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Solvable Complex Structures on a Real Lie-algebra

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Abstract

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Solvable Complex Structures on a Real Lie-algebra.

Daniele Gouthier

July 26, 1999

Abstract: *Let $\mathfrak{g}$ be a real Lie-algebra, endowed with a complex structure J. In the paper we define when J may be said to be* **solvable**. *A criterium for the existence of solvable complex structures is given. Moreover, the 4-dimensional solvable complex structures are classified.*

1 Introduction.

In the study of compact nilmanifolds, Gray introduces, in [1], a class of nilmanifolds which are real parallelizable without being complex parallelizable. Since a compact nilmanifold M is a quotient G/Γ of a nilpotent Lie-group via a discrete subgroup, the class introduced is defined in the terms of invariant complex structures of G. Such a structure is fixed by the corresponding one given on the Lie-algebra $\mathfrak{g} = Lie(G)$.

First of all, let us introduce our notations. Let $\mathfrak{g}$ be a real Lie-algebra, J be an almost-complex structure and $N = N_J$ its Nijenhuis tensor: $N_J(X,Y) := [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$.

A complex structure is an almost-complex structure with vanishing Nijenhuis tensor. Two complex structures J_1 and J_2 are said to be equivalent if there is $\varphi \in Aut\mathfrak{g}$ such that $\varphi J_1 = J_2\varphi$.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$ and τ the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$: $\tau X = \overline{X}$. Then, the subspace $\mathfrak{q} := \{X - iJX : X \in \mathfrak{g}\}$ is a subalgebra such that $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{q}}$.

Given a complex structure J, we focus our mind on the Lie-objects which are J-invariant. In particular, we say that a subalgebra (resp. an ideal) $\mathfrak{h}$ is a J-subalgebra (resp. a J-ideal) whenever it is J-invariant, $J\mathfrak{h} \subseteq \mathfrak{h}$. Classically, J-invariant maps are said to be holomorphic. Thus, $Aut(\mathfrak{g}, J)$ denotes the group of holomorphic Lie-automorphisms of $\mathfrak{g}$, as well as the elements of $Der(\mathfrak{g}, J)$ are the holomorphic derivations.

Obviously, the complexified subalgebra $\mathfrak{h}^{\mathbb{C}}$ splits as $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{q} \oplus \mathfrak{h}^{\mathbb{C}} \cap \overline{\mathfrak{q}}$; and an holomorphic Lie-automorphism of $\mathfrak{g}$ induces a Lie-automorphism of $\mathfrak{g}^{\mathbb{C}}$ which let invariant both $\mathfrak{q}$ and $\overline{\mathfrak{q}}$.

The nilpotent complex structures are defined in the real terms setting $\mathfrak{a}_s(J) = \mathfrak{g}$, where the ascending series is given by $\mathfrak{a}_0(J) = \{0\}$ and $\mathfrak{a}_{i+1}(J) = \{X : [X, \mathfrak{g}] \subseteq \mathfrak{a}_i(J), [JX, \mathfrak{g}] \subseteq \mathfrak{a}_i(J)\}$.

Such a definition is equivalent to the nilpotence of $\mathfrak{q}$. Furthermore, in [1], the authors describe examples of nilpotent Lie-algebras which admit nilpotent complex structures; and examples without nilpotent J's. Recalling the Magnin's classification of six-dimensional Lie-algebras, the first sixteen classes support nilpotent complex structures, while the last sixteen don't.

Following this approach, we define the solvable complex structures on a real Lie-algebra. Thus we have some natural questions on them: which are the algebraic properties of a Lie-algebra endowed with a solvable complex structure? is it possible to classify these structures? is it possible to give a "structure theory" for the complex structures?

First of all, we describe tge principal properties of a solvable complex structure: we prove the existence of a J-radical; we introduce the concept of semisemplicity and we obtain a result on the existence of solvable complex structures (Section 5). Namely, the real Lie-algebra $\mathfrak{g}$ admits a solvable complex structure whenever there is a family $\{\mathfrak{h}_0 = \mathfrak{g}, \mathfrak{h}_1 \dots \mathfrak{h}_n = \{0\}\}$ of subalgebras such that $\mathfrak{h}_i/\mathfrak{h}_{i+1}$ is the abelian 2-dimensional Lie-algebra and $\mathfrak{h}_{n-1}$ supports a complex structure J_{n-1}. The question on the structure theory may be reformulate as follows: let $\mathfrak{r}(J)$ denote the J-radical, then there exists a J-invariant complement of it: $\mathfrak{g} = \mathfrak{r}(J) \oplus V$ and $JV \subseteq V$. Is it possible to select V in such a way that it is a subalgebra (namely, a Levi-J-subalgebra)? if the answer is affermative, then J is forced to be semisimple on V (see Definition 4.10).

Otherwise, there exists a situation in which the answer is no. Take a nilpotent Lie-algebra. Then V should be a nilpotent subalgebra. Hence, Salamon proved that V includes properly $V' := [V, V] + J[V, V]$, [6]. But, if $J|_V$ were semisimple, then V should coincide with V' (Theorem 4.6). Hence, we have a contradiction. So, when $\mathfrak{g}$ is nilpotent there are no Levi-J-subalgebras. In the other cases (:$\mathfrak{g}$ non nilpotent), I have no answer.

The paper is organized as follows: in Section 2 the results obtained in [1] about nilpotent complex structures are recalled. Section 3 is devoted to introduce the definition of solvable complex structure and to show the first natural results. A criterium for solvability is given in Section 5. Moreover, we introduce the J-radical and we define whenever a complex structure is said to be semisimple (Section 4).

The last Section is entirely devoted to describe the solvable complex structure on 4-dimensional Lie-algebras. Furthermore, such a result is used to show that $Lie(T(3, \mathbb{R}))$ is a 6-dimensional Lie algebra which does not admit solvable

complex structures.

2 Nilpotent complex structures.

As we have said, in [1], the authors introduce the nilpotent complex structures on compact nilmanifolds. These structures have been completely classified in the six-dimensional case. Indeed, the nilpotent six-dimensional Lie-algebras have been described by Magnin in [5], and a nilmanifold is just the quotient of a nilpotent Lie group by a discrete subgroup.

In the following, we point our mind only on the Lie-algebra $\mathfrak{g}$. We forget and the associated Lie-group G and the quotient G/Γ.

First of all, let us recall the classical definition of nilpotent Lie-algebras and the new one of nilpotent complex structures, as given in [1]:

Definition 2.1 *The ascending central series is the family of ideals $\{\mathfrak{g}_l\}_{l\geq 0}$ defined by $\mathfrak{g}_0 = \{0\}$ and $\mathfrak{g}_{i+1} = \{X : [X, \mathfrak{g}] \subseteq \mathfrak{g}_i\}$. The Lie algebra $\mathfrak{g}$ is said to be nilpotent, whenever exists an s such that $\mathfrak{g} = \mathfrak{g}_s$.*

Definition 2.2 *The ascending central J-series is the family of J-ideals $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ defined by $\mathfrak{a}_0(J) = \{0\}$ and $\mathfrak{a}_{i+1}(J) = \{X : [X, \mathfrak{g}] \subseteq \mathfrak{a}_i(J), [JX, \mathfrak{g}] \subseteq \mathfrak{a}_i(J)\}$. The complex structure J is said to be nilpotent, whenever exists an s such that $\mathfrak{g} = \mathfrak{a}_s(J)$.*

A direct computation shows that, for any i, $\mathfrak{a}_i(J) \subseteq \mathfrak{g}_i$. So, when J is nilpotent, $\mathfrak{g}$ is nilpotent, too. Obviously, the converse is false. In fact, $\mathfrak{g}$ being nilpotent, we may have that $\mathfrak{a}_s(J) = \mathfrak{a}_{s+1}(J) \neq \mathfrak{g}$.

The two definitions are equivalent in the case of a complex Lie algebra $\mathfrak{g}$, i.e. when ad_X commutes with J, for all $X \in \mathfrak{g}$. In this case, $\mathfrak{a}_i(J)$ coincides with $\mathfrak{g}_i$.

Since the family $\mathfrak{a}_i(J)$ is ascending, there exists an integer s for which $\mathfrak{a}_s(J) = \mathfrak{a}_{s+1}(J)$. By definition, J is nilpotent on $\mathfrak{a}_s(J)$, moreover we have the

Lemma 2.3 *Let $\mathfrak{g}$ be a Lie-algebra with complex structure J. Then the J-ideal $\mathfrak{n}(J) := \mathfrak{a}_s(J) = \mathfrak{a}_{s+1}(J) = \ldots$ is the maximal J-ideal on which J is nilpotent.*

Proof: let $\mathfrak{h}$ be a J-ideal, then by induction it is not difficult to prove that $\mathfrak{a}_i(J) \cap \mathfrak{h} = \mathfrak{a}_i(J_{\mathfrak{h}})$. Hence, $\mathfrak{n}(J)$ is said to be the *J-nilradical.*

In particular, we have $\mathfrak{a}_i(J) \cap \mathfrak{n}(J) = \mathfrak{a}_i(J_{\mathfrak{n}(J)})$, which implies that $J_{\mathfrak{n}(J)}$ is nilpotent.

