partial^2 \tilde{L}^0}{\partial \dot{\varphi}^2} \dot{g} \right\} = 0$$

for $i = 1, 2$, with obviously the periodicity condition on the variations f_1, f_2, g. This system can be rewritten as:

$$A_1 \frac{d^2 \vec{f}}{dt^2} = A_2 \vec{f} + \vec{b}\dot{g}$$

$$\frac{d}{dt}\{C\dot{g}\} = 0$$

The second equation can be integrated one time:

$$C\dot{g} = d \quad d \in \mathbf{R}$$

Then one gets:

$$\frac{d^2 \vec{f}}{dt^2} = A_1^{-1} A_2 \vec{f} + \frac{d}{2C} A_1^{-1} \vec{b}$$

This system has an homogenous part given by:

$$\frac{d^2 \vec{f}}{dt^2} = A\vec{f}$$

where

$$A \doteq \frac{4\pi^2}{{T_0}^2(1+a^3)^3} \begin{pmatrix} 3 - a^3 & 4a^5 \\ a^4 & (3a^3 - 1)a^6 \end{pmatrix}$$

This matrix has eigenvalues with alternate sign since:

$$\det(A) < 0$$

for all $a \in \mathbf{R}$. Using the form of matrix B, the allowed periodic solutions are given by:

$$\vec{f} = \vec{0} \ , \ \ d = 0$$

and hence we conclude:

$$\ker(D^2\mathcal{A}_T^0[\zeta_0]) = \{(0,0,g_0) \ ; \ g_0 \in \mathbf{R}\}$$

which corresponds to the direction tangent to ζ_0. @

Theorem 5.1

There exists $\epsilon_0 \in \mathbf{R}^+$ and an $H^1_{\tilde{Q}}$-neighborhood U_ϵ of $\zeta_0(t)$ such that for all $0 < \epsilon < \epsilon_0$ the Action functional $\mathcal{A}_{T_0}[.,\epsilon]$ has a critical point $\zeta_\epsilon(t)$ in U_ϵ.

Proof

From the definition of $H^1_{\tilde{Q}}$ it turns out that:

$$TH^1_{\tilde{Q}} = \{w = (f_1(t), f_2(t), g(t)) \ / \ f_i, g \in H^1_{T_0}(\mathbf{R}) \ \ f_i(t) = f_i(t+T_0/2) \ \ i = 1,2 \ \ g(t) = g(t+T_0/2)\}$$

we proved that:

$$\ker(D^2\mathcal{A}_{T_0}^0[\zeta_0]) = \{(0,0,g_0) \ ; \ g_0 \in \mathbf{R}\}$$

hence:

$$(\ker(D^2\mathcal{A}_T^0[\zeta_0]))^\perp = \left\{(f_1, f_2, g) \ ; \ \int_0^T dsg(s) = 0\right\}$$

by the symmetry of $D^2\mathcal{A}_T^0[\zeta_0]$ we have:

$$(\ker(D^2\mathcal{A}_{T_0}^0[\zeta]))^\perp \sim rank(D^2\mathcal{A}_{T_0}^0[\zeta_0])$$

Let be $w \in (\ker(D^2\mathcal{A}_{T_0}^0[\zeta_0]))^\perp$, we want to solve

$$D\mathcal{A}_{T_0}[\zeta_0 + \epsilon w, \epsilon] = 0 \tag{58}$$

for ϵ small enough. Now the equation (58) can be rewritten as:

$$D\mathcal{A}_{T_0}^0[\zeta_0] + D^2\mathcal{A}_{T_0}^0[\zeta_0](w) + R(\zeta_0, \epsilon w) = 0$$

with $\lim_{\epsilon\to 0} 1/\epsilon R(\zeta_0, w\epsilon) = 0$ for all w such that $\|w\|_{H^1_{\tilde{Q}}} < \infty$. $D^2\mathcal{A}^0_{T_0}[\zeta_0]$ is invertible on $(\ker(D^2\mathcal{A}^0_{T_0}[\zeta_0]))^\perp$, hence by the Implicit Function Theorem we can deduce that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists a C^1 function $w(\zeta_0, \epsilon)$ such that

$$D\mathcal{A}_{T_0}[\zeta_\epsilon, \epsilon] = 0 \qquad \zeta_\epsilon = \zeta_0 + \epsilon w(\zeta_0, \epsilon)$$

Must be noted that ζ_ϵ depends parametrically on a_1, a_2 which appear in the Lagrangian. The Lagrangian is smooth for $a_1 \neq 0$, $a_2 \neq 0$ hence also ζ_ϵ turns out to be smooth in the parameters.

