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Connection Problem and Monodromy Data for a Class of Solutions the Sixth Painlevé Equation

Davide Guzzetti

S.I.S.S.A. - I.S.A.S., via Beirut 2–4 - 34014 Trieste, Italia.

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Abstract.

In this paper we find a class of solutions of the sixth Painlevé equation appearing in the theory of WDVV equations. We describe the asymptotic behaviour close to the critical points in terms of two parameters and we find the relation among the parameters at the different critical points (connection problem). Results on the subject are already proved by some authors quoted here, who solved the connection problem for a limited range of the parameters. Other approaches allow to find the asymptotic behaviour for a wider range of parameters, but without solving the connection problem. Here we extend the range of the parameters for which we solve the connection problem and give a unified picture of pre-existing results.

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1 Introduction

Let us denote with PVI_μ the sixth Painleve' equation with the following choice of the parameters (in standard notation of [8]):

$$\alpha = \frac{(2\mu-1)^2}{2}, \quad \beta = \gamma = 0, \quad \delta = \frac{1}{2}$$

Namely:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2 + \frac{x(x-1)}{(y-x)^2}\right], \qquad \mu \in \mathbf{C}$$

This choice is important in the theory of WDVV equations of associativity for $n = 3$ fields. The WDVV equations can be reduced to PVI_μ [3]

We study the case $2\mu \notin \mathbf{Z}$ and find solutions to PVI_μ with asymptotic behaviour $a^{(0)}x^{1-\sigma^{(0)}}$ for $x \to 0$, $1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}$ for $x \to 1$, $a^{(\infty)}x^{-\sigma^{(\infty)}}$ for $x \to \infty$, where $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)} \neq 0$ and $\sigma^{(i)} \notin (-\infty, 0) \cap [1, \infty)$ (here (a, b) denotes an open interval and $[a, b]$ a closed one). The convergence of x to the critical points $x = 0, 1, \infty$ is along spiral or radial paths, according to the value of σ. We also describe the analytic continuation of such solutions and solve the *connection problem*, namely we find the relation among the three couples $(\sigma^{(i)}, a^{(i)})$.

The existence of solutions with one of the above asymptotic behaviours at one of the critical points was proved by Shimomura for generic parameters α, β, γ, δ in [17] [9]. The three critical points were treated separately (actually, just $x = 0$ was considered, the others being equivalent by symmetries) and the connection problem was not touched. We quote his result: for any $\sigma \in \mathbf{C}\backslash\{(-\infty, 0] \cap [1, +\infty)\}$ and any $k \in \mathbf{C}$ there exist a solution of PVI on the domain $D(\epsilon) := \left\{x \in \widetilde{\mathbf{C}\backslash\{0\}} \mid 0 < |x| < \epsilon,\ |e^{-k}x^{1-\sigma}| < \epsilon,\ |e^k x^\sigma| < \epsilon\right\}$ behaving in the following way: $y(x) = e^{-k}x^{1-\sigma}(1 + O(|x| + |e^{-k}x^{1-\sigma}| + |e^k x^\sigma|)$. Here $\epsilon = \epsilon(\sigma)$ is sufficently small and $\widetilde{\mathbf{C}\backslash\{0\}}$ is the universal covering of $\mathbf{C}\backslash\{0\}$. In the case $\Re\sigma < 0$ and $\Re\sigma \geq 1$ the shape of the domains above forces x to converge to 0 along spiral paths.

We observe that the case $\sigma = 0$ is not included in Shimomura's picture. Moreover, it is not clear if a solution having the prescribed asymptotic behaviour close to a critical point (say $x = 0$) has a similar behaviour when x approaches another critical point. And if it does, how are the parameters (σ, k) characterizing the behaviour at one critical point related to the parameters at another critical point?

For $0 \leq \Re\sigma < 1$, Jimbo [10] solved the connection problem for solutions having the above asymptotic behaviour along radial paths converging to the critical points. He made use of the isomonodromic deformation theory under some genericity assumption on the parameters α, β, γ, δ. Exploiting the connection between the Painleve' equation and Schlesinger equations, he found the relation between the parameters σ, a and the monodromy data of the fuchsian system associated to PVI. The crucial step in his proof is a result on the asymptotic behaviour of a class of solutions of Schlesinger equations which was proved by Sato, Miwa, Jimbo in [16].

Later, Dubrovin-Mazzocco [4] applied Jimbo procedure to PVI_μ ($2\mu \notin \mathbf{Z}$), which violates some of the genericity assumptions. They classified all the algebric solutions of PVI_μ in terms of monodromy data and gave a geometric interpretation of the data themselves in terms of finite reflection groups.

In this paper we prove that Jimbo [10], and Dubrovin-Mazzocco [4] procedure can be extended to any complex $\sigma^{(i)} \notin (-\infty, 0) \cap [1, \infty)$. While in [10] and [4] the convergence of x to the critical points is along radial paths, the extension of the result to any $\sigma^{(i)} \notin (-\infty, 0) \cap [1, \infty)$ forces us to consider spiral

paths whenever $\Re\sigma < 0$ and $\Re\sigma \geq 1$. Thus it turns out that the solutions we find can be identified with those studied by Shimomura: in this way we give an extended and unified picture of both Shimomura and Jimbo's works, and we solve the connection problem for Shimomura's solutions. Actually, we prove that a solution behaves as stated above *simultaneously* at the three critical points, we give explicit formulae which connect the parameters $(\sigma^{(i)}, a^{(i)})$ with the monodromy data of the fuchsian system associated to PVI_μ by the isomonodromic deformation theory and so we obtain the connection between the parameters themselves. We also describe how these parameters change under analytic continuation of the solution (this point was carefully analyzed in [4] for $0 \leq \Re\sigma < 1$ and is crucial for the classification of algebric solutions).

The notes are organized as follows. In section 2 we give the results with a few comments. In section 3 we derive them. In particular, lemma 1 of section 3 is the generalization to any complex $\sigma \notin (-\infty, 0) \cap [1, \infty)$ of Sato-Miwa-Jimbo result [16] used in [10] [4]. Section 4 is an Appendix on the monodromy data of the fuchsian system associated to PVI_μ by the theory of isomonodromic deformations.

Acknowledgments: I thank prof. B. Dubrovin for being always ready to discuss and give me hints. I'm indebted to M.Mazzocco for many discussions, and for drawing my attention to the works of Shimomura. In this paper I often refer to [4], where the case $1 \leq \Re\sigma < 1$ is studied.

2 Results

2.1 Local Behaviour

In these notes we suppose that

$$2\mu \notin \mathbf{Z}$$

A solution $y(\tilde{x})$ of PVI_μ is a meromorphic function of the point $\tilde{x}$ belonging to the universal covering of $\mathbf{P}^1 \backslash \{0, 1, \infty\}$. Usually, one starts from the local study of a branch $y(x)$ of the solution, where $x \in \mathbf{P}^1 \backslash \{0, 1, \infty\}$. Studying a branch means that x can't describe loops around 1, 0, and ∞. Thus, some branch cuts are necessary (for example $|\arg(x)| < \pi$ and $|\arg(1-x)| < \pi$, see figure 2). The global solution $y(\tilde{x})$ is obtained as the analytic continuation of such a branch along curves encircling the singularities 0, 1, ∞ .

Let σ and a denote two complex numbers, with the restrictions

$$\sigma \in \Omega := \mathbf{C} \backslash \{(-\infty, 0) \cup [1, +\infty)\} \qquad a \neq 0$$

Our firt aim is to show the existence of solutions of PVI_μ which behave like $a\, x^{1-\sigma}$ as $x \to 0$ in a suitable way. Thus, we concentrate on a small punctured neighbourhood of $x = 0$, and the point $\tilde{x}$ can be read as a point in the universal covering of $\mathbf{C}_0 := \mathbf{C} \backslash \{0\}$ with $0 < |x| < \epsilon$ ($\epsilon < 1$). Namely, $\tilde{x} = |x| e^{i \arg(x)}$, where $-\infty < \arg(x) < +\infty$.

In order to specify the way x tends to zero, we introduce a domain contained in the universal covering of $\mathbf{C}_0$ (we denote the universal covering by $\widetilde{\mathbf{C}_0}$). If $\sigma = 0$ we define the domain

$$D(\epsilon; \sigma = 0) = \{\tilde{x} \in \widetilde{\mathbf{C}_0} \text{ s.t. } 0 < |x| < \epsilon\}$$

If $\sigma \neq 0$ we observe that

$$|\tilde{x}^\sigma| = |x|^{\sigma'(x)}, \quad \text{where } \sigma'(x) := \Re\sigma - \frac{\Im\sigma \ \arg(x)}{\log|x|}$$

We define a domain in which $0 < \sigma'(x) < 1$. Let θ_1, $\theta_2 \in \mathbf{R}$, $0 < \tilde{\sigma} < 1$ and:

$$D(\epsilon; \sigma, \theta_1, \theta_2) := \{\tilde{x} \in \widetilde{\mathbf{C}_0} \text{ s.t. } 0 < |x| < \epsilon, \ \ e^{-\theta_1 \Im\sigma} |x|^{\tilde{\sigma}} \leq |\tilde{x}^\sigma| \leq e^{-\theta_2 \Im\sigma} |x|^0, \quad 0 < \tilde{\sigma} < 1\}$$

Here ϵ is small enough, depending on $\tilde{\sigma}$ (in particular, from the proof of theorem 1 which follows, we'll see that $\epsilon \to 0$ if $\tilde{\sigma} \to 1$). The domain can be rewritten as

$$\Re\sigma \log|x| + \theta_2 \Im\sigma \leq \Im\sigma \arg(x) \leq (\Re\sigma - \tilde{\sigma}) \log|x| + \theta_1 \Im\sigma$$

Figure 1 shows the domains. Note that if $0 \leq \Re\sigma < 1$ the domain contains any sector of width less then 2π (provided $|x|$ is sufficiently small). Also note that $D(\epsilon; \sigma, \theta_1, \theta_2) = D(\epsilon; \sigma = 0)$ for real $0 \leq \sigma < 1$.

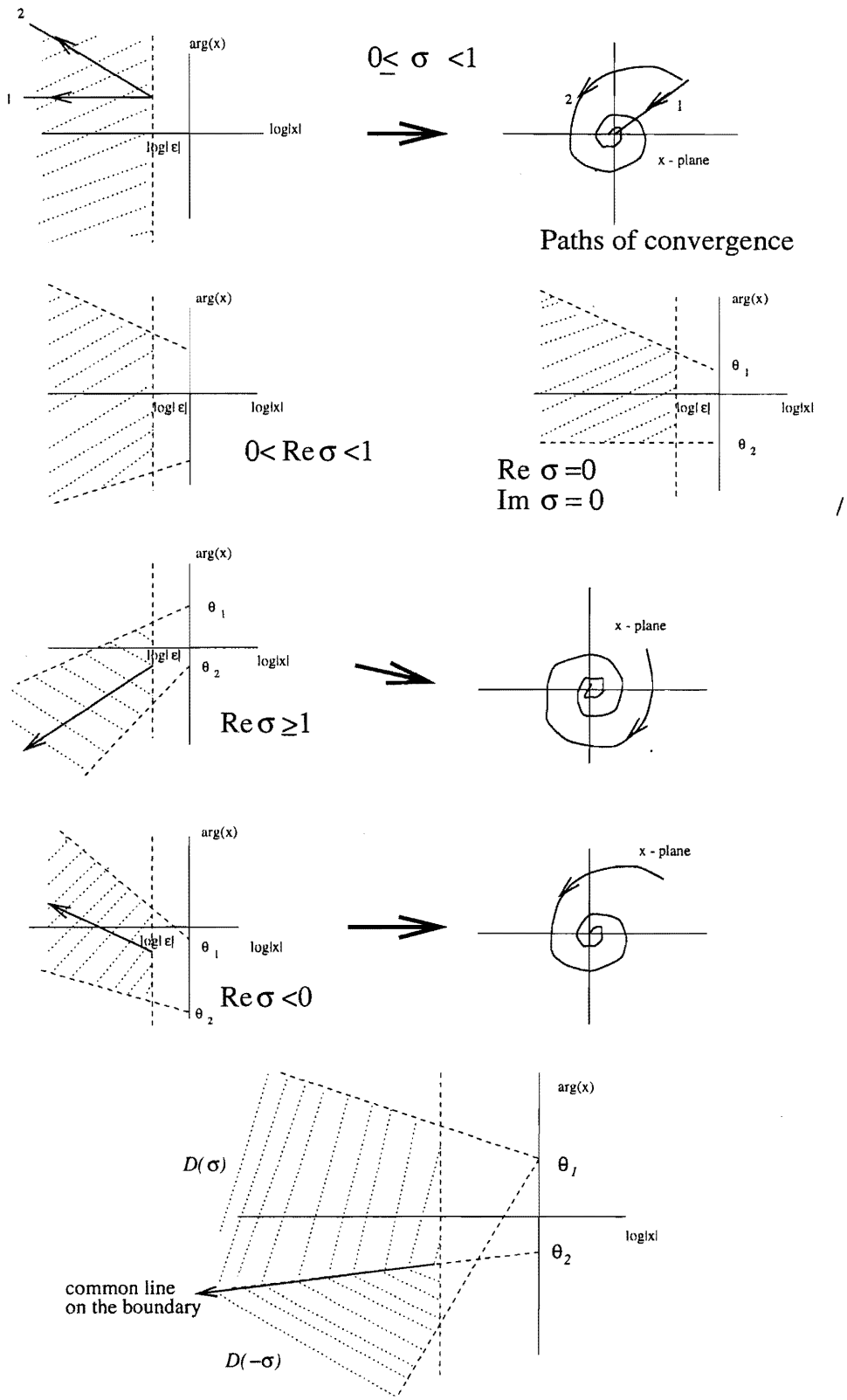

Figure 1: Domains $D(\varepsilon,\sigma)$ and paths of convergence ($\mathrm{Im}\,\sigma>0$ in the figures where it is not zero)

When we consider a branch $y(x)$ of a solution $y(\tilde{x})$ we have to restrict $\arg(x)$ to $a < |\arg(x)| < b$ with $|a-b| < 2\pi$. Thus in the following we'll always understand that θ_1 and θ_2 are chosen in such a way that the values $a < \arg(x) < b$ are contained in $D(\epsilon;\sigma,\theta_1,\theta_2)$

Theorem 1: *Let $2\mu \notin \mathbf{Z}$. For any $\sigma \in \Omega$ and for any $a \in \mathbf{C}$, $a \neq 0$, there exists a branch $y(x;\sigma,a)$ of a solution of PVI_μ whose analytic continuation in $D(\epsilon;\sigma,\theta_1,\theta_2)$ has the asymptotic behaviour*

$$y(\tilde{x};\sigma,a) = a\tilde{x}^{1-\sigma}\left(1+O(|x|^\delta)\right) \quad \text{as } \tilde{x} \to 0 \text{ in } D(\epsilon;\sigma,\theta_1,\theta_2) \tag{1}$$

Here $0 < \delta < 1$ is a small number. If $0 \leq \Re\sigma < 1$ the branch $y(x;\sigma,a)$ itself has the above behaviour as $x \to 0$ in any sector of width less then 2π.