Finally, let us prove the maximality: take $\mathfrak{h}$ on which J is nilpotent, then there exists an index p such that $\mathfrak{h} = \mathfrak{a}_p(J_{\mathfrak{h}}) \subseteq \mathfrak{n}(J) \cap \mathfrak{h} \subseteq \mathfrak{n}(J)$.

□

It is a known fact, [7], that a Lie-algebra $\mathfrak{g}$ is nilpotent if and only if for any X of $\mathfrak{g}$, the endomorphism ad_X is nilpotent: i.e. there exists a k such that $ad_X^k := ad_X \circ \ldots \circ ad_X$ (k times) vanishes. An analogous result is true for the nilpotence of J. First of all, consider the following

Definition 2.4 *An element $X \in \mathfrak{g}$ is J-nilpotent if there exists an index s such that for any element $(X_1, \ldots, X_s)$ in $\{X, JX\}^s$, the derivation $ad_{X_1} \circ \ldots \circ ad_{X_s}$ vanishes.*

Thus, we may formulate the

Proposition 2.5 *The ideal $\mathfrak{a}_i(J)$ coincides the set of the J-nilpotent elements of index i.*

Proof: the statement is true for $i = 1$, since the two definitions coincide. Let us suppose, by induction, that it is true for any integer less then i. Take $X \in \mathfrak{a}_{i+1}(J)$, then $ad_{X_{i+1}}\mathfrak{g}$ is contained in $\mathfrak{a}_i(J)$ and $ad_{X_1} \circ \ldots \circ ad_{X_{i+1}} = 0$, by hypothesis.

Vice versa, if $ad_{X_1} \circ \ldots \circ ad_{X_{i+1}} = 0$, then $(ad_{X_1} \circ \ldots \circ ad_{X_i})ad_{X_{i+1}} = 0$ and $ad_{X_{i+1}}\mathfrak{g}$ is contained in $\mathfrak{a}_i(J)$. Thus $X \in \mathfrak{a}_{i+1}(J)$.

□

A natural consequence of this Proposition is that

Corollary 2.6 *The complex structure J is nilpotent if and only if for any $X \in \mathfrak{g}$, there exists an index k sucht that all the 2^k maps $ad_{X_1} \circ \ldots ad_{X_k}$ vanish.*

3 Solvable complex structures.

In the previous Section, we have remarked that a nilpotent complex structure is necessary given on a nilpotent Lie-algebra. A natural problem is to investigate when a complex structure (of a nilpotent Lie-algebra) is nilpotent itself. In order to give an answer to this question, we introduce the concept of solvable complex structure. Such a concept satisfies all the algebraic properties we may expect; furthermore it is the solution to the above problem: a complex structure J is nilpotent if and only if it is solvable and the Lie-algebra $\mathfrak{g}$ is nilpotent (Theorem 3.6).

The classical definition of solvability for a Lie-algebra $\mathfrak{g}$ is equivalent to the existence of a chain of ideals such that each element contains the first derivative of the previous one. Such a fact suggests the following

Definition 3.1 *The structure J is said to be solvable if there exists a family $\{\mathfrak{g}_i\}$ of ideals such that*

1. $\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_s \supseteq \mathfrak{g}_{s+1} = \{0\}$;

2. $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian;*

3. *each* $\mathfrak{g}_i$ *is a* J*-ideal.*

As well as in the nilpotent case, whenever J is solvable, the Lie-algebra is solvable, too. Moreover, if J is *ad*-invariant, the two conditions coincide: in fact, the derivative series is J-invariant.

Remark 3.2 *Whenever* J *is solvable, the ideal* $\mathfrak{g}' := \mathcal{D}\mathfrak{g} + J\mathcal{D}\mathfrak{g}$ *is strictly contained in* $\mathfrak{g}$*. In fact,* $\mathfrak{g}'$ *is contained in* $\mathfrak{g}_1$ *which is properly contained in* $\mathfrak{g}$*.*

Obviously the link between nilpotence and solvability are quite strong. First of all, we have the

Proposition 3.3 *A nilpotent complex structure* J *is solvable.*

Proof: if $\mathfrak{g} = a_t(J)$, the ideal $\mathfrak{g}_i := a_{t-i}(J)$ satisfy the conditions of Definition 3.1.

□

Let us conclude the present Section, recalling the natural main properties of solvable complex structures.

A direct remark on the definition shows that on any J-subalgebra, of a Lie-algebra with solvable J, the restriction of J is solvable. In particular, whenever $\mathfrak{h}$ is a J-ideal, $J_{\mathfrak{h}}$ is solvable. This fact implies that a "chain condition" is satisfied for J. Moreover, the induced complex structure on the quotient Lie-algebra $\mathbb{J} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ is solvable, too.

Now, we may prove also the converse. In fact, we have the

Theorem 3.4 *Take a* J*-ideal* $\mathfrak{h}$ *such that* $J_{\mathfrak{h}}$ *and* $\mathbb{J} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ *are both solvable. Then* J *is solvable.*

Proof: first of all, notice that $\mathfrak{g}$ is solvable or, equivalentely, $\mathcal{D}\mathfrak{g}$ is nilpotent. Then

$$\rho : \mathcal{D}\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : x \mapsto ad_x$$

is a nilrepresentation. Thus there exists a family of subspaces

$$W_0 = \{0\}$$

$$W_{i+1} = \{w \in \mathfrak{g} : \rho(x)w \in W_i, \rho(x)Jw \in W_i\}.$$

Such subspaces satisfy the conditions

1. $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_s = \mathfrak{g}$

2. $\dim W_i < \dim W_{i+1}$.

3. $\mathcal{D}\mathfrak{g} \subseteq W_{s-1}$.

The last condition implies that W_{s-1} contains $\mathcal{D}\mathfrak{g} + J\mathcal{D}\mathfrak{g}$. Moreover $\mathfrak{g}_1 = \mathcal{D}\mathfrak{g} + J\mathcal{D}\mathfrak{g}$ is an ideal. Substituing $\mathfrak{g}$ with $\mathfrak{g}_1$, we construct $\mathfrak{g}_2$ and hence $\mathfrak{g}_j$. Obviously, $J\mathfrak{g}_i = \mathfrak{g}_i$ and, by construction $\mathcal{D}\mathfrak{g}_i \subseteq \mathfrak{g}_{i+1}$, which means that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

□

Finally, solvable complex structures are characterized by the

Proposition 3.5 *If J is solvable, then there exists a family of J-subalgebras $\mathfrak{h}_1 \supseteq \ldots \supseteq \mathfrak{h}_n = \{0\}$, where $\dim \mathfrak{g} = 2n$, such that $\dim(\mathfrak{h}_i/\mathfrak{h}_{i+1}) = 2$. Moreover, $\mathfrak{h}_{i+1}$ is an ideal of $\mathfrak{h}_i$ and the quotient is abelian.*

Proof: consider a subspace $\mathfrak{h}_1$ such that its codimension is two and it contains $\mathcal{D}\mathfrak{g} + J\mathcal{D}\mathfrak{g}$. Such a $\mathfrak{h}_1$ is an ideal. Moreover, construct $\mathfrak{h}_2$ in the same way, just subsituing $\mathfrak{g}$ with $\mathfrak{h}_1$. Thus, $\mathfrak{h}_2$ is an ideal in $\mathfrak{h}_1$ and a subalgebra of $\mathfrak{g}$.

In order to complete the proof, we have to show that the quotient is abelian. Obviously, $\mathbb{J} : \mathfrak{h}_i/\mathfrak{h}_{i+1} \to \mathfrak{h}_i/\mathfrak{h}_{i+1}$ is solvable. Since the dimension is two, if the quotient is not abelian, then $(\mathfrak{h}_i/\mathfrak{h}_{i+1})'$ coincides with $\mathfrak{h}_i/\mathfrak{h}_{i+1}$, which contradicts Remark 3.2.

□

Now we have all the elements to carachterize the nilpotent comples structures proving that

Theorem 3.6 *Whenever $\mathfrak{g}$ is nilpotent and J is solvable, then J is nilpotent, too.*

Proof: the statement is true if $\dim \mathfrak{g} = 2$, thus we prove the Theorem by induction.