We now prove the nondegenerancy of $D\mathcal{A}_{T_0}[.,\epsilon]$ in ζ_ϵ. Let $v \in TH^1_{\tilde{Q}}$ with $v \in (\ker(D^2\mathcal{A}^0_{T_0}[\zeta_0]))^\perp$ then:

$$\|D^2\mathcal{A}_{T_0}[\zeta_\epsilon, \epsilon](v)\| \geq \|D^2\mathcal{A}^0_{T_0}[\zeta_0](v)\| - \|D^2\mathcal{A}_{T_0}[\zeta_\epsilon, \epsilon](v) - D^2\mathcal{A}^0_{T_0}[\zeta_0](v)\| > 0$$

by means of nondegenerancy of $D^2\mathcal{A}^0_{T_0}[\zeta_0]$ on $(\ker(D^2\mathcal{A}^0_{T_0}[\zeta_0]))^\perp$ and using the regularity properties of $\mathcal{A}_{T_0}[.,\epsilon]$ for ϵ small enough.@

We want to find periodic solutions for the unreduced system the we give a convenient functional description for the periodicity condition on the unreduced configuration space.

Let us define the following functional $\mathcal{P}$ defined on $H^1_{\tilde{Q}}$:

$$\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, a_1, a_2] \doteq \int_t^{T_0/2+t} ds \frac{A_\psi - I_{\psi\varphi}\dot{\varphi}}{I_\psi} + \pi r \doteq \int_t^{T_0/2+t} ds P(\vec{\rho}, \dot{\vec{\rho}}, \varphi, \dot{\varphi}) + \pi r \tag{59}$$

with r rational and $r < 1$, $T_0 > 0$ and $\zeta_\epsilon \in U_\epsilon(\zeta_0)\ \forall \epsilon < \epsilon_0$.

From the definition follows that:

$$\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, a_1, a_2] = \psi(t + T_0/2, a_1, a_2) - \psi(t, a_1, a_2) - r\pi$$

Now if $\epsilon = 0$ one has:

$$\mathcal{P}_{r,T_0/2}[\zeta_0, a_1, a_2] = -\pi \frac{a_1^3 - a_2^3}{a_1^3 + a_2^3} + \pi r$$

this expression vanishes if we take:

$$\frac{a_2}{a_1} = \left[\frac{1-r}{1+r}\right]^{1/3} \tag{60}$$

and this value of the ratio of the masses corresponds to the periodicity condition for $\zeta_0(t)$ in the unreduced configuration space.

In the sequel will be useful to consider the following change of coordinates in the mass parameters space:

$$\begin{cases} a = & a_2/a_1 \\ \alpha = & a_1 + a_2 \end{cases}$$

so we can write $\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, a_1, a_2] = \mathcal{P}_{r,T_0/2}[\zeta_\epsilon, \alpha, a]$.

Theorem 5.2

Given $T_0 > 0$, r rational $r < 1$ and $\alpha \in \mathbf{R}$, there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$ there exists $a(\epsilon)$ continuous function of ϵ, such that for $\zeta_\epsilon(t) \in U_\epsilon(\zeta_0)$

$$\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, \alpha, a(\epsilon)] = 0$$

holds, and $\zeta_\epsilon(t)$ is a periodic solution in the unreduced configuration space Q.

Proof

We have already see that given $T_0 > 0$ and a rational number $r < 1$ one has:

$$\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, \alpha, a^*] = 0$$

for $a^* = [(1-r)/(1+r)]^{1/3}$.

We want to use the Implicit Function Theorem to find a continuous function $a(\epsilon)$ such that $\forall \epsilon < \bar{\epsilon}$, for some $\bar{\epsilon} > 0$:

$$\mathcal{P}_{r,T_0/2}[\zeta_\epsilon, \alpha, a(\epsilon)] = 0$$

with $\zeta_\epsilon \in U_\epsilon(\zeta_0)$.

In order to do that we have to take account that the constant ϵ_0 appearing in **Theorem 4.1** does not depend on the masses a_1, a_2 and so on a. In fact the regularity properties of $\mathcal{A}_{T_0}[., \epsilon]$ as function of ϵ do not depend on a_1, a_2.