Theorem 1 is valid if the convergence of x to zero is along a "regular" path, like a radial path (when it is possible) $\arg(x) = b$ constant, or a spiral-shaped path $\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma$, $0 \leq \Sigma \leq \tilde{\sigma}$, contained in $D(\epsilon;\sigma,\theta_1,\theta_2)$. Along the paths with $\Sigma = 0$ (and $\sigma \neq 0$) the theorem must be slightly modifided by substituting the constant a by a function $a(x) = O(1)$ as $x \to 0$. This function will be explicitely computed in the proof of the theorem.

The solutions $y(x)$ of theorem 1 can be compared with those found by Shimomura [17]. They are actually the same, with a slight change of the domains. In figure 1 we also draw the possible paths along which $x \to 0$.

A remark on the notation: in the following, I will omit θ_1, θ_2 writing the domains. Also, we say that two matrices M and N are *conjugated* if there exists an invertible matrix A such that $M = A^{-1}NA$. Finally, we denote by I the identity matrix.

2.2 Parametrization of a branch through Monodromy Data

As it is well known, in the case $2\mu \notin \mathbf{Z}$ there is a one to one correspondence between a class of equivalence of admissible triples of monodromy data (x_0, x_1, x_∞) of the fuchsian system associated to PVI_μ in the theory of isomonodromic deformation, and a branch of a solution (see section 3 and the Appendix). Two triples (x_0, x_1, x_∞) and (x_0', x_1', x_∞') are equivalent if their elements differ just by the change of two signs. A triple is admissible if just one $x_i = 0$. The following relation holds:

$$x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4\sin^2(\pi\mu) \tag{2}$$

We expect that for any (σ, a) of theorem 1 we can find a set of monodromy data, and conversely for a set of monodromy data satisfying (2), eventually resctricted by some additional conditions, we can find $\sigma = \sigma(x_0, x_1, x_\infty)$ and $a = a(x_0, x_1, x_\infty)$.

Theorem 2: *Let $2\mu \notin \mathbf{Z}$. For any $\sigma \in \Omega$ and for any $a \neq 0$ of theorem 1, there exists a triple of monodromy data (x_0, x_1, x_∞) given by the following formulae:*

i) $\sigma \neq 0, \pm 2\mu + 2m$ for any $m \in \mathbf{Z}$. Let $a =: -\frac{1}{4s}$, $s \neq 0$.

$$\begin{cases} x_0 = 2\sin(\frac{\pi}{2}\sigma) \\ x_1 = -\left[\frac{1}{f(\sigma)}\left(2 + F(\sigma)\ s + \frac{1}{F(\sigma)\ s}\right)\right]^{\frac{1}{2}} \\ x_\infty = -\left[\frac{1}{f(\sigma)}\left(2 - F(\sigma)e^{-i\pi\sigma}\ s - \frac{1}{F(\sigma)e^{-i\pi\sigma}\ s}\right)\right]^{\frac{1}{2}} \end{cases} \tag{3}$$

where

$$f(\sigma) = \frac{2\cos^2(\frac{\pi}{2}\sigma)}{\cos(\pi\sigma) - \cos(2\pi\mu)}, \qquad F(\sigma) = f(\sigma)\frac{16^\sigma \Gamma(\frac{\sigma+1}{2})^4}{\Gamma(1-\mu+\frac{\sigma}{2})^2\Gamma(\mu+\frac{\sigma}{2})^2}$$

ii) $\sigma = 0$

$$\begin{cases} x_0 = 0 \\ x_1^2 = 4\sin^2(\pi\mu)\ (1-a) \\ x_\infty^2 = 4\sin^2(\pi\mu)\ a \end{cases}$$

We can take any sign of the square roots

iii) $\sigma = \pm 2\mu + 2m$. *Let again* $a =: -\frac{1}{4s}$

iii1) $\sigma = 2\mu + 2m$, $m = 0, 1, 2, ...$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = -\frac{16^{\mu+m}\Gamma(\mu+m+\frac{1}{2})^2}{\Gamma(m+1)\Gamma(2\mu+m)}\sqrt{s} \\ x_\infty = i\ x_1\ e^{-i\pi\mu} \end{cases}$$

iii2) $\sigma = 2\mu + 2m$, $m = -1, -2, -3, ...$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = -\frac{\pi^2}{\cos^2(\pi\mu)}\frac{1}{16^{\mu+m}\Gamma(\mu+m+\frac{1}{2})^2\Gamma(-2\mu-m+1)\Gamma(-m)}\frac{1}{\sqrt{s}} \\ x_\infty = ix_1e^{i\pi\mu} \end{cases}$$

iii3) $\sigma = -2\mu + 2m$, $m = 1, 2, 3, ...$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = -\frac{16^{-\mu+m}\Gamma(-\mu+m+\frac{1}{2})^2}{\Gamma(m-2\mu+1)\Gamma(m)}\sqrt{s} \\ x_\infty = ix_1e^{i\pi\mu} \end{cases}$$

iii4) $\sigma = -2\mu + 2m$, $m = 0, -1, -2, -3, ...$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = -\frac{\pi^2}{\cos^2(\pi\mu)}\frac{1}{16^{-\mu+m}\Gamma(-\mu+m+\frac{1}{2})^2\Gamma(2\mu-m)\Gamma(1-m)}\frac{1}{\sqrt{s}} \\ x_\infty = ix_1e^{-i\pi\mu} \end{cases}$$

In all the above formulae the relation $x_0^2 + x_1^2 + x_\infty^2 - x_0x_1x_\infty = 4\sin^2(\pi\mu)$ *is automatically satisfied. Note that* $\sigma \neq 1$ *implies* $x_0 \neq \pm 2$

Conversely, for any set of monodromy data $(x_0,\ x_1,\ x_\infty)$ *such that* $x_0^2+x_1^2+x_\infty^2-x_0x_1x_\infty = 4\sin^2(\pi\mu)$ *there exist parameters* σ, *and* $s := -1/4a$ *(or* a*) obtained as follows:*

I) Generic case

$$\begin{cases} 2(1 - cos(\pi\sigma)) = x_0^2 \\ \frac{1}{f(\sigma)}\left(2 + F(\sigma)\ s + \frac{1}{F(\sigma)\ s}\right) = x_1^2 \\ \frac{1}{f(\sigma)}\left(2 - F(\sigma)e^{-i\pi\sigma}\ s - \frac{1}{F(\sigma)e^{-i\pi\sigma}\ s}\right) = x_\infty^2 \end{cases} \tag{4}$$

Note that $f(\sigma)$ *may be rewritten as*

$$f(\sigma) = \frac{4 - x_0^2}{2 - x_0^2 - 2\cos(2\pi\mu)} = \frac{4 - x_0^2}{x_1^2 + x_\infty^2 - x_0x_1x_\infty}$$

Any solution σ *must satisfy the restriction* $\sigma \neq \pm 2\mu + 2m$ *for any* $m \in \mathbf{Z}$, *otherwise we encounter the singularities in* $F(\sigma)$ *or* $1/F(\sigma)$ *(namely, the zeroes and the poles of* $\Gamma(..)$*) and in* $f(\sigma)$. *If* $x_0^2 = 4$ *the system has solutions* $\sigma = 1 + 2n$, $n \in \mathbf{Z}$, *which do not belong to* Ω. *In order to have solutions* $\sigma = 2n$, $n \in \mathbf{Z}$ *we need* $x_0 = 0$. *We treat this possibility in the following case II).*

II) $x_0 = 0$.

$$\begin{cases} 2(1 - cos(\pi\sigma)) = 0 \\ 4\sin^2(\pi\mu)\ (1 - a) = x_1^2 \\ 4\sin^2(\pi\mu)\ a = x_\infty^2 \end{cases}$$

Note that the only solution in Ω is $\sigma = 0$ and

$$a = \frac{x_\infty}{x_1^2 + x_\infty^2}$$

III) $x_0^2 = 4\sin^2(\pi\mu)$. *Then (2) implies* $x_\infty^2 = -x_1^2 \exp(\pm 2\pi i\mu)$. *Four cases which yield the values of* σ *non included in I) and II) must be considered*

III1) $x_\infty^2 = -x_1^2 e^{-2\pi i\mu}$

$$\sigma = 2\mu + 2m, \qquad m = 0, 1, 2, ...$$

$$s = \frac{\Gamma(m+1)^2\Gamma(2\mu+m)^2}{16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4}\, x_1^2$$

III2) $x_\infty^2 = -x_1^2 e^{2\pi i\mu}$

$$\sigma = 2\mu + 2m, \qquad m = -1, -2, -3, ...$$

$$s = \frac{\pi^4}{\cos^4(\pi\mu)}\left[16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4\Gamma(-2\mu-m+1)^2\Gamma(-m)^2\, x_1^2\right]^{-1}$$

III3) $x_\infty^2 = -x_1^2 e^{2\pi i\mu}$

$$\sigma = -2\mu + 2m, \qquad m = 1, 2, 3, ...$$

$$s = \frac{\Gamma(m-2\mu+1)^2\Gamma(m)^2}{16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4}\, x_1^2$$

III4) $x_\infty^2 = -x_1^2 e^{-2\pi i\mu}$

$$\sigma = -2\mu + 2m, \qquad m = 0, -1, -2, -3, ...$$

$$s = \frac{\pi^4}{\cos^4(\pi\mu)}\left[16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4\Gamma(2\mu-m)^2\Gamma(1-m)^2\, x_1^2\right]^{-1}$$

If $x_0^2 \neq 4$ *(*$\sigma \neq 1 + 2n$, $n \in \mathbf{Z}$*) we can always choose (from I), II), III))* $\sigma \in \Omega$. *Then there exists a branch* $y(x;\sigma,a) \equiv y(x;x_0,x_1,x_\infty)$ *parametrized by the monodromy data and whose analytic continuation in* $D(\epsilon;\sigma,\theta_1,\theta_2)$ *has the asymptotic behaviour of theorem 1.*

Observe that the formulae iii), III) are also obtainable from (4) and (3) respectively in the limit $\sigma \to \pm 2\mu + 2m$. However, we will prove them separately, because in the proof of the general case we will encounter divergences for $\sigma = \pm\mu + 2m$. Three remarks are in order here:

Remark 1: In the generic case *I*), with $x_0^2 \neq 4$, the system (4) in two variables σ, s and three equations is compatible. Actually, the first equation $2(1 - cos(\pi\sigma)) = x_0^2$ has always solutions. Let us choose a solution σ_0 ($\pm\sigma_0 + 2\pi n$, $\forall n \in \mathbf{Z}$ are also solutions). Substitute it in the last two equations. We need to verify they are compatible.