Since J is solvable, the J-ideal $\mathfrak{g}'$ is properly contained in $\mathfrak{g}$, hence, by induction, $J' = J|_{\mathfrak{g}'}$ is nilpotent.

Thus, $\mathfrak{g}'$ is contained in $\mathfrak{n}(J)$ and in particular $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}(J)$, which means that $\mathfrak{g} = \mathfrak{a}_{s+1}(J)$.

□

Such a result suggests us a necessary and sufficient condition for solvability.

Corollary 3.7 *The complex structure J is solvable if and only if J' is nilpotent.*

Proof: the first implication is proved in the above Theorem.

Now, let us take J' nilpotent. Then $\mathfrak{g}' \subset \mathfrak{g}$, otherwise $\mathfrak{g}$ vanishes. Now set $\mathfrak{g}_1 = \mathfrak{g}'$ and $\mathfrak{g}_{i+1} = \mathfrak{a}'_i(J')$. then the family $\{\mathfrak{g}_i\}$ satisfies the conditions of the definition and J is solvable.

□

In order to conclude the present Section, let us recall some "complex" facts. A complex structure J on $\mathfrak{g}$ is equivalent to a complex subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \overline{\mathfrak{q}}$. In particular, $\mathfrak{q} := \{X - iJX : X \in \mathfrak{g}\}$. In these terms, we have the

Theorem 3.8 *A complex structure J is solvable if and only if there exists a chain of ideals $\mathfrak{g}_i^{\mathbb{C}}$ such that $\mathfrak{q}$ induces a complex structure on each $\mathfrak{g}_i^{\mathbb{C}}$. Finally, this fact is equivalent to the solvability of $\mathfrak{q}$.*

Proof: the solvability of J is equivalent to the existence of a chain of J-ideals $\mathfrak{g}_i$. Take their complexified $\mathfrak{g}_i^{\mathbb{C}}$. Then set

$$\mathfrak{q}_i := \{X - iJX : X \in \mathfrak{g}_i\}.$$

Such $\mathfrak{q}_i$ coincide with $\mathfrak{q} \cap \mathfrak{g}_i^{\mathbb{C}}$. A direct computation shows that each $\mathfrak{g}_i^{\mathbb{C}}$ splits as

$$\mathfrak{g}_i^{\mathbb{C}} = \mathfrak{q}_i \oplus \overline{\mathfrak{q}}_i.$$

4 The J-radical.

Consider now a generic Lie algebra $\mathfrak{g}$ endowed with a complex structure J. Take two J-ideal $\mathfrak{h}$ and $\mathfrak{k}$ on which J is solvable. Thus $\mathfrak{h} + \mathfrak{k}$ is a J-ideal on which J is solvable. In fact, $\mathbb{J}(\mathfrak{h} + \mathfrak{k})/\mathfrak{k} \to (\mathfrak{h} + \mathfrak{k})/\mathfrak{k}$ and $J^{@} : \mathfrak{h}/\mathfrak{h} \cap \mathfrak{k} \to \mathfrak{h}/\mathfrak{h} \cap \mathfrak{k}$ are equivalent and $J^{@}$ is solvable. Thus $\mathbb{J}$ and $J_{\mathfrak{k}}$ are solvable and the solvability of $J_{\mathfrak{h}+\mathfrak{k}}$ follows, by Theorem 3.4.

A direct consequence is that any Lie algebra with complex structure J admits a unique *J-radical* $\mathfrak{r}(J)$.

The most important properties of the J-radical are the natural ones.

Proposition 4.1 *The J-radical is invariant by holomorphic derivations.*

Proof: when D is an holomorphic derivation, $\exp tD$ is an holomorphic authomorphism of $\mathfrak{g}$. Hence $(\exp tD)\mathfrak{r}(J)$ is included in $\mathfrak{r}(J)$. So, $D\mathfrak{r}(J) \subseteq \mathfrak{r}(J)$.

□

Remark 4.2 *Moreover, the nilradical $\mathfrak{n}(J)$ is contained in $\mathfrak{r}(J)$ and $[\mathfrak{g}, \mathfrak{r}(J)]$ stays in $\mathfrak{n}(J)$:*

$$[\mathfrak{g}, \mathfrak{r}(J)] \subseteq \mathfrak{n}(J) \subseteq \mathfrak{r}(J).$$

In the term of the J-radical it is a natural fact giving the following

Definition 4.3 *The complex structure J is semisimple whenever $\mathfrak{r}(J)$ vanishes.*

There are two families of complex structures which are trivially semisimple: the complex structures on an algebra without J-ideal; and the complex structures on a semisimple Lie-algebra (In fact, the J-radical is a solvable radical, that is $r(J) \subseteq \mathfrak{r}$). Furthermore, the complex structure $\mathbb{J} : \mathfrak{g}/\mathfrak{r}(J) \to \mathfrak{g}/\mathfrak{r}(J)$ induced by a generic complex structure is semisimple. The converse situation is described by the

Proposition 4.4 *If $\mathfrak{a}$ is a J-ideal such that $\mathbb{J} : \mathfrak{g}/\mathfrak{a} \to \mathfrak{g}/\mathfrak{a}$ is semisimple, then $\mathfrak{r}(J) \subseteq \mathfrak{a}$.*

Proof: consider the natural projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$, then $\pi(\mathfrak{r}(J))$ is a J-ideal on which J is solvable. Thus $\pi(\mathfrak{r}(J)) = \{0\}$ and $\mathfrak{r}(J) \subseteq \mathfrak{a}$.

□

A useful consequence is given by

Proposition 4.5 *Let $\pi : (\mathfrak{g}, J) \to (\mathfrak{g}', J')$ be an holomorphic epimorphism, then $\pi(\mathfrak{r}(J)) = \mathfrak{r}'(J')$.*

Proof: $\pi(\mathfrak{r}(J))$ is a solvable ideal of $\mathfrak{g}'$. Denote with $\pi^* : \mathfrak{g}/\mathfrak{r}(J) \to \mathfrak{g}'/\pi(\mathfrak{r}(J))$ the induced epimorphism. Then $\pi(\mathfrak{r}(J))$ contains $\mathfrak{r}'(J')$, and hence, the statement follows.

□

Moreover, this fact is crucial in proving that

Theorem 4.6 *When J is semisimple, $\mathfrak{g}$ coincides with $\mathfrak{g}'$.*

Proof: the Lie algebra $\mathfrak{g}/\mathfrak{g}'$ is abelian, thus the induced structure $\mathbb{J}$ is solvable: $\mathfrak{r}(\mathbb{J}) = \mathfrak{g}/\mathfrak{g}'$. But, $\mathfrak{r}(\mathbb{J}) = \pi\mathfrak{r}(J) = \{0\}$, and the thesis follows.

□

For reasons of completness we have to show that a J-ideal of a Lie-algebra endowed with a semisimple complex structure supports a semisimple complex structure, the naturally induced one. The first step is given by the

Proposition 4.7 *Let $\mathfrak{h}$ be a J-ideal. Then, the J-radical $\mathfrak{r}(J_{\mathfrak{h}})$ coincides with the intersection $\mathfrak{r}(J) \cap \mathfrak{h}$.*

Proof: the inclusion $\mathfrak{r}(J) \cap \mathfrak{h} \subseteq \mathfrak{r}(J_{\mathfrak{h}})$ is trivial, since $\mathfrak{r}(J) \cap \mathfrak{h}$ is a solvable ideal of $\mathfrak{h}$.

Now, suppose that $\mathfrak{r}(J)$ does not vanish. Then $\mathfrak{h}_1 := \mathfrak{h}/\mathfrak{r}(J) \cap \mathfrak{h}$ is a J_1-ideal of $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{r}(J)$, on which J_1 is semisimple. By induction, the J_1-radical of $\mathfrak{h}_1$ coincides with the intersection $\mathfrak{h}_1 \cap \mathfrak{r}(J_1)$. Hence, it vanishes. Namely, $\mathfrak{r}(J_{\mathfrak{h}})$ is contained in $\mathfrak{r}(J) \cap \mathfrak{h}$, and the Proposition is proved.