In order to apply the Implicit Function Theorem we are left to prove that:

$$\frac{d\mathcal{P}_{T_0/2,r}}{da} = \int_t^{T_0/2+t} ds \frac{dP}{da}(\vec{\rho}^\epsilon, \dot{\vec{\rho}}^\epsilon, \varphi^\epsilon, \dot{\varphi}^\epsilon) \neq 0$$

computed at $\epsilon = 0$ and $a = a^*$.

Now we have:

$$\frac{d\mathcal{P}_{T_0/2,r}}{da} = \int_t^{T_0/2+t} ds \left\{ \frac{\partial P}{\partial \vec{\rho}^\epsilon}\frac{d\vec{\rho}^\epsilon}{da} + \frac{\partial P}{\partial \dot{\vec{\rho}}^\epsilon}\frac{d\dot{\vec{\rho}}^\epsilon}{da} + \frac{\partial P}{\partial \varphi^\epsilon}\frac{d\varphi^\epsilon}{da} + \frac{\partial P}{\partial \dot{\varphi}^\epsilon}\frac{d\dot{\varphi}^\epsilon}{da} \right\}$$

Taking into account that $\zeta_\epsilon(t)$ (from **Theorem 4.1**) is C^1 in a and ϵ, we obtain:

$$\frac{d\mathcal{P}_{T_0/2,r}}{da}|_{a=a^*} = -2\pi\frac{d}{da}\frac{1-a^3}{1+a^3}|_{a=a^*} = -14\pi\frac{a^{*2}}{(1+a^{*3})^2} \neq 0 \quad \forall a^* \neq 0$$

condition $a^* \neq 0$ is equivalent to $r < 1$.

Hence we can now apply the Implicit Function Theorem concluding that given $T_0 > 0$, r rational and less than 1, $\alpha \in \mathbf{R}$ there exists $\bar{\epsilon}(T_0, r, \alpha) \leq \epsilon_0$ such that for all $0 < \epsilon < \bar{\epsilon}$ there exists a continuous function $a(\epsilon)$ such that:

$$\mathcal{P}_{T_0/2.r}[\zeta_\epsilon, \alpha, a(\epsilon)] = 0$$

with $\zeta_\epsilon \in U_\epsilon(\Sigma)$ so $\rho_1^\epsilon(t + T_0/2) = \rho_1(t)$, $\rho_2^\epsilon(t + T_0/2) = \rho_2(t)$, $\varphi^\epsilon(t + T_0/2) - \varphi(t) = \pi$. Let us observe that, considering $r = p/q \quad p, q \in \mathbf{N}$ we have:

$$\varphi^\epsilon(t + qT_0/2) - \varphi^\epsilon(t) = q\pi \quad \psi^\epsilon(t + qT_0/2) - \psi^\epsilon(t) = p\pi$$

therefore the minimal period of $\zeta_\epsilon(t)$ in Q is qT_0.@

6 Non-collision solutions for the perturbed problem

In this section, collecting the preceding results we prove that, for small value of the two masses, the Three-Body Problem reduced on $J = 0$ has non-collision periodic solutions which live in an appropriate $C^0 - compact$ set $\tilde{M}_{k_1,k_2}$.

Now we prove:

1. For some $k_1, k_2 > 0$ $\Sigma \subset \tilde{M}^\epsilon_{k_1,k_2}$ (where $\tilde{M}^\epsilon_{k_1,k_2}$ denotes the set M_{k_1,k_2} whose definition contains $\mathcal{A}_T[., \epsilon]$ in the place of $\bar{\mathcal{A}}_T[.]$).

2. We use the **Theorem 5.1** to *continue* in ϵ the the Critical point ζ_0 into ζ_ϵ which belongs to a neighborhood U_ϵ of ζ_0. Thus we prove the existence of critical points for the functional $\mathcal{A}_T[., \epsilon]$ in $\tilde{M}^\epsilon_{k_1,k_2}$.