Instead of s and $\frac{1}{s}$ write X and Y. We have the linear system in two variable X, Y

$$\begin{pmatrix} F(\sigma_0) & \frac{1}{F(\sigma_0)} \\ F(\sigma_0)\, e^{-i\pi\sigma_0} & \frac{1}{F(\sigma_0)} e^{-i\pi\sigma_0} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} f(\sigma_0)\, x_1^2 - 2 \\ 2 - f(\sigma_0)\, x_\infty^2 \end{pmatrix}$$

The system has a unique solution if and only if $2i\sin(\pi\sigma_0) = \det\begin{pmatrix} F(\sigma_0) & \frac{1}{F(\sigma_0)} \\ F(\sigma_0)\, e^{-i\pi\sigma_0} & \frac{1}{F(\sigma_0)} e^{-i\pi\sigma_0} \end{pmatrix} \neq 0$. This happens for $\sigma_0 \notin \mathbf{Z}$. The condition is not restrictive, because for σ even we turn to the case *II*) of the theorem 2, and σ odd is not in Ω. The solution is then

$$X = \frac{2(1 + e^{-i\pi\sigma_0}) - f(\sigma_0)(x_1^2 + x_\infty^2 e^{-i\pi\sigma_0})}{F(\sigma_0)(e^{-2\pi i\sigma_0} - 1)}$$

$$Y = F(\sigma_0)\, \frac{f(\sigma_0)e^{-i\pi\sigma_0}(e^{-i\pi\sigma_0}x_1^2 + x_\infty^2) - 2e^{-i\pi\sigma_0}(1 + e^{-i\pi\sigma_0})}{e^{-2\pi i\sigma_0} - 1}$$

Compatibility of the system means that $X\,Y \equiv 1$. This is verified by direct computation:

$$XY = \frac{e^{-i\pi\sigma}\ \left[2(1+e^{-i\pi\sigma}) - (x_1^2 + x_\infty^2 e^{-i\pi\sigma})f(\sigma)\right]\ \left[(x_1^2 e^{-i\pi\sigma} + x_\infty^2)f(\sigma) - 2(1+e^{-i\pi\sigma})\right]}{(e^{-2i\pi\sigma}-1)^2}$$

$$= \frac{8\cos^2(\frac{\pi\sigma}{2})(x_1^2+x_\infty^2)f(\sigma) - 4(4-\sin^2(\frac{\pi\sigma}{2})) - ((x_1^2+x_\infty^2)^2 - x_0^2x_1^2x_\infty^2)f(\sigma)^2}{-4\sin^2(\pi\sigma)}$$

Using the relations $\cos^2(\frac{\pi\sigma}{2}) = 1 - x_0^2/4$, $\cos(\pi\sigma) = 1 - x_0^2/2$ and $f(\sigma) = \frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}$ we obtain

$$XY = \frac{1}{x_0^2}\left(-2(x_1^2+x_\infty^2)f(\sigma) + 4 + \frac{(x_1^2+x_\infty^2)^2 - (x_0x_1x_\infty)^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}f(\sigma)\right)$$

$$= \frac{1}{x_0^2}\left(4 - (x_1^2+x_\infty^2-x_0x_1x_\infty)f(\sigma)\right) = \frac{1}{x_0^2}\left(4-(4-x_0^2)\right) = 1$$

It follows from this construction that for any σ solution of the first equation of (4), there always exists a unique s which solves the last two equations.

Remark 2: The first equation of system (4) determines σ up to $\sigma \mapsto \pm\sigma + 2n$, $n \in \mathbf{Z}$. As remarked above, substituting in the last two equations one of the values of $\sigma \in \Omega$ obtained from the first equation, we determine uniquely s. In the different domains $D(\epsilon, \pm\sigma + 2n)$ the spiral-shaped paths along which x tends to zero are different. Thus the same branch $y(x; x_0, x_1, x_\infty)$ has anlytic continuations on different domains with different asymptotic behaviours prescribed by theorem 1. In particular, note that it is always possible to choose $0 \le \Re\sigma < 1$, by the transformation $\sigma \mapsto \pm\sigma + 2n$. In other words, we can always consider radial paths of convergence for the branch $y(x; x_0, x_1, x_\infty)$.

Observe however that for $0 \le \sigma < 1$ the operation $\sigma \mapsto \pm\sigma + 2n$ is not allowed, because σ leaves Ω.

Remark 3: A triple of monodromy data (x_0, x_1, x_∞) determines σ and $-\sigma$ through the equation $2(1 - cos(\pi\sigma)) = x_0^2$. The parameter s corresponding to σ is determined by

$$\frac{1}{f(\sigma)}\left(2 + F(\sigma)\ s + \frac{1}{F(\sigma)\ s}\right) = x_1^2$$

$$\frac{1}{f(\sigma)}\left(2 - F(\sigma)e^{-i\pi\sigma}\ s - \frac{1}{F(\sigma)e^{-i\pi\sigma}\ s}\right) = x_\infty^2$$

We observe that $f(\sigma) = f(-\sigma)$ and that the properties of the Gamma function

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad \Gamma(z+1) = z\Gamma(z)$$

imply

$$F(-\sigma) = \frac{1}{F(\sigma)}$$

Then the value of s corresponding to $-\sigma$ is (uniquely) determined by

$$\frac{1}{f(\sigma)}\left(2 + \frac{s}{F(\sigma)} + \frac{F(\sigma)}{s}\right) = x_1^2$$

$$\frac{1}{f(\sigma)}\left(2 - \frac{s}{F(\sigma)e^{-i\pi\sigma}} - \frac{F(\sigma)e^{-i\pi\sigma}}{s}\right) = x_\infty^2$$

We conclude that $s(-\sigma) = -\frac{1}{s(\sigma)}$.

Theorem 1 describes the asymptotic behaviour of the solution $y(x; x_0, x_1, x_\infty)$ in terms of the parameters $(\sigma, a = -\frac{1}{4s(\sigma)})$ and $(-\sigma, a = -\frac{1}{4s(-\sigma)})$ on the domains (ϵ may not be the same, but we take the smallest)

$$D(\epsilon;\sigma): \qquad \Re\sigma\log|x| + \theta_2\Im\sigma \le \Im\sigma\arg(x) \le (\Re\sigma - \tilde{\sigma})\log|x| + \theta_1\Im\sigma$$

$$D(\epsilon;-\sigma): \qquad (\Re\sigma + \tilde{\sigma})\log|x| + \theta_1\Im\sigma \le \Im\sigma\arg(x) \le \Re\sigma\log|x| + \theta_2\Im\sigma$$

The two domains intersect along the line $\Im\sigma\arg(x) = \Re\sigma\log|x| + \theta_2\Im\sigma$ (see figure 1). This implies that the two different asymptotic representations of theorem 1 on $D(\sigma)$ and $D(-\sigma)$ must become equal along

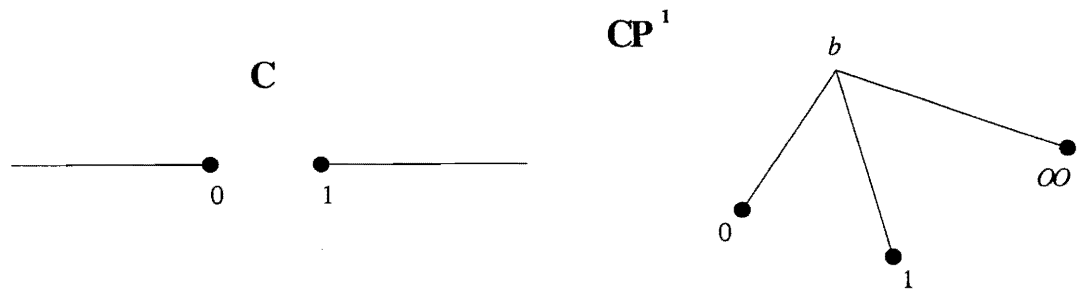

Figure 2: Branch cuts

the line $\Im\sigma \arg(x) = \Re\sigma \log|x| + \theta_2 \Im\sigma$. Actually, after the proof of theorem 1 it will be clear that along the lines $\Im\sigma \arg(x) = \Re\sigma \log|x| + b\Im\sigma$ (b a constant), contained in $D(\sigma)$, the behaviour of $y(x)$ is

$$y(x) = A(x;\sigma, s(\sigma))\ |x|\ \left(1 + O(|x|^{\delta})\right)$$

where δ is a small number between 0 and 1, and

$$A(x;\sigma, s(\sigma)) = \left[-\frac{1}{4}\left(\frac{1}{s\ C}e^{-i\alpha(x)} + s\ C\ e^{i\alpha(x)} - 2\right)\ e^{i\arg(x)}\right]$$

$$\alpha(x) = \frac{1}{\Re\sigma}\left[[(\Re\sigma)^2 + (\Im\sigma)^2]\arg(x) - (\Im\sigma)^2 b\right], \qquad C = e^{-\Im\sigma b}$$

The property $s(-\sigma) = \frac{1}{s(-\sigma)}$ immediately implies that

$$A(x;-\sigma, s(-\sigma)) = A(x;\sigma, s(\sigma))$$

In this way we have proved that the asymptotic behaviour, as prescribed by theorem 1 in $D(\sigma)$ and $D(-\sigma)$, is the same along the common boundary of the two domains.

2.3 Analytic Continuation of a Branch

We describe the analytic continuation of the branch $y(x;\sigma,a) = y(x;x_0,x_1,x_\infty)$, $a < arg(x) < b$, $|a-b| < 2\pi$.

We fix a basis in the fundamental group $\pi(\mathbf{P}^1\backslash\{0,1,\infty\}, b)$, where b is the basepoint (figure 2). The analytic continuation of the branch along paths encircling $x = 0$ and $x = 1$ (a loop around $x = \infty$ is homotopic to the product of two loops around 0 and 1) is given by the action of the group of the pure braids on the monodromy data (see [4]). Namely, for a counter-clockwise loop around 0 we have to transform (x_0, x_1, x_∞) by the action of the braid β_1^2, where

$$\beta_1 : \quad (x_0, x_1, x_\infty) \mapsto (-x_0, x_\infty - x_0x_1, x_1)$$

$$\beta_1^2 : \quad (x_0, x_1, x_\infty) \mapsto (x_0,\ x_1 + x_0x_\infty - x_1x_0^2,\ x_\infty - x_0x_1)$$

For a counter-clockwise loop around 1 we need the braid β_2^2, where

$$\beta_2 : \quad (x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1x_\infty)$$

$$\beta_2^2 : \quad (x_0, x_1, x_\infty) \mapsto (x_0 - x_1x_\infty,\ x_1,\ x_\infty + x_0x_1 - x_\infty x_1^2)$$

Let β be a braid corresponding to a loop in $\mathbf{P}^1\backslash\{0,1,\infty\}$. Solving the system (4) or using the formulae of II), III) in theorem 2, we find the new parameters σ^β, a^β of the new branch $\beta\left(y(x;\sigma,a)\right) = y(x;\sigma^\beta, a^\beta)$.

As an example, consider the loop around $x = 0$. From the formulae in I), II), III) of theorem 2 we derive the transformation

$$\beta_1^2 : \quad (\sigma, a) \mapsto (\sigma, ae^{-2\pi i\sigma})$$

In fact, $x_0 \mapsto x_0$ implies that σ does not change. The transformation of a is proved as follows: For $\sigma = 0$ we have $x_0 = 0$ and $\beta_1^2 : (0, x_1, x_\infty) \mapsto (0, x_1, x_\infty)$. Thus

$$a = \frac{x_\infty}{x_1 + x_\infty} \mapsto \frac{x_\infty}{x_1 + x_\infty} \equiv a$$

For $\sigma = \pm 2\mu + 2m$, we consider the example $\sigma = 2\mu + 2m$, $m = 0, 1, 2, \ldots$. The other cases are analogous. We have $s = x_1^2\ H(\sigma) = -x_\infty^2 H(\sigma) e^{2\pi i\mu}$, where the function $H(\sigma)$ is explicitely given in theorem 2, III). Then

$$\beta_1 : \quad s = -x_\infty^2 H(\sigma) e^{2\pi i\mu} \mapsto -x_1^2 H(\sigma) e^{2\pi i\mu} = -s e^{2\pi i\mu}$$

Then

$$\beta_1^2 : \quad s \mapsto s e^{4\pi i\mu} \quad \Longrightarrow \quad a \mapsto a e^{-4\pi i\mu} \equiv a e^{-2\pi i\sigma}$$

For the generic case I) ($\sigma \notin \mathbf{Z}$, $\sigma \neq \pm 2\mu + 2m$) recall that

$$\begin{cases} F(\sigma)\ s + \frac{1}{F(\sigma)\ s} = x_1^2 f(\sigma) - 2 \\ F(\sigma) e^{-i\pi\sigma}\ s + \frac{1}{F(\sigma) e^{-i\pi\sigma}\ s} = 2 - x_\infty^2 f(\sigma) \end{cases}$$

has a unique solution s. Also observe that $\beta_1 : x_\infty \mapsto x_1$. Then the transformed parameter $\beta_1 : \ s \mapsto s^{\beta_1}$ satisfies the equation

$$\begin{aligned} F(\sigma) e^{-i\pi\sigma}\ s^{\beta_1} + \frac{1}{F(\sigma) e^{-i\pi\sigma}\ s^{\beta_1}} &= 2 - x_1^2 f(\sigma) \\ &\equiv -\left(F(\sigma)\ s + \frac{1}{F(\sigma)\ s} \right) \end{aligned}$$

Thus $s^{\beta_1} = -e^{i\pi\sigma}\ s$. This implies

$$\beta_1^2 : \ s \mapsto s e^{2\pi i\sigma} \quad \Longrightarrow \quad a \mapsto a\ e^{-2\pi i\sigma}$$

We know that the branch $y(x; \sigma, a)$, and the branch $y(x; \sigma, a e^{-2\pi i\sigma})$, which correspond to different monodromy data (x_0, x_1, x_∞) and $(x_0^{\beta_1^2}, x_1^{\beta_1^2}, x_\infty^{\beta_1^2})$ respectively, and with the same $a < \arg(x) < b$, $|a - b| < 2\pi$, are local representations for the same solution of PVI_μ. On the other hand theorem 1 tells us that the analytic continuations of the two branches have the behaviour

$$\begin{aligned} y(\tilde{x}; \sigma, a) &= a \tilde{x}^{1-\sigma} (1 + O(|x|^\delta)) \\ y(\tilde{x}; \sigma, a e^{-2\pi i\sigma}) &= a e^{-2\pi i\sigma} \tilde{x}^{1-\sigma} (1 + O(|x|^\delta)) \equiv a (\tilde{x} e^{2\pi i})^{1-\sigma} (1 + O(|x|^\delta)) \end{aligned} \tag{5}$$

as $x \to 0$ in $D(\epsilon; \sigma, \theta_1, \theta_2)$. For suitable θ_1, θ_2 and for sufficiently small $|x|$ both $\tilde{x}$ and $\tilde{x}' := \tilde{x} e^{2\pi i}$ are contained in $D(\epsilon; \sigma, \theta_1, \theta_2)$ and thus $y(\tilde{x}; \sigma, a)$ and $y(\tilde{x}; \sigma, a e^{-2\pi i\sigma}) \equiv y(\tilde{x}'; \sigma, a)$ are actually the analytic continuations of the branch $y(x; \sigma, a)$ to the points $\tilde{x}$ and $\tilde{x}'$. Thus theorem 1 is in accordance with the analytic continuation obtained by the action of the braid group.