Whenever $\mathfrak{r}(J)$ vanishes, consider the Lie-algebra $\mathfrak{k} := \mathfrak{h} \oplus_{ad} \mathfrak{g}$, together with the canonical projections π_1 and π_2. If $\pi_1(\mathfrak{r}(J_{\mathfrak{k}})) = \mathfrak{r}(J_{\mathfrak{h}})$ does not vanish, then the J-radical of $\mathfrak{k}$ is the nonvanishing J-ideal $\mathfrak{r}(J_{\mathfrak{k}}) = \mathfrak{r}(J_{\mathfrak{h}}) \oplus \{0\}$. Take, now, the J-ideal $\mathfrak{h} \oplus_{ad} \mathfrak{h}$ and apply the proof for the nonvanishing case. Then we have that $\mathfrak{r}(J_{\mathfrak{h} \oplus_{ad} \mathfrak{h}}) = \mathfrak{r}(J_{\mathfrak{k}}) \cap \mathfrak{h} \oplus_{ad} \mathfrak{h} = \mathfrak{r}(J_{\mathfrak{h}}) \oplus \{0\}$. Otherwise, $\mathfrak{r}(J_{\mathfrak{h}}) \oplus_{ad} \mathfrak{r}(J_{\mathfrak{h}})$ is a J-ideal of $\mathfrak{h} \oplus_{ad} \mathfrak{h}$ on which J is solvable: a contradiction.

□

Corollary 4.8 *Let $\mathfrak{g}$ be a Lie-algebra on which J is semisimple, then J is semisimple on each J-ideal.*

In order to conclude this Section, let us expose a result envolving the linear endomorphism J:

Lemma 4.9 *Any almost-complex structure $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is semisimple, in the sense of the matrix theory.*

Proof: take the unique decomposition $J = S + N$, where S is the semisimple part of J and N the nilpotent one. Now, compute $J^2 = S^2 + 2SN + N^2 = -I$. Since S and N commute and I is semisimple, we have $S^2 = -I$ and $(2S+N)N = 0$. Moreover, the equation $J^3 = -J$ becomes

$$S^3 + 3S^2N + 3SN^2 + N^3 = -S - N$$

and hence, $S^3 = -S$ and $3S^2N + 3SN^2 + N^3 = -N$. Obvious sobstituitions imply that

$$N^3 = -4N.$$

Thus, since N is nilpotent, N has to vanish, and J is a semisimple isomorphism.

□

In particular, there exists a J-invariant linear subspace V such that $\mathfrak{g} = \mathfrak{r}(J) \oplus V$. This situation suggests us the obvious

Definition 4.10 *Whenever V is a subalgebra, V is said to be a Levi J-subalgebra and J_V is semisimple (in fact, $\mathfrak{g}/\mathfrak{r}(J)$ is holomorphically isomorphic to V).*

Thus, we have all the elements to suspect that a J-Levi-Mal'cev decomposition exists. If this is the case, we are able to decompose canonically a generic complex structure into the direct sum of a solvable and a semisimple one. And we may limit our study to the two distinct cases.

Let us conclude observing that the cathegory of complex structures which are solvable is more interesting and quite smaller than the one of solvable Lie-algebras. Thus we have a nontrivial chance to classify such complex structures. Future papers will be devoted to the classification of solvable complex structures; and, moreover, to the study of the possible rigidity of a solvable complex structure in the moduli space of complex structures on a solvable Lie-algebras.

5 On the existence of solvable complex structures.

In Section 3, we have proved that, whenever J is a solvable complex structure, then there exists a codimension two subspace $\mathfrak{h}$ containing $\mathcal{D}\mathfrak{g}$ such that $J|_{\mathfrak{h}}$ is solvable.

In the following we shall consider a pair $(\mathfrak{h}, J)$ such that

1. $codim\mathfrak{h} = 2$
2. $\mathfrak{h} \subseteq \mathcal{D}\mathfrak{g}$
3. $N_J = 0$
4. J is solvable,

and we prove that

Theorem 5.1 *Let $(\mathfrak{h}, J)$ satisfy the above four conditions. Then almost any extension $\mathbb{J}$ of J to $\mathfrak{g}$ is a solvable complex structure.*

The proof will be given in three steps. First of all, let us show that there exists an extension which is integrable. In fact, in correspondence of the pair $(\mathfrak{h}, J)$, we may set

$$\mathbb{J}|_{\mathfrak{h}} = J \qquad \mathbb{J}Z = W \qquad \mathbb{J}W = -Z$$

(where $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}Z \oplus \mathbb{R}W$). Such a $\mathbb{J}$ is, by definition an almost-complex structure. The first step consists in the

Proposition 5.2 *There exist Z, W for which $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}Z \oplus \mathbb{R}W$ and $\mathbb{J}$ (defined as above) is integrable.*

Proof: take an almost-complex extension $\mathbb{J}$ of J and consider the structure $\mathbb{J}_1$ (extension of J) defined setting

$$\mathbb{J}_1 Z := H_1 + aZ + b\mathbb{J}Z,$$

where H_1 does not vanish and $Z \notin \mathfrak{h}$. Compute now the following formulas

$$-Z = JH_1 + aH_1 + a^2 Z + ab\mathbb{J}Z + b\mathbb{J}_1\mathbb{J}Z$$

$$\mathbb{J}_1\mathbb{J}Z = -\frac{1}{b}(aH_1 + \jmath H_1) - \frac{1+a^2}{b}Z - a\mathbb{J}Z$$

$$\mathbb{J}\mathbb{J}_1 Z = JH_1 - bZ + a\mathbb{J}Z.$$

Let us verify the condition of linear independence for $\mathbb{J}_1 Z, \mathbb{J}_1 \mathbb{J} Z$ and $\mathbb{J}\mathbb{J}_1 Z$: $\alpha \mathbb{J}\mathbb{J}_1 Z + \beta \mathbb{J}_1 \mathbb{J} Z + \gamma \mathbb{J}_1 Z = 0$ is equivalent to

$$\begin{cases} aJH_1 - \frac{\beta}{b}(aH_1 + JH_1) + \gamma H_1 = 0 \\ -\alpha b - \beta \frac{1+a^2}{b} + a\gamma = 0 \\ \alpha a - \beta a + \gamma b = 0 \end{cases}$$

which is

$$\begin{cases} \alpha = \frac{\beta}{b} \\ \gamma = a\frac{\beta}{b} \\ (b+1)\beta = 0. \end{cases}$$

When $b \neq -1$, then $\beta = \alpha = \gamma = 0$ and the vectors are linearly independents

If $b = -1$, then $\alpha = -\beta, \gamma = -\beta a, \alpha a = 0$. In this case, when a does not vanish, they are independents. Then, the only case in which they are linearly dependent is $a^2 + (b+1)^2 = 0$.

In the following we set $a^2 + (b+1)^2 \neq 0$ and hence there exists a nonvanishing element of $\mathfrak{h}$

$$H_2 = \alpha \mathbb{J}\mathbb{J}_1 Z + \beta \mathbb{J}_1 \mathbb{J} Z + \gamma \mathbb{J}_1 Z.$$

This means that the following conditions are true

$$\begin{cases} \gamma a = \alpha b + \beta \frac{1+a^2}{b} \\ \gamma b = (\beta - \alpha) a. \end{cases}$$

Now, let us compute the expression of $N_{\mathbb{J}_1}(H, H_2)$ which vanishes since coincides with $N_J(H, H_2)$:

$$\begin{aligned} N_{\mathbb{J}_1}(H, H_2) &= \alpha N_{\mathbb{J}_1}(H, \mathbb{J}\mathbb{J}_1 Z) - \beta J N_{\mathbb{J}_1}(H, \mathbb{J} Z) - \gamma N_{\mathbb{J}_1}(H, Z) \\ &= -(\alpha b I + \gamma J) N_{\mathbb{J}_1}(H, Z) + (\alpha a I - \beta J) N_{\mathbb{J}_1}(H, \mathbb{J} Z) \\ &= \{-(\alpha b I + \gamma J) + (\alpha a I - \beta J)(-\frac{a}{b} I - \frac{1}{b} J)\} N_{\mathbb{J}_1}(H, Z) = 0. \end{aligned}$$

Let us suppose that $\{-(\alpha b I + \gamma J) + (\alpha a I - \beta J)(-\frac{a}{b} I - \frac{1}{b} J)\}$ vanishes. Then, we have

$$\begin{cases} \alpha(a^2 + b^2) + \beta = 0 \\ \gamma b = (\alpha - \beta) a. \end{cases}$$

Take now an extension $\mathbb{J}_1$ such that $a \neq 0$ and remind that if H_2 is an element of $\mathfrak{h}$, then $\gamma b = (\beta - \alpha) a$. Summarizing the two conditions, we have that $\alpha = \beta$. Thus we may rewrite the global system as

$$\begin{cases} \gamma a = \alpha \frac{1+a^2+b^2}{b} \\ \gamma b = 0 \\ \alpha(1+a^2+b^2) = 0 \\ \alpha = \beta. \end{cases}$$

The third one says that α vanishes, and hence γ does. So we have proved that, if the $\{\}$ vanishes, then H_2 does, which is a contradiction. Hence, it is $N_{\mathbb{J}_1}(H, Z) = 0$. Globally, we have shown that

Proposition 5.3 *Let $\mathbb{J}$ be a generic almost-complex extension of J and let $\mathbb{J}_1 Z := H_1 + aZ + b\mathbb{J}Z$ (with $aH_1 \neq 0$) complete the definition of a different extension of J. Then $\mathbb{J}_1$ is integrable.*

□

A direct corollary of this Proposition is given by the

Theorem 5.4 *Almost any extension of J to $\mathfrak{g}$ is integrable.*

Proof: once proved the above Proposition, take $\mathbb{J}_1$ as (almost-)complex structure and write $\mathbb{J}Z$ as

$$\mathbb{J}Z = -\frac{1}{b}H_1 - \frac{a}{b}Z - \frac{1}{b}\mathbb{J}_1 Z.$$

We have just changed their roles. When $b^2 + (a+1)^2 \neq 0$, $\mathbb{J}Z$ is in the hypothesis and then $\mathbb{J}$ is integrable, too.