Proposition 6.1

There exits $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$ the circular orbit $\zeta_0(t)$ are such that $\zeta_0(t) \in \tilde{M}^\epsilon_{k_1,k_2}$ with

$$k_1 < \min_{i,j} \frac{4\rho_i^*}{[2(\rho_i^{*2} + \rho_j^{*2} - \rho_i^*\rho_j^*)]^{1/2}}$$

$$k_2 = 3(2\pi)^{2/3}[T_0(1+a^3)]^{1/3}$$

Proof

We show that $\zeta_0(t) \in \tilde{M}_{k_1,k_2}$ using transformation $\mathcal{T}$. Indeed, for a trajectory in $\tilde{M}_{k_1,k_2}$ one needs that there exists $t^* \in [0, T_0/2]$ $z(t^*) = -z(t^* + T_0/2)$ and $\rho_i(t^*) = \rho_i(t^* + T_0/2)$ for $i = 1,2,3$. Now let $\varphi(0)$ be in $[0,\pi]$ so $z(0) = 1$. Now there exists $t^* \in [0, T_0/2]$ such that $\varphi(t^*) = \pi/2$ so $z(t^*) = 1$. For ζ_0 we have:

$$\varphi(t + T_0/2) = \varphi(t) + \pi$$

hence $\varphi(t^* + T_0/2) \in (\pi, 2\pi)$ and thus $z(t^* + T_0/2) = -1$ and $\rho_i(t^*) = \rho_i(t^* + T_0/2)$ for $i = 1,2,3$ so we have $\zeta_0(t) \in \tilde{M}_{k_1,k_2}$.

The first estimation comes from the definition of the set $\tilde{M}^{\epsilon}_{k_1,k_2}$. The term

$$\sup_{s\in[0,T_0]} g(\vec{\rho}(s)) = \sup_{s\in[0,T_0]} \frac{1}{\|\vec{\rho}(s)\|} \min_{i,j,k}{}' \left\{(\rho_i(s) + \rho_j(s) - \rho_k(s))^2\right\}$$

Computing this expression on $\zeta_0(t;\varphi(0)) = (\rho_1^0, \rho_2^0, \omega^0 t + \varphi(0))$ one obtains a smooth function on S^1 which can be minimized uniformly in ϵ getting the first estimation.

The second estimation can be found evaluating $\mathcal{A}_{T_0}[.,\epsilon]$ on a generic element of Σ, then one finds :

$$\mathcal{A}_{T_0}[\zeta_0,\epsilon] = \frac{3}{2}(2\pi)^{2/3}[T_0(1+a^3)]^{1/3} + \mathcal{A}^{\epsilon}_{T_0}[\zeta_0]$$

by means of the properties of the term depending on ϵ we can find ϵ_1 such that for all $\epsilon < \epsilon_1$

$$\mathcal{A}_{T_0}[\zeta_0,\epsilon] < k_2 \doteq 3(2\pi)^{2/3}[T_0(1+a^3)]^{1/3}$$

From this estimations one can also conclude that $\zeta_0 \notin \partial\tilde{M}^{\epsilon}_{k_1,k_2}$.@

Theorem 6.1

Let $\mathcal{A}_{T_0}[.,\epsilon] = \mathcal{A}^0_{T_0}[.] + \mathcal{A}^{\epsilon}_v[.]$ the Action functional for the Three-Body Problem reduced on $J = 0$ with two small masses $m_1 = m$ $m_2 = ma_1\epsilon$ $m_3 = ma_2\epsilon$.

There exist $k_1, k_2 > 0$ $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$ the set $\tilde{M}^{\epsilon}_{k_1,k_2}$ contains a neighborhood U_ϵ of ζ_0 (critical point for $\mathcal{A}^0_{T_0}[.]$. It turns out that the Action functional $\mathcal{A}_{T_0}[.,\epsilon]$ has a critical point ζ_ϵ in U_ϵ. This critical point gives a periodic non-collision strong solution of the Lagrange equations in the unreduced Configuration space Q.

Proof

In section 4 we proved that for $\mathcal{A}^0_{T_0}$ ζ_0 is critical and non-degenerate. Moreover in **Proposition 5.1** we proved that for small values of ϵ (say $\epsilon < \epsilon_1$) there exists a neighborhood of ζ_0 which contains one critical point ζ_ϵ of the Action functional. As it was recalled in section 5, collision orbits are not regular critical points of the Action functional. Therefore, since $\mathcal{A}_{T_0}[.,\epsilon]$ is of class C^2 on the *continued* critical point, ζ_ϵ is a non-collision critical points the Action.

In **Theorem 5.2** we proved that one can choose ϵ so small that there exist continuous functions $a_1(\epsilon), a_2(\epsilon) > 0$ such that the continued solution ζ_ϵ is periodic of period qT_0 ($q \in \mathbf{N}$) in the unreduced configuration space Q.