For a loop around 1 the action of β_2^2 changes x_0 to $x_0 - x_1 x_\infty$. Thus the new branch has new exponent $\sigma^{\beta_2^2} \neq \sigma$ and new coefficient $a^{\beta_2^2}$ obtained from I) II), III) of theorem 1.

2.4 Singular Points $x = 1$, $x = \infty$ (Connection Problem)

In this subsection we use the notation $\sigma^{(0)}$ and $a^{(0)}$ to denote the parameters of theorem 1 near the critical point $x = 0$. We describe now the analogous of theorem 1 near $x = 1$ and $x = \infty$. The three critical points 0, 1, ∞ are equivalent thanks to the symmetries of the PVI_μ equation.

a) Let

$$x = \frac{1}{t} \qquad y(x) := \frac{1}{t}\ \hat{y}(t) \tag{6}$$

Then $y(x)$ is a solution of PVI_μ (variable x) if and only if $\hat{y}(t)$ is a solution of PVI_μ (variable t). The singularities 0 and ∞ are exchanged. Now, we can prove theorem 1 near $t = 0$. We go back to $y(x)$ and find a branch $y(x; \sigma^{(\infty)}, a^{(\infty)})$ whose analytic continuation to

$$D(M; \sigma^{(\infty)}) := \{\tilde{x} \in \widetilde{\mathbf{C} \backslash \{\infty\}} \text{ s.t. } |x| > M,\ e^{\theta_2 \Im \sigma^{(\infty)}} |x|^{-\tilde{\sigma}} \leq |\tilde{x}^{-\sigma^{(\infty)}}| \leq e^{\theta_1 \Im \sigma^{(\infty)}}$$

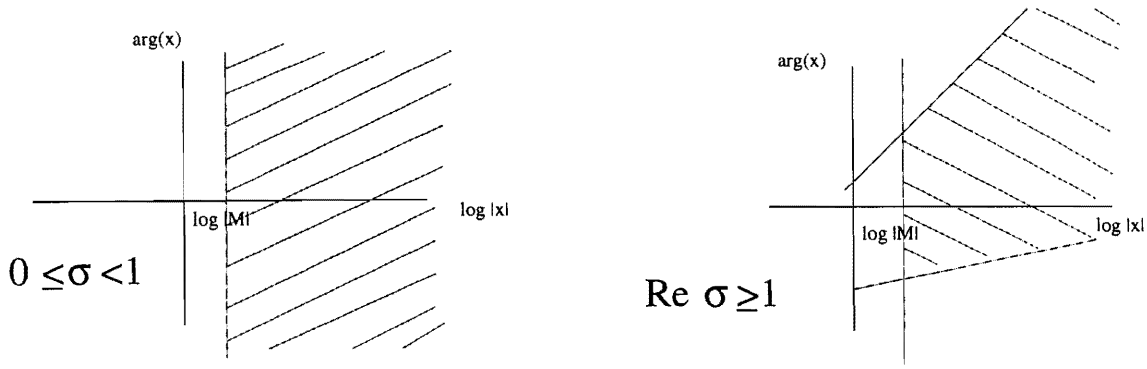

Figure 3: Some examples of the domain D(M, σ^{∞})

$$0 \leq \tilde{\sigma} < 1\}$$

has the asymptotic behaviour

$$y(\tilde{x}; \sigma^{(\infty)}, a^{(\infty)}) = a^{(\infty)} \tilde{x}^{\sigma^{(\infty)}} \left(1 + O(\frac{1}{|x|^{\delta}})\right) \qquad \tilde{x} \to \infty \tag{7}$$

where $M > 0$ is sufficiently big and $0 < \delta < 1$ is small (figure 3).

b) Let

$$x = 1 - t, \qquad y(x) = 1 - \hat{y}(t) \tag{8}$$

$y(x)$ satisfies PVI_{μ} if and only if $\hat{y}(t)$ satisfies PVI_{μ}. Theorem 1 holds for $\hat{y}(t)$ near $t = 0$ with coefficients $\sigma^{(1)}$ and $a^{(1)}$. Going back to $y(x)$ we obtain a branch $y(x; \sigma^{(1)}, a^{(1)})$ whose analytic continuation to

$$D(\epsilon; \sigma^{(1)}) := \{\tilde{x} \in \widetilde{\mathbf{C}\backslash\{1\}} \text{ s.t. } 0 < |1 - x| < \epsilon, \quad e^{-\theta_1 \Im\sigma} |1 - x|^{\tilde{\sigma}} \leq |(1 - \tilde{x})^{\sigma^{(1)}}| \leq e^{-\theta_2 \Im\sigma},$$

$$0 \leq \tilde{\sigma} < 1\}$$

has behaviour

$$y(\tilde{x}, \sigma^{(1)}, a^{(1)}) = 1 - a^{(1)} (1 - \tilde{x})^{1 - \sigma^{(1)}} (1 + O(|1 - x|^{\delta})) \qquad \tilde{x} \to 1 \tag{9}$$

Let us fix branch cuts in $\mathbf{C}\backslash\{0, 1\}$ (for example $|\arg(x)| < \pi$, $|\arg(1 - x)| < \pi$), or equivalently in $\mathbf{P}^1\backslash\{0, 1, \infty\}$ (figure 2).

Then we choose a (class of equivalence of) triple of monodromy data (x_0, x_1, x_{∞}) satisfying (2). They correspond to a unique branch $y(x; x_0, x_1, x_{\infty})$ whose asymptotic behaviour at $x = 0$ is given by theorems 1 and 2.

After the transformation (6) the solution $\hat{y}(t)$ corresponds to a solution of Schlesinger equations whose class of equivalence of monodromy data is represented by $(x_{\infty}, -x_1, x_0 - x_1 x_{\infty})$ [4]. In the same way, for (8) the monodromy data are $(x_1, x_0, x_0 x_1 - x_{\infty})$. It follows that the analogous of theorem 2 holds at $x = \infty$ and $x = 1$. It determines $\sigma^{(\infty)}$, $a^{(\infty)}$ and $\sigma^{(1)}$, $a^{(1)}$ in terms of monodromy data, provided we make the substitutions

$$(x_0, x_1, x_{\infty}) \mapsto (x_{\infty}, -x_1, x_0 - x_1 x_{\infty}) \qquad \text{at } x = \infty$$

$$(x_0, x_1, x_{\infty}) \mapsto (x_1, x_0, x_0 x_1 - x_{\infty}) \qquad \text{at } x = 1$$

We have thus solved the connection problem: given (x_0, x_1, x_{∞}) we can always find $(\sigma^{(0)}, a^{(0)})$, $(\sigma^{(1)}, a^{(1)})$, $(\sigma^{(\infty)}, a^{(\infty)})$ from theorem 2 with the above substitutions. There is a unique branch $y(x; x_0, x_1, x_{\infty})$ whose analytic continuations on the domains $D(\epsilon^{(0)}; \sigma^{(0)})$, $D(\epsilon^{(1)}; \sigma^{(1)}, a^{(1)})$ and $D(M; \sigma^{(\infty)}, a^{(\infty)})$ have asymptotic behaviours at $x = 0$, $x = 1$, $x = \infty$ given by (1), (9), (7) respectively.

As a last remark, note that, for $2\mu \notin \mathbf{Z}$, the case $\sigma^{(0)} = \sigma^{(1)} = \sigma^{(\infty)} = 0$ is forbidden because otherwise $x_0 = x_1 = x_{\infty} = 0$. Also, the cases $\sigma^{(0)} = \pm 2\mu + 2m^{(0)}$, $\sigma^{(1)} = \pm 2\mu + 2m^{(1)}$, $\sigma^{(\infty)} = \pm 2\mu + 2m^{(\infty)}$ are forbidden, otherwise $x_0^2 = x_1^2 = x_{\infty}^2 = 4\sin^2(\pi\mu)$ and (2) is violated.

3 Derivation of the Results

3.1 Proof of Theorem 1

In order to prove theorem 1 we have to recall the connection between PVI_μ and Schlesinger equations for 2×2 matrices $A_0(x)$, $A_x(x)$, $A_1(x)$ (x in A_x is a label, not a variable !!)

$$\begin{aligned} \frac{dA_0}{dx} &= \frac{[A_x, A_0]}{x} \\ \frac{dA_1}{dx} &= \frac{[A_1, A_x]}{1-x} \\ \frac{dA_x}{x} &= \frac{[A_x, A_0]}{x} + \frac{[A_1, A_x]}{1-x} \end{aligned} \tag{10}$$

We look for solutions satisfying

$$A_0(x) + A_x(x) + A_1(x) = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix} := -A_\infty \quad \mu \in \mathbf{C}, \quad 2\mu \notin \mathbf{Z}$$

$$\mathrm{tr}(A_i) = \det(A_i) = 0$$

Let

$$A(z, x) := \frac{A_0}{z} + \frac{A_x}{z - x} + \frac{A_1}{z - 1}$$

Then $y(x)$ is a solution of PVI_μ if and only if $A(y(x), x)_{12} = 0$.

The system (10) is a particular case of the system

$$\begin{aligned} \frac{dA_\mu}{dx} &= \sum_{\nu=1}^{n_2} [A_\mu, B_\nu]\ f_{\mu\nu}(x) \\ \frac{dB_\nu}{dx} &= -\frac{1}{x} \sum_{\nu'=1}^{n_2} [B_\nu, B_{\nu'}] + \sum_{\mu=1}^{n_1} [B_\nu, A_\mu]\ g_{\mu\nu}(x) + \sum_{\nu'=1}^{n_2} [B_\nu, B_{\nu'}]\ h_{\nu\nu'}(x) \end{aligned} \tag{11}$$

where the funcions $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are meromorphic with poles at $x = 0, 1, \infty$ and $\sum_\nu B_\nu + \sum_\mu A_\mu = -A_\infty$ (here the subscript μ is a label, not the eigenvalue of A_∞ !). System (10) is obtained for $f_{\mu\nu} = g_{\mu\nu} = b_\nu/(a_\mu - xb_\nu)$, $h_{\mu\nu} = 0$, $n_1 = 1$, $n_2 = 2$, $a_1 = b_2 = 1$, $b_1 = 0$ and $B_1 = A_0$, $B_2 = A_x$, $A_1 = A_1$.

We prove the analogous result of [16], page 262, for the domain $D(\epsilon; \sigma)$:

Lemma 1: *Consider matrices B^0_ν ($\nu = 1, .., n_2$), A^0_μ ($\mu = 1, .., n_1$) and Λ, independent of x and such that*

$$\sum_\nu B^0_\nu + \sum_\mu A^0_\mu = -A_\infty$$

$$\sum_\nu B^0_\nu = \Lambda, \qquad \textit{eigenvalues}(\Lambda) = \frac{\sigma}{2}, \ -\frac{\sigma}{2}, \quad \sigma \in \Omega$$

Consider the domain $D(\epsilon; \sigma, \theta_1, \theta_2)$ (or $D(\epsilon; \sigma = 0)$) introduced in section 2, depending also on $0 < \tilde{\sigma} < 1$. Also, suppose that $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are holomorphic for $|x| < \epsilon$.

For a fixed $\tilde{\sigma}$ and a sufficiently small $\epsilon < 1$, the system (11) has holomorphic solutions $A_\mu(\tilde{x})$, $B_\nu(\tilde{x})$ in $D(\epsilon; \sigma, \theta_1, \theta_2)$, satisfying:

$$||A_\mu(\tilde{x}) - A^0_\mu|| \leq C\ |x|^{1-\sigma_1}$$

$$||\tilde{x}^{-\Lambda} B_\nu(\tilde{x})\ \tilde{x}^\Lambda - B^0_\nu|| \leq C\ |x|^{1-\sigma_1}$$

Here C is a positive constant and $\tilde{\sigma} < \sigma_1 < 1$

Important remark: There is no need to assume here that $2\mu \notin \mathbf{Z}$. The theorem holds true for any value of μ. If in the system (11) the functions $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are chosen in such a way to yield Schlesinger equations for the fuchsian system of PVI_μ, the assumption $2\mu \notin \mathbf{Z}$ is still not necessary, provided the matrix R (see the Appendix) is considered as a monodromy datum independent of the deformation parameter x.