□

Thus, we have proved the first part of Theorem 5.1: in correspondence of $(\mathfrak{h}J)$ almost any extension $\mathbb{J}$ of J is a complex structure of $\mathfrak{g}$.

The third and last step of the proof is given by the

Theorem 5.5 *Whenever J is solvable, $\mathbb{J}$ is solvable, too.*

Proof: since J is solvable, $\mathfrak{h}$ is a $\mathbb{J}$-ideal on which $\mathbb{J}$ is solvable. In particular, $\mathfrak{h} \subseteq \mathfrak{r}(\mathbb{J})$. Two cases are possible: either $\mathfrak{r}(\mathbb{J}) = \mathfrak{g}$ (which means that $\mathbb{J}$ is solvable!) or $\mathfrak{r}(\mathbb{J}) = \mathfrak{h}$. In this case, the quotient complex structure on $\mathfrak{g}/\mathfrak{r}(\mathbb{J})$ has to be semisimple, while the quotient is an abelian Lie-algebra, a contradiction.

□

Finally, we may give a sufficient condition for the existence of a solvable complex structure. Such a condition is exactly the converse of Proposition 3.5.

Proposition 5.6 *Let $\mathfrak{g}$ be a Lie-algebra and let $\mathfrak{h}_0 = \mathfrak{g} \supseteq \mathfrak{h}_1 \supseteq \cdots \supseteq \mathfrak{h}_n = \{0\}$ be subalgebras such that $\dim \mathfrak{h}_i/\mathfrak{h}_{i+1} = 2$ and $\mathfrak{h}_i/\mathfrak{h}_{i+1}$ is abelian. Furthermore, suppose that $\mathfrak{h}_{n-1}$ supports a complex structure J_{n-1} (which is solvable since $\mathfrak{h}_{n-1}$ is abelian). Then $\mathfrak{g}$ admits a solvable complex structure.*

The proof is given by the iterated application of the results of the present Section.

6 The 4-dimensional case.

In conclusion we study the 4-dimensional Lie-algebrasw which admit a solvable complex structure. Let $\mathfrak{g}$ be a 4-dimensional real Lie-algebra endowed with a complex structure J. Then J is solvable if and only if there exists an abelian 2-dimensional J-ideal $\mathfrak{g}_1$ containing $\mathcal{D}\mathfrak{g}$. (This is just the reformulation of the definition).

In order to classify these complex structures, take a basis e_1, e_2, e_3, e_4 of the linear space $\mathfrak{g}$ satisfying

$$\mathfrak{g}_1 =< e_1, e_2 >$$

$$Je_1 = e_2 \qquad\qquad Je_3 = e_4.$$

Now we have to consider all the possible skewsymmetric bilinear maps $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. the Jacobi identity holds for $[,]$;
2. $\mathfrak{g}_1$ is abelian;
3. $\mathcal{D}\mathfrak{g} \subseteq \mathfrak{g}_1$;
4. the Nijenhuis tensor vanishes identically.

The conditions "$\mathfrak{g}_1$ abelian" and $\mathcal{D}\mathfrak{g} \subseteq \mathfrak{g}_1$ are equivalent to

$$\begin{cases} [e_1, e_2] = 0 \\ [e_1, e_3] = B^1 e_1 + B^2 e_2 \\ [e_1, e_4] = C^1 e_1 + C^2 e_2 \\ [e_2, e_3] = D^1 e_1 + D^2 e_2 \\ [e_2, e_4] = E^1 e_1 + E^2 e_2 \\ [e_3, e_4] = F^1 e_1 + F^2 e_2. \end{cases}$$

A direct computation proves that

Lemma 6.1 *The bilinear form $[,]$ is a Lie bracket (i.e. satisfies the Jacobi condition) if and only if*

$$\begin{cases} B^1 C^2 + B^2 E^2 - C^1 B^2 - C^2 D^2 = 0 \\ B^2 E^1 - C^2 D^1 = 0 \\ D^1 C^1 + D^2 E^1 - E^1 B^1 - E^2 D^1 = 0. \end{cases}$$

and that

Lemma 6.2 *The almost complex structure J is integrable if and only if*

$$\begin{cases} E^1 = B^1 - C^2 - D^2 \\ E^2 = B^2 + C^1 + D^1. \end{cases}$$

Now, three cases are possible:

(A) the Lie-algebra is abelian. Thus any complex structure is solvable.

(B) $\dim \mathcal{D}\mathfrak{g} = 1$, i.e. $\mathcal{D}\mathfrak{g} =< A^1 e_1 + A^2 e_2 >$

(C) $\dim \mathcal{D}\mathfrak{g} = 2$, i.e. $\mathcal{D}\mathfrak{g} =< e_1, e_2 >$

Case (A) is trivial; then we start with the case (B) and set

$$B^i = \beta A^i \quad C^i = \gamma A^i \quad D^i = \delta A^i \quad E^i = \varepsilon A^i \quad F^i = \varphi A^i.$$

Then the sistem becomes

$$\begin{cases} \varepsilon A^1 = \beta A^1 - \gamma A^2 - \delta A^2 \\ \varepsilon A^2 = \beta A^2 + \gamma A^1 + \delta A^1 \\ \beta A^2(\beta A^2 + \delta A^1) = \gamma A^2(\delta A^2 - \beta A^1) \\ \beta A^2(\beta A^1 - \gamma A^2 - \delta A^2) = \gamma \delta A^1 A^2 \\ \delta A^2(2\beta A^1 - \gamma A^2 - \delta A^2) = \beta^2 (A^1)^2 + \delta^2 (A^1)^2 - \beta\gamma A^1 A^2 + \beta\delta A^1 A^2. \end{cases}$$

First of all, set $A^1 = 0$. Thus ths system reduces to

$$\begin{cases} \gamma = -\delta \\ \varepsilon = \beta \\ \gamma\delta - \beta^2 = 0 \end{cases}$$

which has the unique solution $\beta = \gamma = \delta = \varepsilon = 0$. Thus the unique nonvanishing bracket is $[e_3, e_4] = \lambda e_2, \lambda \neq 0$, and hence we denote the Lie-algebra as $\mathfrak{g}(\lambda)$. Moreover, let $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda'}$ two different Lie-algebras. Then they are not isomorphic, even in the real sense.

The only fact we have to prove is the non existence of an isomorphism. An isomorphism φ should send $< e_2 >$ in $< e_2 >$. Thus, φ has the form

$$\begin{cases} \varphi e_1 = a e_1 + b e_2 + c e_3 + d e_4 \\ \varphi e_2 = e e_2 \\ \varphi e_1 = f e_1 + g e_2 + h e_3 + i e_4 \\ \varphi e_1 = l e_1 + m e_2 + n e_3 + p e_4. \end{cases}$$

Since we want that φ be a Lie-isomorphism, we obtain $(ci - dh)\gamma' = (cp - dn)\gamma' = 0$ and $(hp - in)\gamma' = \gamma e \neq 0$, which is a contradiction.