In **Proposition 6.1** we showed that there are $k_1, k_2 > 0$ such that $\zeta_0 \in \tilde{M}^\epsilon_{k_1,k_2}$ and $\zeta_0 \notin \partial\tilde{M}^\epsilon_{k_1,k_2}$ for $\epsilon < \epsilon_2$. Hence chosen $\epsilon < \min\{\epsilon_1, \epsilon_2\}$ we can find U_ϵ, a neighborhood of ζ_0 such that:

$$U_\epsilon \subset \tilde{M}^\epsilon_{k_1,k_2}$$

$$U_\epsilon \cap \partial\tilde{M}^\epsilon_{k_1,k_2} = \emptyset$$

thus we can conclude that all the perturbed critical point ζ_ϵ of the Action functional belong to $\tilde{M}^\epsilon_{k_1,k_2}$.@

7 Behaviour of the Action functional on unbounded trajectories

In this last section we study the behaviour of the full reduced Action functional on sequence of functions which converge to unbounded trajectory. Let us observe that the Action functional $\mathcal{A}_T[.,\epsilon]$ defined on $H^1_{\tilde{Q}}$ (with (48)), loses coercivity only on sequence of trajectories $\zeta^{(n)}$ for which:

i) $\exists i \ / \ \sup_t \rho_i^{(n)}(t) \in (0,\infty)$

ii) $\int_0^T ds(\dot{\varphi}^{(n)}(s))^2 < \infty$

This follows from the following inequality :

$$\mathcal{A}_T[\zeta,\epsilon] \leq \sum_i \|\dot{\rho}_i\|^2_{L^2} + \frac{a_1 a_2 (\sup_t \rho_1)^2 (\sup_t \rho_2)^2}{a_1(\sup_t \rho_1)^2 + a_2(\sup_t \rho_2)^2}\|\dot{\varphi}\|^2_{L^2} + O(1/\rho_1, 1/\rho_2)$$

then one finds:

$$\lim_{n\to\infty} \mathcal{A}_T[\zeta^{(n)},\epsilon] < \infty$$

We will therefore consider only configurations in which goes to infinity the relative distance of one body with respect to the center of mass of the other two.

In order to do that we rewrite $\mathcal{A}_T[.]$ using (24) as:

$$\mathcal{A}_T[\zeta] = \mathcal{A}_T^{\infty}[\zeta] + \mathcal{A}_T^{res}[\zeta] \tag{61}$$

Some modifications are needed in the transformation by which the Reduction "a la Routh" was made. Since we want to consider the case when one of the three particles escapes to infinity, it is natural to use a system of coordinates which contains, as a coordinate, the distance of one particle from the center of the mass of the other two. Hence we substitute $\mathcal{T}_1$ and define $\mathcal{T}_1' : Q \to Q$ as:

$$\begin{cases} y_1 = q_2 - q_1 \\ y_2 = q_3 - (m_1 q_1 + m_2 q_2)/(m_1 + m_2) \\ y_3 = (m_1 q_1 + m_2 q_2 + m_3 q_3)/(m_1 + m_2 + m_3) \end{cases} \tag{62}$$

Remark 7.1

In this coordinates the third body (with mass m_3) will be the "leaving body" and in $H^1_{\tilde{Q}}$ we will consider sequence of trajectories for which $\inf_t \|y_2^{(n)}(t)\| \to \infty$ *when* $n \to \infty$.

Then we apply transformation $\mathcal{T}_2$ with the substitutions:

$$x_i \to y_i \quad r_i \to \rho_i \quad \theta_i \to \Theta_i$$

Therefore the Lagrangian (1) becomes:

$$L = \frac{\mu}{2}\|\dot{y}_3\|^2 + \frac{1}{2}\sum_{i=0}^{2} \tilde{M}_i(\dot{r}_i^2 + {r_i}^2\dot{\Theta}_i^2) + V(r_1, r_2, \Theta_1 - \Theta_2) \tag{63}$$

where:

$$M_1 \doteq \frac{m_2 m_1}{(m_1 + m_2)} \qquad M_2 \doteq \frac{m_3(m_1 + m_2)}{\mu} \qquad \mu \doteq \sum_{i=1}^{3} m_i \tag{64}$$

and in this coordinates one finds that the potential V becomes:

$$\begin{aligned} V(r_1, r_2, \Theta_1 - \Theta_2) &= m_1 m_2 / r_1 + \\ &+ m_1 m_3 [r_2^2 + 2 m_2 r_1 r_2 \cos(\Theta_1 - \Theta_2)/(m_1 + m_2) + m_2^2 r_1^2/(m_1 + m_2)^2]^{1/2} + \\ &+ m_2 m_3 [r_2^2 - 2 m_2 r_1 r_2 \cos(\Theta_1 - \Theta_2)/(m_1 + m_2) + m_2^2 r_1^2/(m_1 + m_2)^2]^{1/2} \end{aligned} \tag{65}$$