Proof: To simplify notations, we write x instead of $\tilde{x}$. Let $A(x)$ and $B(x)$ be 2×2 matrices holomorphic on $D(\epsilon; \sigma)$ and such that

$$||A(x)|| \leq C_1, \qquad ||B(x)|| \leq C_2 \quad \text{on } D(\epsilon; \sigma)$$

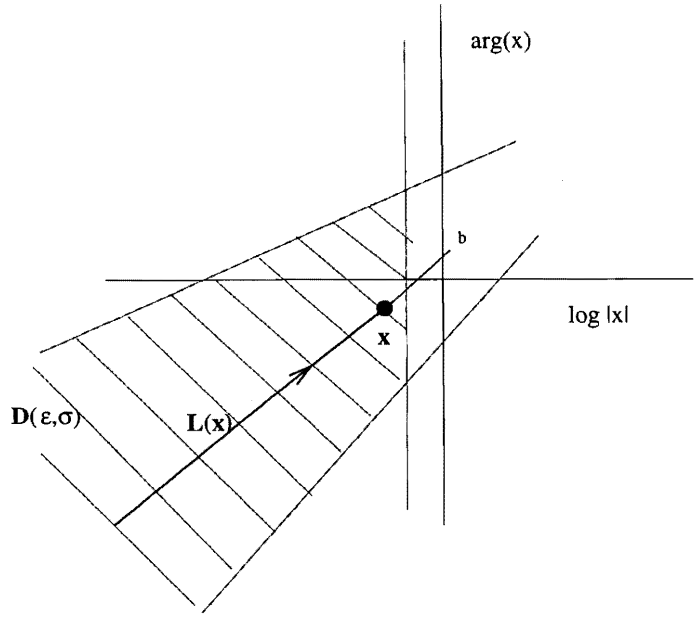

Figure 4: Path of integration.

Let $f(x)$ be a holomorphic function for $|x| < \epsilon$. Let σ_2 be a real number such that $\tilde{\sigma} < \sigma_2 < 1$. Then, for sufficently small ϵ the following estimates are valid for $x \in D(\epsilon;\sigma)$

$$||x^{\pm\Lambda}\ A(x)\ x^{\mp\Lambda}|| \leq C_1|x|^{-\sigma_2}$$

$$||x^{\pm\Lambda}\ B(x)\ x^{\mp\Lambda}|| \leq C_2|x|^{-\sigma_2}$$

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}\right|\right| \leq C_1C_2\ |x|^{1-\sigma_2}$$

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ A(s)\ f(s)\ x^{\Lambda}\right|\right| \leq C_1C_2\ |x|^{1-\sigma_2}$$

where $L(x)$ is a path in $D(\epsilon;\sigma)$ joining 0 to x. To prove the estimates, we observe that

$$||x^{\Lambda}|| = ||x^{\mathrm{diag}(\frac{\sigma}{2},-\frac{\sigma}{2})}|| = \max\{|x|^{\frac{\sigma'(x)}{2}}, |x|^{-\frac{\sigma'(x)}{2}}\} \leq e^{\theta_1\Im\sigma}\ |x|^{\frac{-\tilde{\sigma}}{2}}, \qquad \text{in } D(\epsilon;\sigma)$$

Then

$$\begin{aligned}||x^{\Lambda}\ A(x)\ x^{-\Lambda}|| &\leq ||x^{\Lambda}||\ ||A(x)||\ ||\ x^{-\Lambda}|| \leq e^{2\theta_1\Im\sigma}\ C_1\ |x|^{-\tilde{\sigma}}\\ &= \left(e^{2\theta_1\Im\sigma}\ |x|^{\sigma_2-\tilde{\sigma}}\right)\ C_1\ |x|^{-\sigma_2}\end{aligned}$$

Thus, if ϵ is small enough (we require $\epsilon^{\sigma_2-\tilde{\sigma}} \leq e^{-2\theta_1\Im\sigma}$) we obtain $||x^{\Lambda}\ A(x)\ x^{-\Lambda}|| \leq C_1|x|^{-\sigma_2}$.

We turn to the integrals. The integrands are holomorphic on $D(\epsilon;\sigma)$, then they do not depend on the choice of the path, but only on the point x. Thus we choose the path

$$\arg(x) = a\log|x| + b, \quad a = \frac{\Re\sigma - \tilde{\sigma}}{\Im\sigma}, \quad 0 \leq \bar{\sigma} \leq \tilde{\sigma}, \quad \text{or } \arg(x) = b \text{ if } \Im\sigma = 0$$

where b is chosen appropriately such that $L(x)$ stays in $D(\epsilon;\sigma)$. See figure 4. Then we compute

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}\right|\right| = \left|\left|\int_{L(x)} ds\ x^{-\Lambda}\ A(s)\ x^{\Lambda}\ \left(\frac{s}{x}\right)^{\Lambda}\ B(s)\ \left(\frac{s}{x}\right)^{-\Lambda}\ f(s)\right|\right|$$

$$\leq e^{2\theta_1\Im\sigma}x^{-\tilde{\sigma}}\ C_1C_2\ \max_{|x|<\epsilon}|f(x)|\ \int_{L(x)}|ds|\ \frac{|s|^{-\bar{\sigma}}}{|x|^{-\bar{\sigma}}}$$

where by $|ds|$ I mean $d\vartheta\ |ds(\vartheta)/d\vartheta|$, and ϑ is a parameter on the curve $L(x)$. The last step in the inequality follows from

$$\left|\left|\left(\frac{s}{x}\right)^{\Lambda}\right|\right| = \left|\left|\mathrm{diag}(\frac{s^{\frac{\sigma}{2}}}{x^{\frac{\sigma}{2}}}, \frac{s^{-\frac{\sigma}{2}}}{x^{-\frac{\sigma}{2}}})\right|\right| = \max_L \left\{\frac{|s^{\frac{\sigma}{2}}|}{|x^{\frac{\sigma}{2}}|}, \frac{|s^{-\frac{\sigma}{2}}|}{|x^{-\frac{\sigma}{2}}|}\right\}$$

and the observation that, on L, $|s^{\sigma}| = |s|^{\bar{\sigma}} e^{-b\Im\sigma}$. Thus

$$\max\left\{\frac{|s^{\frac{\sigma}{2}}|}{|x^{\frac{\sigma}{2}}|}, \frac{|s^{-\frac{\sigma}{2}}|}{|x^{-\frac{\sigma}{2}}|}\right\} = \max\left\{\frac{|s|^{\frac{\bar{\sigma}}{2}}}{|x|^{\frac{\bar{\sigma}}{2}}}, \frac{|s|^{-\frac{\bar{\sigma}}{2}}}{|x|^{-\frac{\bar{\sigma}}{2}}}\right\} = \frac{|s|^{-\frac{\bar{\sigma}}{2}}}{|x|^{-\frac{\bar{\sigma}}{2}}}, \qquad |s| \le |x|$$

The parameter s on $L(x)$ is

$$s(\vartheta) := e^{\frac{\vartheta - b}{a}}\ e^{i\vartheta}$$

where $\vartheta \in (-\infty, \arg(x)]$ or $\in [\arg(x), +\infty)$. Then

$$\int_{L(x)} |ds|\ |s|^{-\bar{\sigma}} = \int_{L(x)} d\vartheta\ \left|\frac{ds}{d\vartheta}\right|\ |s(\vartheta)|^{-\bar{\sigma}} = \frac{|a|}{1-\bar{\sigma}}\left|i + \frac{1}{a}\right| e^{\frac{\arg(x)-b}{a}(1-\bar{\sigma})} = \frac{|a|}{1-\bar{\sigma}}\left|i + \frac{1}{a}\right|\ |x|^{1-\bar{\sigma}}$$

Then, the initial integral is less or equal to

$$e^{2\theta_1 \Im\sigma} \max_{|x|<\epsilon} |f(x)|\ C_1 C_2 \text{ constant } |x|^{1-\bar{\sigma}}$$

Now, we write $|x|^{1-\bar{\sigma}} = |x|^{\sigma_2 - \bar{\sigma}}\ |x|^{1-\sigma_2}$ and we obtain, for sufficiently small ϵ:

$$e^{2\theta_1 \Im\sigma} \max_{|x|<\epsilon} |f(x)|\ C_1 C_2 \text{ constant } |x|^{1-\bar{\sigma}} \le C_1 C_2\ |x|^{1-\sigma_2}$$

We remark that for $\sigma = 0$ the above extimates are still valid. Actually $||x^{\Lambda}|| \equiv ||x^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}||$ diverges like $|\log x|$, $||x^{\Lambda} A(x) x^{-\Lambda}||$ are less or equal to $C_1\ |\log(x)|^2$, and finally $||x^{-\Lambda} \int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}||$ is less or equal to $C_1 C_2 \max|f|\ |\log(x)|^2 \int_{L(x)} |ds|\ |\log s|^2$. We can choose $L(x)$ to be a radial path $s = \rho\exp(i\alpha)$, $0 < \rho < |x|$, α fixed. Then the integral is $|x|(\log|x|^2 - 2\log|x| + 2 + \alpha^2)$. The factor $|x|$ does the job, because we rewrite it as $|x|^{\sigma_2}\ |x|^{1-\sigma_2}$ (here σ_2 is any number between 0 and 1) and we proceed as above to choose ϵ small enough in such a way that $\left(\max|f|\ |x|^{\sigma_2} \times \text{function diverging like } \log^2|x|\right) \le 1$.

The estimates above are in a sense enough to prove the lemma.

We solve the Schlesinger equations by successive approximations, as in [16]: let $\tilde{B}_\nu(x) := x^{-\Lambda} B_\nu(x) x^{\Lambda}$. The Schlesinger equations are re-written as

$$\frac{dA_\mu}{dx} = \sum_{\nu=1}^{n_2} [A_\mu, x^{\Lambda}\tilde{B}_\nu x^{-\Lambda}]\ f_{\mu\nu}(x)$$

$$\frac{d\tilde{B}_\nu}{dx} = \frac{1}{x}[\tilde{B}_\nu, \sum_\mu x^{-\Lambda}(A_\mu(x) - A_\mu^0)x^{\Lambda}] + \sum_{\mu=1}^{n_1}[\tilde{B}_\nu, x^{-\Lambda}A_\mu x^{\Lambda}]\ g_{\mu\nu}(x) + \sum_{\nu'=1}^{n_2}[\tilde{B}_\nu, \tilde{B}_{\nu'}]\ h_{\nu\nu'}(x)$$

Then, by successive approximations:

$$A_\mu^{(k)}(x) = A_\mu^0 + \int_{L(x)} ds\ \sum_\nu [A_\mu^{(k-1)}(s), s^{\Lambda}\tilde{B}_\nu^{(k-1)}(s)s^{-\Lambda}]\ f_{\mu\nu}(s)$$

$$\tilde{B}_\nu^{(k)}(x) = B_\nu^0 + \int_{L(x)} ds\ \left\{\frac{1}{s}[\tilde{B}_\nu^{(k-1)}(s), \sum_\mu s^{-\Lambda}(A_\mu^{(k-1)}(s) - A_\mu^0)s^{\Lambda}] + \right.$$

$$\left. + \sum_\mu [\tilde{B}_\nu^{(k-1)}(s), s^{-\Lambda}A_\mu^{(k-1)}(s)x^{\Lambda}]\ g_{\mu\nu}(s) + \sum_{\nu'}[\tilde{B}_\nu^{(k-1)}(s), \tilde{B}_{\nu'}^{(k-1)}(s)]\ h_{\nu\nu'}\right\}$$

The functions $A_\mu^{(k)}(x)$, $B_\nu^{(k)}(x)$ are holomorphic in $D(\epsilon;\sigma)$, by construction. Observe that $||A_\mu^0|| \leq C$, $||B_\nu^0|| \leq C$ for some constant C. We claim that for $|x|$ sufficiently small

$$\begin{aligned} ||A_\mu^{(k)}(x) - A_\mu^0|| &\leq C|x|^{1-\sigma_1} \\ \left|\left|x^{-\Lambda}\left(A_\mu^{(k)}(x) - A_\mu^0\right)x^\Lambda\right|\right| &\leq C^2|x|^{1-\sigma_2} \\ ||\tilde{B}_\nu^{(k)}(x) - B_\nu^0|| &\leq C|x|^{1-\sigma_1} \end{aligned} \tag{12}$$

where $\tilde{\sigma} < \sigma_2 < \sigma_1 < 1$. Note that the above inequalities imply $||A_\mu^{(k)}|| \leq 2C$, $||\tilde{B}_\nu^{(k)}|| \leq 2C$. Moreover we claim that

$$\begin{aligned} ||A_\mu^{(k)}(x) - A_\mu^{(k-1)}(x)|| &\leq C\ \delta^{k-1}\ |x|^{1-\sigma_1} \\ \left|\left|x^{-\Lambda}\left(A_\mu^{(k)}(x) - A_\mu^{(k-1)}(x)\right)x^\Lambda\right|\right| &\leq C^2\ \delta^{k-1}\ |x|^{1-\sigma_2} \\ ||\tilde{B}_\nu^{(k)}(x) - B_\nu^{(k-1)}(x)|| &\leq C\ \delta^{k-1}|x|^{1-\sigma_1} \end{aligned} \tag{13}$$

where $0 < \delta < 1$.