Remark 6.3 *Now, set $A^2 = 0$ and in an obvious way we obtain $\varepsilon = \beta = \gamma = \delta = 0$. Hence, the Lie-bracket is given by $[e_3, e_4] = \mu e_1, \mu \neq 0$.*

Finally, when $A^1 A^2 \neq 0$, set $a := A^1/A^2$. Then the first two equations become

$$\begin{cases} \gamma + \delta = (\beta - \varepsilon)a \\ (\gamma + \delta)a = \varepsilon - \beta, \end{cases}$$

whose solutions satisfy $\varepsilon = \beta$ and $\gamma = -\delta$. Making these sobstitutions in the fourth equation we obtain $\varepsilon^2 = -\delta^2$. Thus, the unique nonvanishing coefficient is φ. It is not difficult to prove that two of these "generic" Lie-algebras are nonisomorphic. The case (B) may be summarized as follows

Theorem 6.4 *Let $\mathfrak{b}$ be a 4-dimensional Lie-algebra and J denote a solvable complex structure. If there exists a basis such taht $Je_1 = e_2, Je_3 = e_4$ and $\mathcal{D}\mathfrak{b} =< \lambda e_1 + \mu e_2 >$, with $\lambda^2 + \mu^2 \neq 0$. Then the unique nonvanishing Lie-bracket is $[e_3, e_4] = \lambda e_1 + \mu e_2$. Different pairs of (λ, μ) define nonisomorphic $(\mathfrak{b}, J)$. Thus we set $\mathfrak{b} = \mathfrak{b}(\lambda, \mu)$.*

Finally, the case (C) is divided in two subcases: $B^2 = 0$ and $B^2 \neq 0$. Let us set $B^2 = 0$. Then the conditions become

$$\begin{cases} E^1 = B^1 - C^2 - D^2 \\ E^1 = C^1 + D^1 \\ C^2(D^2 - B^1) = 0 \\ C^2 D^1 = 0 \\ D^2(2B^1 - C^2 - D^2) = (B^1)^2 + (D^1)^2 - B^1 C^2. \end{cases}$$

One solution is given if C^2 vanishes. Then, the last equation is

$$(D^2 - B^1)^2 = -(D^1)^2$$

and hence $D^1 = 0$ and $B^1 = D^2$, which implies that

Proposition 6.5 *The (C1) Lie-algebras are given by $[e_1, e_3] = \lambda e_1$, $[e_1, e_4] = \mu e_1$, $[e_2, e_3] = \lambda e_2$, $[e_2, e_4] = \mu e_2$, $[e_3, e_4] = \pi e_1 + \xi e_2$, $\lambda^2 + \mu^2 \neq 0$.*

Moreover, we have that

Theorem 6.6 *Two (C1) algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are mutually isomorphic.*

Proof: an holomorphic isomorphism φ sends $< e_1, e_2 >$ in itself and commutes with J, thus it has the form

$$\begin{cases} \varphi e_1 = ae_1 + be_2 \\ \varphi e_2 = -be_1 + ce_2 \\ \varphi e_3 = ee_1 + fe_2 + ge_3 + he_4 \\ \varphi e_4 = -fe_1 + ee_2 - he_3 + ge_4. \end{cases}$$

Now, let us impose that it is Lie's:

$$[\varphi e_1, \varphi e_3] = ag\lambda' e_1 + ah\mu' e_1 + bg\lambda' e_2 + bh\mu' e_2 = a\lambda e_1 + b\lambda e_2$$

$$[\varphi e_1, \varphi e_4] = -ah\lambda' e_1 + ag\mu' e_1 - bh\lambda' e_2 + bg\mu' e_2 = a\mu e_1 + b\mu e_2$$

$$\begin{aligned} [\varphi e_3, \varphi e_4] &= (-eh + fg)\lambda' e_1 + (eg + fh)\mu' e_1 - (fh + eg)\lambda' e_2 \\ &+ (fg - eh)\mu' e_2 + (g^2 + h^2)(\pi' e_1 + \xi' e_2) \\ &= \pi(ae_1 + be_2) - \xi(be_1 - ae_2). \end{aligned}$$

So, we have

$$\begin{cases} g\lambda' + h\mu' = \lambda \\ -h\lambda' + g\mu' = \mu \\ e\mu + f\lambda + (g^2 + h^2)\pi' = a\pi - b\xi \\ f\mu - e\lambda + (g^2 + h^2)\xi' = b\pi + a\xi \end{cases}$$

which has solution for any choice of φ invertible (i.e. $(a^2+b^2)(g^2+h^2) \neq 0$.). In particular, the choice $a = 1, b = 0, g = \mu, h = \lambda$ defines an isomorphism between the generic (C1) Lie algebra $\mathfrak{c}^1(\lambda, \mu, \pi, \xi)$ and $\mathfrak{c}^1 := \mathfrak{c}^1(0, 1, 0, 0)$.

□

Let us return to the sistem. The other solution is provided by $D^1 = 0$, in correspondence of which we obtain $E^1 = -C^2, D^2 = B^1$ and $E^2 = C^1$.

Proposition 6.7 *A (C2) Lie-algebra is given by the bracket* $[e_1, e_3] = \lambda e_1$, $[e_1, e_4] = \mu e_1 + \nu e_2$, $[e_2, e_3] = \lambda e_2$, $[e_2, e_4] = -\nu e_1 + \mu e_2$, $[e_3, e_4] = \pi e_1 + \xi e_2$, *with* $\lambda^2 + \mu^2 + \nu^2 \neq 0$.

Theorem 6.8 *A (C2) Lie-algebra is isomorphic either to* $\mathfrak{c}^2 := \mathfrak{c}^2(1, 0, 0, 0, 0)$ *whenever* $\lambda^2 + \mu^2 \neq 0 = \nu$, *or to* $\mathfrak{c}^2(0, 0, 1, 0, 0) = \mathfrak{c}^1$ *whenever* $\lambda^2 + \mu^2 = 0 = \nu$, *or to* $\mathfrak{c}^2(\frac{\lambda}{\nu}, \frac{\mu}{\nu}, 1, 0, 0)$, *whenever* $(\lambda^2 + \mu^2)\nu \neq 0$.

Proof: let φ be an isomorphism, described by the usual equations. Then we have the following conditions

$$[\varphi e_1, \varphi e_3] = ag\lambda' e_1 + ah(\mu' e_1 + \nu' e_2) + bg\lambda' e_2 + bh(-\nu' e_1 + \mu' e_2) = a\lambda e_1 + b\lambda e_2$$

$$[\varphi e_1, \varphi e_4] = -bg\lambda' e_1 - bh(\mu' e_1 + \nu' e_2) + ag\lambda' e_2 + ah(-\nu' e_1 + \mu' e_2) = \lambda(-be_1 + ae_2)$$

$$\begin{array}{rcl} [\varphi e_3, \varphi e_4] & = & (-eh + fg)\lambda' e_1 + (eg + fh)(\mu' e_1 + \nu' e_2) + \\ & - & (fh + eg)\lambda' e_2 + (fg - eh)(-\nu' e_1 + \mu' e_2) + \\ & + & (g^2 + h^2)(\pi' e_1 + \xi' e_2) = \pi(ae_1 + be_2) - \xi(be_1 - ae_2). \end{array}$$

which are equivalent to

$$\begin{cases} ag\lambda' + ah\mu' - bh\nu' = a\lambda \\ ah\nu' + bg\lambda' + bh\mu' = b\lambda \\ -ah\lambda' + ag\mu' - bg\nu' = a\mu - b\nu \\ ag\nu' - bh\lambda' + bg\mu' = b\mu + a\nu \\ (eh - fg)(\nu' - \lambda') + (eg + fh)\mu' + (g^2 + h^2)\pi' = a\pi - b\xi \\ (eg + fh)(\nu' - \lambda') + (fg - eh)\mu' + (g^2 + h^2)\xi' = b\pi - a\xi. \end{cases}$$

Now, set $(\lambda', \mu', \nu', \pi', \xi') = (1, 0, 0, 0, 0)$. Whenever $\lambda^2 + \mu^2 \neq 0$ and $\nu = 0$, the choice $a = 1$, $b = 0$, $g = \lambda$ and $h = -\mu$ defines an isomorphism between $\mathfrak{c}^2(\lambda, \mu, 0, \pi, \xi)$ and $\mathfrak{c}^2(1, 0, 0, 0, 0)$, which is not isomorphic to $\mathfrak{c}^2(0, 0, \nu, \pi, \xi)$.

Setting $\lambda^2 + \mu^2 = 0$, we obtain the equations

$$\begin{cases} g\lambda' + h\mu' = 0 \\ h\lambda' - g\mu' = 0 \\ h\nu' = 0 \\ \nu = g\nu'. \end{cases}$$

Since, $\nu \neq 0$, it follows that $g \neq 0$ and $\nu' \neq 0$. So $\lambda' = \mu' = 0$ and $h = 0$. In this case, the choiche $a = 1$, $b = 0$, $g = \frac{\nu}{\nu'}$, $h = 0$ defines an isomorphism between $\mathfrak{c}^2(0, 0, \nu, \pi, \xi)$ and $\mathfrak{c}^2(0, 0, 1, 0, 0)$.