At this point we use the transformation $\mathcal{T}_3$ with $\theta_i \to \Theta_i \quad \varphi \to \Phi \quad \psi \to \Psi$ and the Reduction "a la Routh" and therefore we obtain the reduced Lagrangian whose form is:

$$L_0 = \frac{1}{2}\sum_{i=1}^{2} M_i \dot{r}_i^{\,2} + \frac{\dot{\Phi}^2}{2}\frac{M_1 M_2 r_1^2 r_2^2}{M_1 r_1^2 + M_2 r_2^2} + V(r_1, r_2, \Phi) \tag{66}$$

We can now write explicitly the terms appearing in (61):

$$\mathcal{A}_T^{\infty}[\zeta] \doteq \int_0^T dt \left\{ \frac{M_1}{2}[\dot{r}_1^2 + r_1^2\dot{\Phi}^2] + \frac{m_1 m_2}{r_1} \right\}$$

$$\mathcal{A}_T^{res}[\zeta] \doteq \int_0^T dt \left\{ \frac{M_2}{2}\dot{r}_2^2 + V(r_1, r_2, \Phi) - \frac{M_1^2 (r_1^2\dot{\Phi})^2}{2\sum_{i=1,2} M_i r_i^2} \right\}$$

Remark 7.2

A rapid inspection to the two coordinates systems $(\rho_1, \rho_2, \varphi)$ and (r_1, r_2, Φ) shows that the passage between the two is give by:

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 - m_2 x_1/(m_1 + m_2) \\ y_3 = x_3 \end{cases} \tag{67}$$

this transformation implies correspondence the periodicity conditions: (r_1, r_2, Φ) have the same periodicity condition of $(\rho_1, \rho_2, \varphi)$; therefore we consider again the space $H^1_{\tilde{Q}}$ whose elements will be denoted with $\zeta = (r_1, r_2, \Phi)$.

We study $\mathcal{A}_T[.]$ on $H^1_{\tilde{Q}}$ with the usual condition (48), in the sequel we often use the canonical embedding of H^1 into C^0.

Firstly we study the behaviour of the Action on a particular set of unbounded sequence.

Proposition 7.1

Let $\{\zeta^{(k)}(t)\}_{k=0}^{\infty} \in H^1_{\tilde{Q}}$ be a sequence of continuous functions satisfying (48) for all k, and such that:

$$\lim_{k\to\infty} \inf_t r_i^{(k)} = \infty \quad i = 1, 2$$

then:

$$\lim_{k\to\infty} \mathcal{A}_T[\zeta^{(k)}(t)] = +\infty$$

Proof

All the terms in:

$$\mathcal{A}_T[\zeta] = \int_0^T dt L_0(\zeta(t))$$

are positive, hence we have:

$$\mathcal{A}_T[\zeta] \geq \int_0^T dt \frac{\dot{\Phi}^2}{2} \frac{M_1 M_2 r_1^2 r_2^2}{M_1 r_1^2 + M_2 r_2^2}$$

Now since:

$$r_i(t) \geq \inf_t r_i(t) \quad \forall i$$

implies:

$$\frac{M_1}{r_2^2} + \frac{M_2}{r_1^2} \leq \frac{M_1}{(\inf_t r_2)^2} + \frac{M_2}{(\inf_t r_1)^2}$$

Therefore:

$$\mathcal{A}_T[\zeta] \geq \frac{M_1 M_2}{2} \|\dot{\Phi}\|_{L^2}^2 \left[\frac{M_1}{(\inf_t r_2)^2} + \frac{M_2}{(\inf_t r_1)^2} \right]^{-1}$$

Now we have only to remember that by means of (48) for Φ one can prove that:

$$\|\dot{\Phi}\|_{L^2}^2 \geq \frac{4\pi^2}{T}$$

Now evaluating the estimation found for the Action on the sequence prescribed by hypothesis, the Proposition is proved. @

In the next Proposition we will use the decomposition (61).