For $k = 1$ the above inequalities are proved using the simple methods used in the estimates at the beginning of the proof. Then we proceed by induction, still using the same estimates. We leave this technical point to the reader, but we give at least one example of how to proceed. As an example, we prove the $(k+1)^{th}$ step of the first of (13):

$$||A_\mu^{(k+1)}(x) - A_\mu^{(k)}(x)|| \leq C\ \delta^k\ |t|^{1-\sigma_1}$$

supposing the kth step of (13) is true. For simplicity, we take $\theta_1 = \theta_2 = 0$. Let us proceed using the integral equations:

$$\begin{aligned} ||A_\mu^{(k+1)}(x) - A_\mu^{(k)}(x)|| = \Bigg|\Bigg|\int_{L(x)} ds \sum_{\nu=1}^{n_2}\Big(&A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda} + \\ &+ s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}A_\mu^{(k-1)} - s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}A_\mu^{(k)}\Big)\ f_{\mu\nu}(s)\Bigg|\Bigg| \leq \\ \leq \int_{L(x)} d|s| \sum_{\nu=1}^{n_2}&\left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right|\ |f_{\mu\nu}(s)| + \\ + \int_{L(x)} d|s| \sum_{\nu=1}^{n_2}&\left|\left|s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}A_\mu^{(k-1)} - s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}A_\mu^{(k)}\right|\right|\ |f_{\mu\nu}(s)| \end{aligned}$$

Now we estimate

$$\begin{aligned} &\left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right| \leq \\ &\leq \left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}\right|\right| + \\ &+ \left|\left|A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right| \\ &\leq ||A_\mu^{(k)} - A_\mu^{(k-1)}||\ ||s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}|| + ||A_\mu^{(k-1)}||\ ||s^\Lambda||\ ||\tilde{B}_\nu^{(k)} - \tilde{B}_\nu^{(k-1)}||\ ||s^{-\Lambda}|| \end{aligned}$$

By induction then:

$$\leq \left(C\ \delta^{k-1}\ |s|^{1-\sigma_1}\right)\ 2C\ |s|^{-\tilde{\sigma}} + 2C\ \left(C\ \delta^{k-1}\ |s|^{1-\sigma_1}\right)\ |s|^{-\tilde{\sigma}}$$

The other term is estimated in an analogous way. Then

$$||A_\mu^{(k+1)} - A_\mu^{(k)}|| \leq \text{constant } 8n_2C^2 \max|f_{\mu\nu}|\ \delta^{k-1}\ |x|^{1-\tilde{\sigma}}\ |x|^{1-\sigma_1}$$

We choose ϵ small enough to have const $8n_2C^2\max|f|\ |x|^{1-\tilde{\sigma}} \leq \delta$. Note that the choice of ϵ is independent of k. In the case $\sigma = 0$, $|x|^{1-\tilde{\sigma}}$ is subsituted by $|x|(\log^2|x| + O(\log|x|))$.

The inequalities (12) (13) imply the convergence of the successive approximations to a solution of the Schlesinger equations, satisfying the assertion of the lemma, plus the additional inequality

$$||x^{-\Lambda}(A_\mu(x) - A_\mu^0)x^\Lambda|| \leq C^2|x|^{1-\sigma_2}$$

□

We turn to the case in which we are concerned: we consider three matrice A_0^0, A_x^0, A_1^0 such that

$$A_0^0 + A_x^0 = \Lambda, \quad A_0^0 + A_x^0 + A_1^0 = \mathrm{diag}(-\mu, \mu)$$

$$\mathrm{tr}(A_i^0) = \det(A_i^0) = 0, \quad i = 0, x, 1$$

Lemma 2: *Let r and s be two complex numbers not equal to 0 and ∞. Let T be the matrix which brings Λ to the Jordan form:*

$$T^{-1}\Lambda T = \begin{cases} diag(\frac{\sigma}{2}, -\frac{\sigma}{2}), & \sigma \neq 0 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma = 0 \end{cases}$$

The general solution of

$$A_0^0 + A_x^1 + A_1^0 = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}, \qquad \mathrm{tr}(A_i) = \det(A_i) = 0, \qquad A_0^0 + A_x^0 = \Lambda$$

is the following:

For $\sigma \neq 0, \pm 2\mu$:

$$\Lambda = \frac{1}{8\mu}\begin{pmatrix} -\sigma^2 - (2\mu)^2 & (\sigma^2 - (2\mu)^2)r \\ \frac{(2\mu)^2 - \sigma^2}{r} & \sigma^2 + (2\mu)^2 \end{pmatrix} \qquad A_1^0 = \frac{\sigma^2 - (2\mu)^2}{8\mu}\begin{pmatrix} 1 & -r \\ \frac{1}{r} & -1 \end{pmatrix}$$

$$A_0^0 = T\begin{pmatrix} \frac{\sigma}{4} & \frac{\sigma}{4}s \\ -\frac{\sigma}{4}\frac{1}{s} & -\frac{\sigma}{4} \end{pmatrix} T^{-1}, \qquad A_x^0 = T\begin{pmatrix} \frac{\sigma}{4} & -\frac{\sigma}{4}s \\ \frac{\sigma}{4}\frac{1}{s} & -\frac{\sigma}{4} \end{pmatrix} T^{-1}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ \frac{(\sigma+2\mu)^2}{\sigma^2-(2\mu)^2}\frac{1}{r} & \frac{(\sigma-2\mu)^2}{\sigma^2-(2\mu)^2}\frac{1}{r} \end{pmatrix}$$

For $\sigma = -2\mu$: A_0^0 and A_x^0 as above, but

$$\Lambda = \begin{pmatrix} -\mu & r \\ 0 & \mu \end{pmatrix} \qquad A_1^0 = \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & \frac{2\mu}{r} \end{pmatrix} \tag{14}$$

or

$$\Lambda = \begin{pmatrix} -\mu & 0 \\ r & \mu \end{pmatrix} \qquad A_1^0 = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ -\frac{r}{2\mu} & 1 \end{pmatrix} \tag{15}$$

For $\sigma = 2\mu$: A_0^0 and A_x^0 as above, but

$$\Lambda = \begin{pmatrix} -\mu & r \\ 0 & \mu \end{pmatrix} \qquad A_1^0 = \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ \frac{2\mu}{r} & 0 \end{pmatrix} \tag{16}$$

or

$$\Lambda = \begin{pmatrix} -\mu & 0 \\ r & \mu \end{pmatrix} \qquad A_1^0 = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{r}{2\mu} \end{pmatrix} \tag{17}$$

For $\sigma = 0$:

$$A_0^0 = T\begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} T^{-1} \qquad A_x^0 = T\begin{pmatrix} 0 & 1-s \\ 0 & 0 \end{pmatrix} T^{-1}$$

$$\Lambda = \begin{pmatrix} -\frac{\mu}{2} & -\frac{\mu^2}{4}r \\ \frac{1}{r} & \frac{\mu}{2} \end{pmatrix} \qquad A_1^0 = \begin{pmatrix} -\frac{\mu}{2} & \frac{\mu^2}{4}r \\ -\frac{1}{r} & \frac{\mu}{r} \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ -\frac{2}{\mu r} & -2\frac{\mu+2}{\mu^2}\frac{1}{r} \end{pmatrix}$$

We leave the proof as an exercise for the reader.

We are ready to prove theorem 1:

Theorem 1: *The solutions of PVI_μ, corresponding to the solutions of Schlesinger equations (10) obtained in lemma 1, have the following behaviour for $\tilde{x} \to 0$ along a path $\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma$, $0 \le \Sigma \le \tilde{\sigma}$, contained in $D(\epsilon; \sigma, \theta_1, \theta_2)$:*

$$y(x) = a(x)\ \tilde{x}^{1-\sigma}(1 + O(|x|^\delta)), \qquad \sigma \neq 0$$

$$y(\tilde{x}) = s\tilde{x}\ (1 + O(|x|^\delta)), \qquad \sigma = 0$$

where $0 < \delta < 1$ is a small number, and a can be computed as a function of s. Namely

$$a = -\frac{1}{4s}$$

along any path, except for the paths $\Im\sigma \arg(x) = \Re\sigma \log|x| + b\Im\sigma$, along which $|\tilde{x}^\sigma| = Constant$ as $x \to 0$. In this case

$$a(x) = -\frac{1}{4s}\ \left(1 - 2\ s\ Ce^{i\alpha(x)} + s^2\ C^2 e^{2i\alpha(x)}\right) = O(1),\ for\ x \to 0$$

$$\alpha(x) = \frac{1}{\Re\sigma}\ \left[\ \left[(\Re\sigma)^2 + (\Im\sigma)^2\right]\arg(x) - (\Im\sigma)^2 b\ \right], \qquad C = e^{-\Im\sigma b}$$

Proof: We write x for $\tilde{x}$ to simplify notations. $y(x)$ can be computed in terms of the $A_i(x)$ from $A(y(x), x)_{12} = 0$:

$$y(x) = \frac{x(A_0)_{12}}{(1+x)(A_0)_{12} + (A_x)_{12} + x(A_1)_{12}} \equiv \frac{x(A_0)_{12}}{x(A_0)_{12} - (A_1)_{12} + x(A_1)_{12}}$$

$$= -x\frac{(A_0)_{12}}{(A_1)_{12}}\ \frac{1}{1 - x((A_0)_{12} + (A_1)_{12})}$$

As a consequence of lemmas 1 and 2 it follows that $|x\ (A_1)_{12}| \le c\ |x|\ (1 + O(|x|^{1-\sigma_1}))$ and $|x\ (A_0)_{12}| \le c\ |x|^{1-\tilde{\sigma}}\ (1 + O(|x|^{1-\sigma_1}))$, where c is a constant. Then

$$y(x) = -x\frac{(A_0)_{12}}{(A_1)_{12}}\ (1 + O(|x|^{1-\tilde{\sigma}}))$$

From lemma 2 we find, for $\sigma \neq 0, \pm 2\mu$:

$$(A_0)_{12} = -r\frac{\sigma^2 - 4\mu^2}{32\mu}\left[\frac{x^{-\sigma}}{s}(1 + O(|x|^{1-\sigma_1})) + s\ x^\sigma\ (1 + O(|x|^{1-\sigma_1})) - 2(1 + O(|x|^{1-\sigma_1}))\right]$$

$$(A_1)_{12} = -r\frac{\sigma^2 - 4\mu^2}{8\mu}(1 + O(|x|^{1-\sigma_1}))$$

Then (recall that $\tilde{\sigma} < \sigma_1$)

$$y(x) = -\frac{x}{4}\left[\frac{x^{-\sigma}}{s}(1 + O(|x|^{1-\sigma_1})) + s\ x^\sigma\ (1 + O(|x|^{1-\sigma_1})) - 2(1 + O(|x|^{1-\sigma_1}))\right]\ (1 + O(|x|^{1-\sigma_1}))$$

Now $x \to 0$ along a path

$$\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma$$

for a suitable b and $0 \le \Sigma \le \tilde{\sigma}$. Along this path we rewrite:

$$x^\sigma = C\ |x|^\Sigma\ e^{i\alpha(x)}$$

$$\alpha(x) = \left[\Re\sigma + \frac{(\Im\sigma)^2}{\Re\sigma - \Sigma}\right] \arg(x) - \frac{(\Im\sigma)^2}{\Re\sigma - \Sigma} b, \qquad C = e^{-\Im\sigma\ b}$$

Then

$$y(x) = -\frac{x^{1-\sigma}}{4}\left[\frac{1}{s} - 2Ce^{i\alpha(x)}|x|^{\Sigma}(1 + O(|x|^{1-\sigma_1})) + sC^2 e^{2i\alpha(x)}\ |x|^{2\Sigma}\ (1 + O(|x|^{1-\sigma_1}))\right]\ (1 + O(|x|^{1-\sigma_1}))$$

For $\Sigma \neq 0$ the above expression becomes

$$y(x) = -\frac{1}{4s} x^{1-\sigma}\ \ (1 + O(|x|^{1-\sigma_1}) + O(|x|^{\Sigma}))$$

We collect the two $O(..)$ contribution in $O(|x|^{\delta})$ where $\delta = \min\{1 - \sigma_1, \Sigma\}$ is a small number between 0 and 1. We take the occasion here to remark that in the case of real $0 < \sigma < 1$, if we consider $x \to 0$ along a radial path (i.e. $\arg(x) = b$), then $\Sigma = \bar{\sigma} = \sigma$ and thus:

$$y(x) = \begin{cases} \frac{1}{4s} x^{1-\sigma}(1 + O(|x|^{\sigma})) \text{ for } 0 < \sigma < \frac{1}{2} \\ \frac{1}{4s} x^{1-\sigma}(1 + O(|x|^{1-\sigma_1})) \text{ for } \frac{1}{2} < \sigma < 1 \end{cases}$$

Finally, along the path with $\Sigma = 0$ we have:

$$y(x) = -\frac{x^{1-\sigma}}{4}\ \left(\frac{1}{s} - 2Ce^{i\alpha(x)} + s\ C^2 e^{2i\alpha(x)}\right)\ \ (1 + O(|x|^{1-\sigma}))$$

Namely

$$y(x) = a(x)x^{1-\sigma}\ \ (1 + O(|x|^{1-\sigma}))\,, \qquad a(x) := -\frac{1}{4s}\ \left(1 - 2\ s\ Ce^{i\alpha(x)} + s^2\ C^2 e^{2i\alpha(x)}\right) = O(1) \text{ for } x \to 0$$

We observe that $y(x)$ is

$$y(x) = \left[-\frac{1}{4\ s\ C} e^{-i\alpha(x)}\ e^{i\arg(x)}\right]\ |x|^{1-\Sigma}\ \ (1 + O(|x|^{\delta}))\,, \qquad \Sigma \neq 0$$

$$y(x) = \left[-\frac{1}{4}\ \left(\frac{1}{s\ C} e^{-i\alpha(x)} + s\ C\ e^{i\alpha(x)} - 2\right)\ e^{i\arg(x)}\right]\ |x|\ \ (1 + O(|x|^{\delta}))\,, \qquad \Sigma = 0$$

We let the reader verify that also in the cases $\sigma = \pm 2\mu$ the behaviour of $y(x)$ is as above (use the matrices (14)and (16)) and that for $\sigma = 0$

$$y(x) = s\ x\ \left(1 + O(|x|^{1-\sigma_1})\right)$$

For $\sigma = 0$, we recall that $0 < \sigma_1 < 1$ is arbitrarily small.