In conclusion, consider the case $(\lambda^2 + \mu^2)\nu \neq 0$. Thus the system becomes

$$\begin{cases} g\lambda' + h\mu' = \lambda \\ h\nu' = 0 \\ g\mu' - h\lambda' = \mu \\ g\nu' = \nu, \end{cases}$$

and then

$$\begin{cases} h = 0 \\ g\nu' = \nu \\ g\lambda' = \lambda \\ g\mu' = \mu. \end{cases}$$

So, we have that $\mathfrak{c}^2(\lambda, \mu, \nu, \pi, \xi) \equiv \mathfrak{c}^2(\frac{\lambda}{\nu}, \frac{\mu}{\nu}, 1, 0, 0)$. Now, we have completed the case $B^2 = 0$.

□

In the following B^2 will be always nonvanishing. Whenever $C^2 = 0$, the only solution is given by $E^1 = 0$, $E^2 = C^1$, $B^2 = -D^1$ and $B^1 = D^2$, which implies the

Proposition 6.9 *The (C3) Lie-algebras are given by the Lie bracket* $[e_1, e_3] = \lambda e_1 + \mu e_2$, $[e_1, e_4] = \nu e_1$ $[e_2, e_3] = -\mu e_1 + \lambda e_2$ $[e_2, e_4] = \nu e_2$ $[e_3, e_4] = \pi e_1 + \xi e_2, \lambda^2 + \mu^2 + \nu^2 \neq 0$.

Let us compute the isomorphisms between $\mathfrak{c}^3(\lambda, \mu, \nu, \pi, \xi)$ and $\mathfrak{c}^3(1, 0, 0, \pi', \xi')$:

$$\begin{aligned}
[\varphi e_1, \varphi e_3] &= ahe_1 + bhe_2 = \lambda(ae_1 + be_2) + \mu(-be_1 + ae_2) \\
[\varphi e_1, \varphi e_4] &= age_1 + bge_2 = \nu(ae_1 + be_2) \\
[\varphi e_2, \varphi e_3] &= -bhe_1 + ahe_2 = -\mu(ae_1 + be_2) + \lambda(-be_1 + ae_2) \\
[\varphi e_2, \varphi e_4] &= -bge_1 + age_2 = \nu(-be_1 + ae_2) \\
[\varphi e_1, \varphi e_3] &= (eg + fh)e_1 + (fg - eh)e_2 + (g^2 + h^2)(\pi' e_1 + \xi' e_2) = \pi(ae_1 + be_2) + \xi(-be_1 + ae_2),
\end{aligned}$$

which imply $h = \lambda$, $\mu = 0$ and $g = \nu$: i.e. $\mathfrak{c}^3(\lambda, 0, \nu, \pi, \xi) \equiv \mathfrak{c}^3(1, 0, 0, 0, 0)$.

Hence, we study the isomorphism in the case $\mu\mu' \neq 0$. The usual computations show that

$$\begin{cases} g\lambda' + h\nu' = \lambda \\ g\mu' = \mu \\ h\mu' = 0 \\ \nu + h\lambda' = g\mu' \end{cases}$$

and we obtain $h = 0$, $g \neq 0$ and $(\lambda, \mu, \nu) = g(\lambda', \mu', \nu')$. Summarizing all the cases, we have the

Theorem 6.10 *A Lie algebra* $\mathfrak{c}^3(\lambda, \mu, \nu, \pi, \xi)$ *is isomorphic to* $\mathfrak{c}^3(1, 0, 0, 0, 0)$ *when* $\mu = 0$, *to* $\mathfrak{c}^3(\frac{\lambda}{\mu}, 1, \frac{\nu}{\mu}, 0, 0)$, *if* $\mu \neq 0$. *The Lie-algebras* $\mathfrak{c}^3(\lambda, 1, \mu, \pi, \xi)$ *and* $\mathfrak{c}^3(\lambda', 1, \mu', \pi', \xi')$ *are isomorphic if and only if* $(\lambda, \mu) = (\lambda'.\mu')$.

Finally, we have to consider the case $B^2C^2 \neq 0$. First of all, let us remark that D^1 can not vanish. In fact, if it vanishes, we have $(B^2)^2 = -(C^2)^2$, which is a contradiction.

Now, let us write the third and the fourth equations as

$$D^2 - B^1 = \frac{B^2}{C^2}(B^2 + D^1)$$
$$B^1 - D^2 = \frac{C^2}{B^2}(B^2 + D^1)$$

which imply $B^1 = D^2$, $B^2 = -D^1$ and hence
4

Proposition 6.11 *The (C4) Lie-algebras are given by the bracket* $[e_1, e_3] = \lambda e_1 + \mu e_2$ $[e_1, e_4] = \nu e_1 + \pi e_2$ $[e_2, e_3] = -\mu e_1 + \lambda e_2$ $[e_2, e_4] = -\pi e_1 + \nu e_2$ $[e_3, e_4] = \xi e_1 + \rho e_2, \lambda^2 + \mu^2 + \nu^2 + \pi^2 \neq 0$.

The matrix of an isomorphism φ between two (C4) Lie-algebras satisfies the relations

$$\begin{cases} ag\lambda' + ah\nu' - bg\mu' - bh\pi' = a\lambda - b\mu \\ ag\mu' + ah\pi' + bg\lambda' + bh\nu' = b\lambda + a\mu \\ -ah\lambda' + ag\nu' + bh\mu' - bg\pi' = a\nu - b\pi \\ -ah\mu' + ag\pi' - bh\lambda' + bg\nu' = b\nu + a\pi. \end{cases}$$

Now, let us set $(\lambda', \mu', \nu', \pi') = (1, 0, 0, 0)$. Then it follows that $g = \lambda$, $h = -\nu$ and $\mu = \pi = 0$. So, we have the isomorphism $\mathfrak{c}^4(\lambda, 0, \nu, 0, \xi, \rho) \equiv \mathfrak{c}^4(1, 0, 0, 0, 0, 0) = \mathfrak{c}^1$.
Moreover, we may rewrite the above relations as

$$\begin{cases} g\lambda' + h\nu' = \lambda \\ g\mu' + h\pi' = \mu \\ g\nu' - h\lambda' = \nu \\ h\mu' - g\pi' = -\pi \end{cases}$$

which say that $(\lambda, \nu) = (0, 0)$ if and only if $(\lambda', \nu') = (0, 0)$. The same result holds for the pair (μ, π). Hence, we may write

$$\mathfrak{c}^4(\lambda, 0, \nu, 0, \xi, \rho) \simeq \mathfrak{c}^4(1, 0, 0, 0, 0, 0) = \mathfrak{c}^1$$
$$\mathfrak{c}^4(0, \mu, 0, \pi, \xi, \rho) \simeq \mathfrak{c}^4(0, 1, 0, 0, 0, 0) =: \mathfrak{c}^4$$

which are not equivalent. The last case is provided by the relation $(\lambda'^2 + \nu'^2)(\mu'^2 + \pi'^2) \neq 0$. In this hypothesis, we have that, fixed the 4-tuple $(\lambda', \mu', \nu', \pi')$ the corresponding (C4) Lie-algebra is holomorphically isomorphic to the Lie

algebras $\mathfrak{c}^4(\lambda,\mu,\nu,\pi,\xi,\rho)$ where (λ,μ,ν,π) is a point of $\mathbb{R}^4$ contained in the bidimensional plane of equations

$$\frac{\lambda'}{\lambda'^2+\nu'^2}\lambda+\frac{\nu'}{\lambda'^2+\nu'^2}\nu-\frac{\mu'}{\mu'^2+\pi'^2}\mu-\frac{\pi'}{\mu'^2+\pi'^2}\pi=0$$

$$\frac{\nu'}{\lambda'^2+\nu'^2}\lambda-\frac{\lambda'}{\lambda'^2+\nu'^2}\nu-\frac{\pi'}{\mu'^2+\pi'^2}\mu\frac{\mu'}{\mu'^2+\pi'^2}\pi=0.$$

The plane is defined by the points $P'=(\lambda',\mu',\nu',\pi')$, $Q'=(\nu',\pi',-\lambda',-\mu')$, $O=(0,0,0,0)$. Vice versa, if P and P'' are equivalent, even P'' stays on the same plane.

It is not difficult to verify that any of these complex structures is not equivalent to those determined by $(\lambda,\nu)=(0,0)$ or by $(\mu,\pi)=(0,0)$. Otherwise, on any plane we can find an element of the form $(\lambda,\mu,\nu,0)$.