Proposition 7.2

Let $\{\zeta^{(k)}(t)\}_{k=0}^{\infty} \in H_{\bar{Q}}^1$ be a sequence of continuous functions satisfying (48) for all k, and such that:

$$\lim_{k\to\infty} \inf_t r_2^{(k)} = \infty$$

$$\lim_{k\to\infty} \|\dot{r}_2^{(k)}\|_{L^2} = 0$$

while uniformly in k:

$$\|r_1^{(k)}\|_\infty < C_1 \quad \|\dot{r}_1^{(k)}\|_{L^2} < C_2$$

and

$$\|\dot{\Phi}^{(k)}\|_{L^2} < C_3$$

with $C_1, C_2, C_3 > 0$, then:

$$\lim_{k\to\infty} |\mathcal{A}_T[\zeta^{(k)}] - \mathcal{A}_T^\infty[\zeta^{(k)}]| = 0$$

and

$$\lim_{k\to\infty} | < D\mathcal{A}_T[\zeta^{(k)}], \vec{v} > - < D\mathcal{A}_T^{\infty}[\zeta^{(k)}], \vec{v} > | = 0$$

for all variations $\vec{v} = (f_1(t), f_2(t), w(t))$ *in* $H^1_{\dot{Q}}$.

Proof

By means of (61) in order to prove the thesis we have to prove:

$$\lim_{k\to\infty} |\mathcal{A}_T^{res}[\zeta^{(k)}]| = 0 \tag{68}$$

and

$$\lim_{k\to\infty} | < D\mathcal{A}_T^{res}[\zeta^{(k)}], \vec{v} > | = 0 \tag{69}$$

(68) comes from the following estimations

$$\int_0^T dt \frac{M_2}{2}\dot{r}_2^2 = \frac{M_2}{2}\|\dot{r}_2\|^2_{L^2}$$

$$\int_0^T dt \frac{M_1^3(r_1\dot{\Phi})^2}{2\sum_{i=1,2} M_i r_i^2} \leq \frac{M_1^3(\|r_1\|_\infty \|\dot{\Phi}\|_{L^2})}{2\sum_{i=1,2} M_i(\inf_t r_i)^2}$$

and using that $ax^2 + yb^2 \pm cxy\cos\varphi \geq ax^2 + yb^2 - cxy \quad x, y, a, b, c > 0$ one has:

$$\int_0^T dt V(r_1, r_2, \Phi) \leq T(m_1 m_3 + m_2 m_3)\left[(\inf_t r_2)^2 - \frac{2m_2 \inf_t r_1 \inf_t r_2}{(m_1+m_2)} + \frac{m_2^2(\inf_t r_1)^2}{(m_1+m_2)^2}\right]^{-1/2}$$

all these terms go to zero when evaluated on the sequence defined in the hypothesis.

For (69) we have:

$$\begin{aligned} | < D\mathcal{A}_T^{res}[\zeta^{(k)}], \vec{v} > | &\leq \textstyle\int_0^T dt |M_2 \dot{r}_2 \dot{f}_2| + \\ &+ \textstyle\int_0^T dt \left|\frac{\partial V}{\partial \Phi} w + \sum_i \frac{\partial V}{\partial r_i} f_i\right| + \\ &+ \textstyle\int_0^T dt \frac{M_1^3 r_1^2}{\sum_i M_i r_i^2} |\dot{\Phi}\dot{w}| + \\ &+ \textstyle\int_0^T dt \frac{M_1^3 M_2 (\dot{\Phi})^2 r_1 r_2}{(\sum_i M_i r_i^2)^2} |r_1 f_2 - r_2 f_1| \end{aligned} \tag{70}$$

Now using the following relations

$$(\sum_i M_i r_i^2)^2 = r_2^4\left[M_1\left(\frac{r_1}{r_2}\right)^2 + M_2\right]^2 \geq (\inf_t r_2)^2 r_2^2\left[M_1\left(\frac{\inf_t r_1}{\sup_t r_2}\right)^2 + M_2\right]^2$$

$$(\sum_i M_i r_i^2)^2 \geq (\inf_t r_2)^3 r_2 \left[M_1 \left(\frac{\inf_t r_1}{\sup_t r_2} \right)^2 + M_2 \right]^2$$

and the usual estimations, one has:

$$\begin{aligned}
| < D\mathcal{A}_T^{res}[\zeta^{(k)}], \vec{v} > | \leq M_2 \|\dot{r}_2\|_{L^2} \|\dot{f}_2\|_{L^2} + \\
+ \left\| \frac{\partial V}{\partial \Phi} \right\|_{L^2} \|w\|_{L^2} + \sum_i \left\| \frac{\partial V}{\partial r_i} \right\|_{L^2} \|f_i\|_{L^2} + \\
+ \frac{M_1^3 \|r_1\|_\infty^2}{\sum_i M_i (\inf_t r_i)^2} \|\dot{\Phi}\|_{L^2} \|\dot{w}\|_{L^2} + \\
+ M_1^3 M_2 \|f_1\|_\infty \|\dot{\Phi}\|_{L^2}^2 \|r_1\|_\infty (\inf_t r_2)^2 \left[M_1 \left(\frac{\inf_t r_1}{\sup_t r_2} \right)^2 + M_2 \right]^{-2} + \\
+ M_1^3 M_2 \|f_2\|_\infty \|\dot{\Phi}\|_{L^2}^2 (\|r_1\|_\infty)^2 (\inf_t r_2)^3 \left[M_1 \left(\frac{\inf_t r_1}{\sup_t r_2} \right)^2 + M_2 \right]^{-2}
\end{aligned} \tag{71}$$

now, evaluating this estimation on sequences defined in the hypothesis one get the thesis.@

Let us call Ξ^∞ the the set of trajectories in $H^1_{\tilde{Q}}$, satisfying (48), which fulfil the hypothesis of the previous Proposition. Now we show that this kind of trajectories lives out of any L^2-compact $\tilde{M}_{k_1,k_2}$ for all $k_2 > 0$.

Proposition 7.3

There exist $k_1 > 0$ such that:

$$\Xi^\infty \cap \tilde{M}_{k_1,k_2} = \emptyset$$

for all $k_2 > 0$.

Proof

The constant k_1 is computed, by means of the preceding Proposition, using $\mathcal{A}_T^\infty$ evaluated on the Kepler System which remain in the finite part of the configuration space.

Given a sequence of continuous function in Ξ^∞, in $(\rho_1, \rho_2, \varphi)$ coordinates, we are left to prove that:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \min_{i,j,k}{}' \frac{(\rho_i^{(n)} + \rho_j^{(n)} - \rho_k^{(n)})^2}{(\sum_l^3 (\rho_l^{(n)})^2)^{1/2}} = 0 \tag{72}$$

where obviously: $\rho_3 = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\varphi}$.

It is easy to see that if $\inf_t r_2 \to \infty \quad \|r_1\|_\infty \leq C_1$ then, $\inf_t \rho_2 \to \infty \quad \|\rho_1\|_\infty \leq C_1$. Now since $\rho_2^{(n)}$ and $\rho_3^{(n)}$ have a divergent sup-norm and $\rho_1^{(n)}$ is bounded, one has that

$$\sup_{t\in[0,T]} \min_{i,j,k}{}' \frac{(\rho_i^{(n)} + \rho_j^{(n)} - \rho_k^{(n)})^2}{(\sum_l^3 (\rho_l^{(n)})^2)^{1/2}}$$

this is equal to:

$$\sup_t \frac{(\rho_1^{(n)} + \rho_2^{(n)} - \rho_3^{(n)})^2}{(\sum_l^3 (\rho_l^{(n)})^2)^{1/2}}$$

Using the expression for ρ_3 one finds that the last term is equal to:

$$\sup_{t\in[0,T]}\{(\rho_1^{(n)})^2[(1 + (|\rho_1^{(n)}/\rho_2^{(n)} - 2\cos\varphi^{(n)}|)/(1 + (1 + (\rho_1^{(n)}/\rho_2^{(n)})^2 - 2(\rho_1^{(n)}/\rho_2^{(n)})\cos\varphi^{(n)})^{1/2})]^2$$
$$1/[\rho_2^{(n)}(1 + (\rho_1^{(n)}/\rho_2^{(n)})^2 - 2(\rho_1^{(n)}/\rho_2^{(n)})\cos\varphi^{(n)})^{1/2})]\} \tag{73}$$

but $\rho_1^{(n)}/\rho_2^{(n)} \leq \sup_t \rho_1^{(n)}/\inf_t \rho_2^{(n)}$ and $\sup_t \rho_1^{(n)} \leq C_1$ one can see that the last term goes to zero as $O(1/\inf_t \rho_2^{(n)})$. @

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