3.2 Proof of Theorem 2

We are interested in lemma 1 when

$$f_{\mu\nu} = g_{\mu\nu} = \frac{b_\nu}{a_\mu - xb_\nu}, \qquad h_{\mu\nu} = 0$$

$$a_\mu, b_\nu \in \mathbf{C}, \qquad a_\mu \neq 0 \quad \forall \mu = 1, ..., n_1$$

Equations (11) are the isomonodromy deformation equations for the fuchsian system

$$\frac{dY}{dz} = \left[\sum_{\mu=1}^{n_1} \frac{A_\mu(\tilde{x})}{z - a_\mu} + \sum_{\nu=1}^{n_2} \frac{B_\nu(\tilde{x})}{z - xb_\nu}\right]\ Y$$

As a corollary of lemma 1, for a fundamental matrix solution $Y(z, x)$ of the fuchsian system the limits

$$\hat{Y}(z) := \lim_{\tilde{x} \to 0} Y(z, \tilde{x})$$

$$\tilde{Y}(z) := \lim_{\tilde{x}\to 0} \tilde{x}^{-\Lambda} Y(\tilde{x}z, \tilde{x})$$

exist. Here $\tilde{x} \to 0$ in $D(\epsilon;\sigma)$. They satisfy

$$\frac{d\hat{Y}}{dz} = \left[\sum_{\mu=1}^{n_1} \frac{A_\mu^0}{z-a_\mu} + \frac{\Lambda}{z}\right] \hat{Y}$$

$$\frac{d\tilde{Y}}{dz} = \sum_{\nu=1}^{n_2} \frac{B_\nu(x)}{z-b_\nu} \tilde{Y}$$

In our case, the last three systems reduce to

$$\frac{dY}{dz} = \left[\frac{A_0(\tilde{x})}{z} + \frac{A_x(\tilde{x})}{z-x} + \frac{A_1(\tilde{x})}{z-1}\right] Y \tag{18}$$

$$\frac{d\hat{Y}}{dz} = \left[\frac{A_1^0}{z-1} + \frac{\Lambda}{z}\right] \hat{Y} \tag{19}$$

$$\frac{d\tilde{Y}}{dz} = \left[\frac{A_0^0}{z} + \frac{A_x^0}{z-1}\right] \tilde{Y} \tag{20}$$

Before taking the limit $\tilde{x} \to 0$, let us choose

$$Y(z,\tilde{x}) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-A_\infty}, \quad z \to \infty \tag{21}$$

Thus we have

$$\begin{aligned} \hat{Y}(z) &= \left(I + O\left(\frac{1}{z}\right)\right) z^{-A_\infty}, \quad z \to \infty \\ &= (I + O(z))\, z^{\Lambda}\, \hat{C}_0, \quad z \to 0 \\ &= \hat{G}_1 (I + O(z-1))\, (z-1)^J\, \hat{C}_1, \quad z \to 1 \end{aligned} \tag{22}$$

Where $\hat{G}_1^{-1} A_1^0 \hat{G}_1 = J$, $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\hat{C}_0$, $\hat{C}_1$ are *conection matrices.* If we choose a fundamental matrix solution for (20) with the following behaviours

$$\begin{aligned} \tilde{Y}(z) &= \left(I + O\left(\frac{1}{z}\right)\right) z^{\Lambda}, \quad z \to \infty \\ &= \tilde{G}_0 (I + O(z))\, z^J\, \tilde{C}_0, \quad z \to 0 \\ &= \tilde{G}_1 (I + O(z-1))\, (z-1)^J\, \tilde{C}_1, \quad z \to 1 \end{aligned} \tag{23}$$

then

$$\lim_{\tilde{x}\to 0} \tilde{x}^{-\Lambda} Y(\tilde{x}z, \tilde{x}) = \tilde{Y}(z)\, \hat{C}_0 \tag{24}$$

Here $\tilde{G}_0^{-1} A_0^0 \tilde{G}_0 = J$, $\tilde{G}_1^{-1} A_x^0 \tilde{G}_1 = J$. We choose a monodromy repersentation for (18) by fixing a basepoint and a basis in the fundamental group of $\mathbf{P}^1$ as in figure 5. Let M_0, M_1, M_x, M_∞ be the monodromy matrices for the solution (21) corresponding to the loops γ_i $i = 0, x, 1, \infty$. Note that $M_\infty M_1 M_x M_0 = I$. The dependence of (18) on $\tilde{x}$ is isomonodromic, because the coefficients satisty the Schlesinger equations. For this reason, since the system (19) is obtained from (18) when x and 0 merge. and the singular point $z = 1$ does not move, we have

$$M_1 = \hat{M}_1 = \hat{C}_1^{-1} e^{2\pi i J} \hat{C}_1 \tag{25}$$

where $\hat{M}_1$ is the monodromy matrix of (22) for the loop $\hat{\gamma}_1$ in the basis of figure 6. Moreover, observe that

$$\frac{d(x^{-\Lambda} Y(xz,x))}{d(xz)} = \left[\frac{x^{-\Lambda} A_0 x^{\Lambda}}{z} + \frac{x^{-\Lambda} A_x x^{\Lambda}}{z-1} + \frac{x^{-\Lambda} A_1 x^{\Lambda}}{z-\frac{1}{x}}\right] x^{-\Lambda} Y(xz,x) \tag{26}$$

and the system (20) is obtained as $\frac{1}{x}$ and ∞ merge (figure 7). Also, observe that the singularities $z=0$, $z=1$, $z=1/x$ of (26) correspond to $z=0$, $z=x$, $z=1$ of (18). The poles $z=0$ and $z=1$ of (26) do not move as $x \to 0$. By isomonodromicity we conclude that

$$M_0 = \hat{C}_0^{-1}\tilde{M}_0\hat{C}_0 \equiv \hat{C}_0^{-1}\tilde{C}_0^{-1}e^{2\pi i J}\tilde{C}_0\hat{C}_0 \tag{27}$$

$$M_x = \hat{C}_0^{-1}\tilde{M}_1\hat{C}_0 \equiv \hat{C}_0^{-1}\tilde{C}_1^{-1}e^{2\pi i J}\tilde{C}_1\hat{C}_0 \tag{28}$$

where $\tilde{M}_{0,1}$ are the monodromy matrices for (23) in the basis of figure 6. Here $\hat{C}_0$ comes from (24).

The monodromy group of (18) is the group generated by the linear operators represented by M_i, $i=0,x,1$. Thus, the matrices M_i and KM_iK^{-1} are equivalent, for any non degenerate 2×2 matrix K. If follows that the monodromy group is completely caracterized by the three numbers x_i^2, $i=0,1,\infty$ given by

$$2-x_0^2 := \operatorname{tr}(M_0M_x); \quad 2-x_1^2 := \operatorname{tr}(M_xM_1); \quad 2-x_\infty^2 := \operatorname{tr}(M_0M_1);$$

Taking square roots we define the triples (x_0,x_1,x_∞) of section 2. The sign of the roots are chosen in such a way that

$$x_0^2+x_1^2+x_\infty^2-x_0x_1x_\infty = 4\sin^2(\pi\mu) \tag{29}$$

this last condition coming from $M_1M_xM_0 = M_\infty^{-1}$. For details see the Appendix and [4]. We only recall that two equivalent triples (as defined in section 2) represent the same monodromy group, and in the case $2\mu \notin \mathbf{Z}$ at most one x_i may be zero.

In order to find the parametrization $y(\tilde{x};\sigma,a)$ in terms of (x_0,x_1,x_∞) we have to compute the monodomy matrices M_0, M_1, M_∞ in terms of σ and a, and then take the traces of their products. In order to do this we use the formulae (25), (27),(28). In fact, the matrices $\tilde{M}_i$ $(i=0,1)$ and $\hat{M}_1$ can be computed explicitely because a 2×2 fuchsian system with three singular points can be reduced to the hypergeometric equation, whose monodromy is completely known.

Lemma 3: *The Gauss hypergeometric equation*

$$z(1-z)\,\frac{d^2y}{dz^2} + [\gamma_0-(\alpha_0+\beta_0+1)]\,\frac{dy}{dz} - \alpha_0\beta_0\; y = 0 \tag{30}$$

is equivalent to the system

$$\frac{d\Psi}{dz} = \left[\frac{1}{z}\begin{pmatrix} 0 & 0 \\ -\alpha_0\beta_0 & -\gamma_0\end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} 0 & 1 \\ 0 & \gamma_0-\alpha_0-\beta_0\end{pmatrix}\right]\Psi \tag{31}$$

where $\Psi = \begin{pmatrix} y \\ (z-1)\frac{dy}{dz}\end{pmatrix}$.

Lemma 4: *Let B_0 and B_1 be matrices of eigenvalues $0, 1-\gamma$, and $0, \gamma-\alpha-\beta-1$ respectively, such that*

$$B_0+B_1 = \operatorname{diag}(-\alpha,-\beta), \quad \alpha\neq\beta$$

Then

$$B_0 = \begin{pmatrix} \frac{\alpha(1+\beta-\gamma)}{\alpha-\beta} & \frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta}\,r \\ \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\,\frac{1}{r} & \frac{\beta(\gamma-\alpha-1)}{\alpha-\beta}\end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -(B_0)_{12} \\ -(B_0)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\end{pmatrix}$$

for any $r\neq 0$.

We leave the proof as an exercise. The following lemma connects lemmas 3 and 4:

Lemma 5: *The system (31) with*

$$\alpha_0=\alpha, \quad \beta_0=\beta+1, \quad \gamma_0=\gamma, \quad \alpha\neq\beta$$

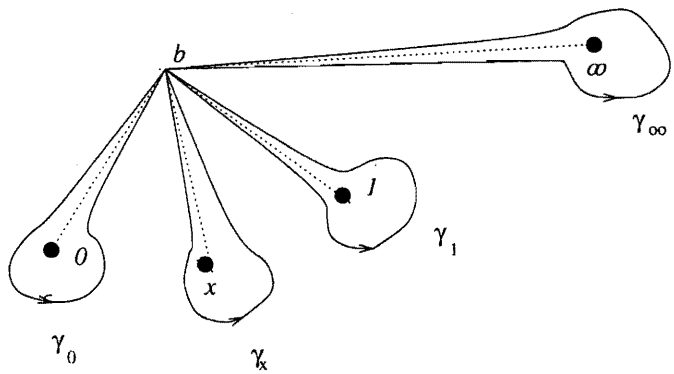

Figure 5

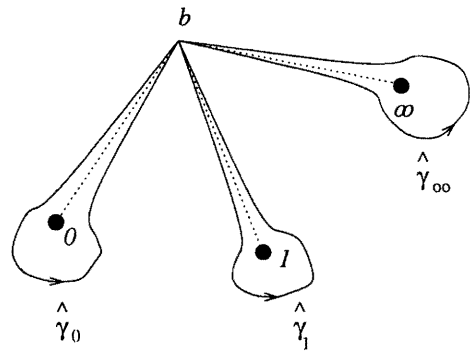

Figure 6

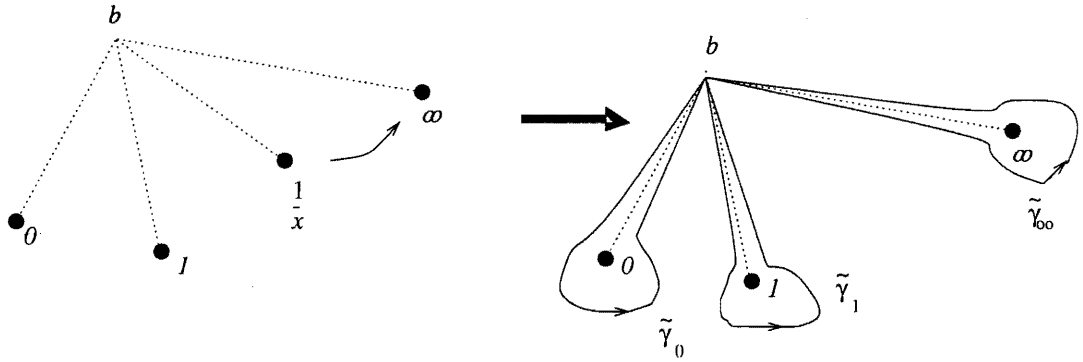

Figure 7

Figures 5: Branch cuts and loops for the fuchsian system associated with PVI_{μ}

Figure 6: Branch cuts and loops when $x \to 0$

Figure 7: Branch cuts and loops for the rescaled system before and after $x \to 0$

is gauge-equivalent to the system

$$\frac{dX}{dz} = \left[\frac{B_0}{z} + \frac{B_1}{z-1}\right] X \tag{32}$$

where B_0, B_1 are given in lemma 4. This means that there exists a matrix

$$G(z) := \begin{pmatrix} 1 & 0 \\ \frac{\alpha-\beta}{(\beta+1-\gamma)\beta} \left[\alpha\, z + \frac{\alpha(\beta+1-\gamma)}{\alpha-\beta}\right] \frac{1}{r} & z\, \frac{\alpha-\beta}{(\beta+1-\gamma)\beta}\, \frac{1}{r} \end{pmatrix}$$

such that $X(z) = G(z)\ \Psi(z)$. It follows that (32) and the corresponding hypergeometric equation (30) have the same fuchsian singularities 0,1,∞ and the same monodromy group.