Proposition 6.12 *The (C4) Lie-algebras are isomorphic to*

$$\mathfrak{c}^1=\mathfrak{c}^4(1,0,0,0,0,0)\quad \textit{whenever}\quad \mu^2+\pi^2=0$$

$$\mathfrak{c}^4=\mathfrak{c}^4(0,1,0,0,0,0)\quad \textit{whenever}\quad \lambda^2+\nu^2=0$$

$$\mathfrak{c}^4(\lambda,\mu,\nu,\pi,0,0)\quad \textit{whenever}\quad (\lambda^2+\nu^2)(\mu^2+\pi^2)\neq 0.$$

In the last case, we have that $\mathfrak{c}^4(\lambda,\mu,\nu,\pi,0,0)\equiv\mathfrak{c}^4(\lambda',\mu',\nu',\pi',0,0)$ *if and only if* $P=\alpha P'+\beta Q'$, *where* P,P' *and* Q' *are defined as above. Thus, we may find a (C4) Lie-algebra of this last type of the form* $\mathfrak{c}^4(\lambda,1,\nu,0,0,0)$.

Let us summarize all the results about 4-dimensional Lie-algebras in the following

Theorem 6.13 *A 4-dimensional Lie-algebra* $\mathfrak{g}$ *with solvable complex structure* J *is isomorphic to one of the following Lie-algebras, where the (nonvanishing) Lie-brackets are described with respect to the basis* $(e_1, e_2:=Je_1, e_3, e_4:=Je_3)$*:*

$\mathfrak{a}$	*abelian*
$\mathfrak{b}(\lambda,\mu)$	$[e_3,e_4]=\lambda e_1+\mu e_2,\ \lambda^2+\mu^2\neq 0;$
$\mathfrak{c}^1$	$[e_1,e_4]=e_1,\ [e_2,e_4]=e_2$
$\mathfrak{c}^2$	$[e_1,e_3]=e_1,\ [e_2,e_3]=e_2$
$\mathfrak{c}^2(\lambda,\mu)$	$[e_1,e_3]=\lambda e_1,\ [e_1,e_4]=\mu e_1+e_2,\ [e_2,e_3]=\lambda e_2,\ [e_2,e_4]=-e_1+\mu e_2,$ $\lambda^2+\mu^2\neq 0;$

$\mathfrak{c}^3(\lambda,\nu)$ $\quad [e_1,e_3]=\lambda e_1+e_2,\ [e_1,e_4]=\nu e_1,\ [e_2,e_3]=-e_1+\lambda e_2,\ [e_2,e_4]=\nu e_2$
$\lambda^2+\nu^2\neq 0;$

$\mathfrak{c}^4$ $\quad [e_1,e_3]=e_2,\ [e_2,e_3]=-e_1.$

All these Lie-algebras are not isomorphic each other. Moreover, the Lie-algebras of type (A) or (B) are nilpotent, while no (C) Lie-algebra is nilpotent.

Thus we have that the 4-dimensional Lie-algebras with solvable J are given by threee families depending on two parameters $(\mathfrak{b}(\lambda,\mu),\mathfrak{c}^2(\lambda,\mu),\mathfrak{c}^3(\lambda,\nu))$; and four special Lie-algebras $(\mathfrak{a},\mathfrak{c}^1,\mathfrak{c}^2,\mathfrak{c}^4)$.

As an application of the above classification, let us prove that we may have obstructions on a Lie-algebra with a solvable complex structure. Such obstructions are given by the existence of a 4-dimensional subalgebra in the studied Lie-algebra. An example of this situation is the following

Proposition 6.14 *The Lie algebra $\mathfrak{g}$ of the Lie group $T(3,\mathbb{R})$ does not admit solvable complex structures.*

Proof: let us denote with

$$U=\begin{pmatrix} u & t & r\\ 0 & v & s\\ 0 & 0 & l\end{pmatrix}$$

the generic element of $\mathfrak{g}$. Consider now the basis $E_1=A_{11},E_2=A_{22},E_3=A_{33},E_4=A_{12},E_5=A_{23},E_6=A_{13}$, where $(A_{ij})_{hk}=\delta_{ih}\delta_{jk}$. Then the bracket is given by $[E_1,E_4]=E_4$, $[E_1,E_6]=E_6$, $[E_2,E_4]=-E_4$, $[E_2,E_5]=E_5$, $[E_3,E_5]=E_5$, $[E_3,E_6]=-E_6$, $[E_4,E_5]=E_6$. Hence, we have $\mathcal{D}\mathfrak{g}=<E_4,E_5,E_6>$.

First of all, let us recall that Gradali proves, in [4], that the complex structure of $\mathfrak{g}$ are divided in two families, described setting

$$JU=\begin{pmatrix} f_{11} & f_{12} & \eta s-\lambda r\\ 0 & f_{2}2 & \lambda s+\mu r\\ 0 & 0 & f_{33}\end{pmatrix}$$

$$J^*U=\begin{pmatrix} f^*_{11} & f^*_{12} & f^*_{13}\\ 0 & f^*_{22} & f^*_{23}-\lambda r\\ 0 & 0 & f^*_{33}\end{pmatrix}$$

where f_{ij} and f^*_{ij} are suitable functions.

In particular, Gradali proves that f_{ij} and f^*_{ij} vanish in correspondence of E_6, thus we may set

$$JE_6 = -\lambda r E_6 + \mu E_5 \quad \text{and} \quad J^* E_6 = -\lambda r E_5.$$

In any case, the plane $< E_5, E_6 >$ is invariant. Now, let us suppose to have a solvable complex structure J on $\mathfrak{g}$. This implies that J' is nilpotent and $\mathfrak{g}' = \mathcal{D}\mathfrak{g} + J\mathcal{D}\mathfrak{g}$ takes the form $< A, E_4, E_5, E_6 >$, with $(A^1)^2 + (A^2)^2 + (A^3)^2 \neq 0$. Hence, by Theorem 6.13, $\mathfrak{g}'$ is equal to $\mathfrak{b}(\lambda, \mu)$ for suitable λ and μ. Then $\mathcal{D}\mathfrak{g}'$ is one-dimensional. Thus, it is $A^1 = A^2 = A^3$, and (I, E_4, E_5, E_6) is a basis of $\mathfrak{g}'$.

Let us write the matrix of an isomorphism $\varphi : \mathfrak{g}' \to \mathfrak{b}$ with respect to the bases (I, E_4, E_5, E_6) and (e_1, e_2, e_3, e_4)

$$\varphi = \begin{pmatrix} a & b & c & d \\ f & g & h & i \\ l & m & n & p \\ q & r & s & t \end{pmatrix}.$$

Since the unique nonvanishing bracket of $\mathfrak{g}'$ (resp. of $\mathfrak{b}$) is $[E_4, E_5] = E_6$ (resp. $[e_3, e_4] = \lambda e_1 + \mu e_2$), and φ is a Lie isomorphism, then

$$(nt - ps)[e_3, e_4] = f e_1 + g e_2 + h e_3 + i e_4.$$

In particular, $h = i = 0$. Moreover,

$$0 = \varphi[I, JI] = [\varphi I, KJ\varphi I] = (c^2 + d^2)[e_3, e_4]$$

and hence, $c = d = 0$.

Then $\varphi I, \varphi E_6 \in < e_1, e_2 >$ which is both the center of $\mathfrak{b}$ and a J-ideal. Thus, JI, JE_6 are in the center of $\mathfrak{g}'$ which is $< I, E_6 >$ and

$$< E_6, JE_6 > = < I, E_6 > .$$

Otherwise, Gradali showed that $< E_6, JE_6 >$ coincides with $< E_5, E_6 >$, which is a contradiction. Hence there are no solvable complex structures on the Lie group $T(3, \mathbb{R})$.

□

References

[1] L. Cordero, M. Fernandez, A. Gray, L. Ugarte, *Nilpotent complex structures on compact nilmanifolds*, Rend. Circ. Mat. di Palermo, **49** (1997), 83-100.

[2] D. Gouthier, *Solvable and Levi-flat CR-structures on a real Lie-algebra*, to appear on Ann. di Mat Pura ed Appl.

[3] D. Gouthier, *LCR-structures and LCR-algebras*, PhD thesi, SISSA 1996.

[4] P. Gradali, *Strutture complesse su solvmanifolds*, Tesi di Laurea, Parma 1997.

[5] L. Magnin, *Sur les algebres de Lie nilpotentes de dimension* ≤ 7, J. Geom. Phis, **3** (1986), 119-144.

[6] S.M. Salamon, *Complex structures on nilpotent Lie algebras*, mathDG/9808025, 5 Aug 1988.

[7] V.S. Varadarajan, *Lie groups, Lie algebras and their Rewpresentations*, Prentice-Hall, Englewood Cliffs, NJ, 1974.