Proof: By direct computation.

Note that the form of $G(z)$ ensures that if y_1, y_2 are independent solutions of the hypergeopmetric equation, then a fundamental matrix of (32) may be chosen a $X(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ * & * \end{pmatrix}$

Now we compute the monodromy matrices for the systems (19), (20) by reduction to an hypergeometric equation. We first study the case $\sigma \notin \mathbf{Z}$. Let us start with (19). With the gauge

$$Y^{(1)}(z) := z^{-\frac{\sigma}{2}}\ \hat{Y}(z)$$

we transform (19) in

$$\frac{dY^{(1)}}{dz} = \left[\frac{A_1^0}{z-1} + \frac{\Lambda - \frac{\sigma}{2}I}{z}\right] Y^{(1)}$$

We identify the matrices B_0, B_1 with A_1^0 and $\Lambda - \frac{\sigma}{2}I$ with eigenvalues 0, 0 and 0, $-\sigma$ respectively. Moreover $A_1^0 + \Lambda - \frac{\sigma}{2}I = \mathrm{diag}(-\mu - \frac{\sigma}{2}, \mu - \frac{\sigma}{2})$. Thus:

$$\alpha = \mu + \frac{\sigma}{2}, \quad \beta = -\mu + \frac{\sigma}{2}, \quad \gamma = \sigma + 1$$

$$\alpha - \beta = 2\mu \neq 0 \quad \text{by hypothesis}$$

The parameters of the correspondent hypergeometric equation are

$$\begin{cases} \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_0 = 1 - \mu + \frac{\sigma}{2} \\ \gamma_0 = \sigma + 1 \end{cases}$$

¿From them we deduce the nature of two linearly independent solutions at $z = 0$. Since $\gamma_0 \notin \mathbf{Z}$ ($\sigma \notin \mathbf{Z}$) the solutions are expressed in terms of hypergeometric functions. On the other hands, the significative parameters at $z = 1$ and $z = \infty$ are respectively:

$$\begin{cases} \alpha_1 := \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_1 := \beta_0 = 1 - \mu + \frac{\sigma}{2} \\ \gamma_1 := \alpha_0 + \beta_0 - \gamma_0 + 1 = 1 \end{cases}$$

$$\begin{cases} \alpha_\infty := \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_\infty := \alpha_0 - \gamma_0 + 1 = \mu - \frac{\sigma}{2} \\ \gamma_\infty = \alpha_0 - \beta_0 + 1 = 2\mu \end{cases}$$

Since $\gamma_1 = 1$, at least one solution has a logaritmic singularity at $z = 1$. The condition $2\mu \notin \mathbf{Z}$ ensures that no logaritmic singularities appear at $z = \infty$.

For the derivations which follows, we use the notations of the fundamental paper by Norlund [15]. To derive the connection formulae we use the paper of Norlund when logarithms are involved. Otherwise, in the generic case, any textbook of special functions (like [13]) may be used.

First case: $\alpha_0, \beta_0 \notin \mathbf{Z}$. This means

$$\sigma \neq \pm 2\mu + 2m, \quad m \in \mathbf{Z}$$

We can choose the following independent solutions of the hypergeometric equation:

At $z=0$

$$y_1^{(0)}(z) = F(\alpha_0, \beta_0, \gamma_0; z)$$

$$y_2^{(0)}(z) = z^{1-\gamma_0}\ F(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1, 2 - \gamma_0; z)$$

where $F(\alpha, \beta, \gamma; z)$ is the well known hypergeometric function (see [15]).

At $z=1$

$$y_1^{(1)}(z) == F(\alpha_1, \beta_1, \gamma_1; 1-z)$$

$$y_2^{(1)}(z) = g(\alpha_1, \beta_1, \gamma_1; 1-z)$$

Here $g(\alpha, \beta, \gamma; z)$ is a logarithmic solution introduced in [15].

At $z=\infty$

$$y_1^{(\infty)} = z^{-\alpha_0}\ F(\alpha_\infty, \beta_\infty, \gamma_\infty; \frac{1}{z})$$

$$y_2^{(\infty)} = z^{-\beta_0}\ F(\beta_0, \beta_0 - \gamma_0 + 1, \beta_0 - \alpha_0 + 1; \frac{1}{z})$$

Then, from the connection formulas between $F(...; z)$ and $g(...; z)$ of [13] and [15] we derive

$$[y_1^{(\infty)}, y_2^{(\infty)}] = [y_1^{(0)}, y_2^{(0)}]\ C_{0\infty}$$

$$C_{0\infty} = \begin{pmatrix} e^{-i\pi\alpha_0}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(1-\gamma_0)}{\Gamma(1-\beta_0)\Gamma(1+\alpha_0-\gamma_0)} & e^{-i\pi\beta_0}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(1-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1+\beta_0-\gamma_0)} \\ e^{i\pi(\gamma_0-\alpha_0-1)}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(\gamma_0-1)}{\Gamma(\alpha_0)\Gamma(\gamma_0-\beta_0)} & e^{i\pi(\gamma_0-\beta_0-1)}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(\gamma_0-1)}{\Gamma(\beta_0)\Gamma(\gamma_0-\alpha_0)} \end{pmatrix}$$

$$[y_1^{(0)}, y_2^{(0)}] = [y_1^{(1)}, y_2^{(1)}]\ C_{01}$$

$$C_{01} = \begin{pmatrix} 0 & -\frac{\pi\sin(\pi(\alpha_0+\beta_0))}{\sin(\pi\alpha_0)\sin(\pi\beta_0)}\frac{\Gamma(2-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_0)} \\ -\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_0-\alpha_0)\Gamma(\gamma_0-\beta_0)} & -\frac{\Gamma(2-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1-\beta_0)} \end{pmatrix}$$

We observe that

$$Y^{(1)}(z) = \left(I + \frac{F}{z} + O\left(\frac{1}{z^2}\right)\right)\ z^{\mathrm{diag}(-\mu-\frac{\sigma}{2}, \mu-\frac{\sigma}{2})}, \quad z \to \infty$$

$$= \hat{G}_0(I + O(z))\ z^{\mathrm{diag}(0,-\sigma)}\ \hat{G}_0^{-1}\hat{C}_0, \quad z \to 0$$

$$= \hat{G}_1(I + O(z-1))\ (z-1)^J\ \hat{C}_1, \quad z \to 1$$

where $\hat{G}_0 \equiv T$ of lemma 2; namely $\hat{G}_0^{-1}\Lambda\hat{G}_0 = \mathrm{diag}(\frac{\sigma}{2}, -\frac{\sigma}{2})$. By direct substitution in the differential equation we compute the coefficient F

$$F = -\begin{pmatrix} (A_1^0)_{11} & \frac{(A_1^0)_{12}}{1-2\mu} \\ \frac{(A_1^0)_{21}}{1+2\mu} & (A_1^0)_{22} \end{pmatrix}, \quad \text{where } A_1^0 = \frac{\sigma^2 - (2\mu)^2}{8\mu}\begin{pmatrix} 1 & -r \\ \frac{1}{r} & -1 \end{pmatrix}$$

Thus, from the asymptotic behaviour of the hypergometric function ($F(\alpha, \beta, \gamma; \frac{1}{z}) \sim 1,\ z \to \infty$) we derive

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(\infty)}(z) & \frac{\sigma^2-(2\mu)^1}{8\mu(1-2\mu)}\ y_2^{(\infty)} \\ * & * \end{pmatrix}$$

¿From

$$Y^{(1)}(z) \sim \begin{pmatrix} 1 & z^{-\sigma} \\ * & * \end{pmatrix}\ \hat{G}_0^{-1}\hat{C}_0, \quad z \to 0$$

we derive

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(1)}(z) & y_2^{(1)}(z) \\ * & * \end{pmatrix}\hat{G}_0^{-1}\hat{C}_0$$

Finally, observe that $\hat{G}_1 = \begin{pmatrix} a & \frac{a}{\omega} + br \\ \frac{a}{r} & b \end{pmatrix}$ for arbitrary $a, b \in \mathbf{C}$, $a \neq 0$, and $\omega := \frac{\sigma^2-(2\mu)^2}{8\mu}$. We recall that $y_2^{(1)} = g(\alpha_1, \beta_1, 1; 1-z) \sim \psi(\alpha_1) + \psi(\beta_1) - 2\psi(1) - i\pi + \log(z-1)$, $|\arg(1-z)| < \pi$, as $z \to 1$.

We can choose $a = 1$ and a suitable b, in such a way that the asymptotic behaviour of $Y^{(1)}$ for $z \to 1$ is precisely realized by

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(1)}(z) & y_2^{(1)}(z) \\ * & * \end{pmatrix} \hat{C}_1$$

The we conclude that the connection matrices are:

$$\hat{C}_0 = \hat{G}_0 \begin{pmatrix} (C_{0\infty})_{11} & r\,\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12} \\ (C_{0\infty})_{21} & r\,\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22} \end{pmatrix}$$

$$\hat{C}_1 = C_{01}\ (\hat{G}_0^{-1}\hat{C}_0) = C_{01} \begin{pmatrix} (C_{0\infty})_{11} & r\,\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12} \\ (C_{0\infty})_{21} & r\,\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22} \end{pmatrix}$$

Second case: $\alpha_0, \beta_0 \in \mathbf{Z}$, namely

$$\sigma = \pm 2\mu + 2m, \qquad m \in \mathbf{Z}$$

The formulae are almost identical to the first case, but $C_{0\infty}$ changes. To see this, we need to distinguish four cases.

i) $\sigma = 2\mu + 2m$, $m = -1, -2, -3, \ldots$. We choose

$$y_2^{(1)}(z) = g_0(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Here $g_0(z)$ is another logaritmic solution of [15]. Thus

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-m)\Gamma(-2\mu-m+1)}{\Gamma(-2\mu-2m)} & 0 \\ 0 & -\frac{\Gamma(1-2\mu-2m)}{\Gamma(1-m-2\mu)\Gamma(-m)} \end{pmatrix}$$

As usual, the matrix is computed from the connection formulas between the hypergeometric functions and g_0 that the reader can find in [15].

ii) $\sigma = 2\mu + 2m$, $m = 0, 1, 2, \ldots$. We choose

$$y_1^{(2)} = g(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} 0 & \frac{\Gamma(m+1)\Gamma(2\mu+m)}{\Gamma(2\mu+2m)} \\ -\frac{\Gamma(2\mu+2m+1)}{\Gamma(2\mu+m)\Gamma(m+1)} & 0 \end{pmatrix}$$

iii) $\sigma = -2\mu + 2m$, $m = 0, -1, -2, \ldots$. We choose

$$y_2^{(1)}(z) = g_0(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} \frac{\Gamma(1-m)\Gamma(2\mu-m)}{\Gamma(2\mu-2m)} & 0 \\ 0 & -\frac{\Gamma(1+2\mu-2m)}{\Gamma(2\mu-m)\Gamma(1-m)} \end{pmatrix}$$

iv) $\sigma = -2\mu + 2m$, $m = 1, 2, 3, \ldots$. We choose

$$y_2^{(1)}(z) = g(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} 0 & \frac{\Gamma(m)\Gamma(m+1-2\mu)}{\Gamma(2m-2\mu)} \\ -\frac{\Gamma(2m+1-2\mu)}{\Gamma(m+1-2\mu)\Gamma(m)} & 0 \end{pmatrix}$$

Note that this time $F = \begin{pmatrix} 0 & \frac{r}{1-2\mu} \\ 0 & 0 \end{pmatrix}$ in the case $\sigma = \pm 2\mu$ (i.e. $m = 0$) because A_1^0 has a special form in this case. Then in $\hat{C}_0$ the elements $\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12}$, $\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22}$ must be substituted, for $m = 0$, with $\frac{1}{1-2\mu}(C_{0\infty})_{12}$, $\frac{1}{1-2\mu}(C_{0\infty})_{22}$.

We turn to the system (20). Let $\tilde{Y}$ be the fundamental matrix (23). With the gauge

$$Y^{(2)}(z) := \hat{G}_0^{-1} \left(\tilde{Y}(z)\hat{G}_0\right)$$

we have

$$\frac{dY^{(2)}}{dz} = \left[\frac{\tilde{B}_0}{z} + \frac{\tilde{B}_1}{z-1}\right] Y^{(2)}$$

$$\tilde{B}_0 = \hat{G}^{-1} A_0^0 \hat{G}_0 = \begin{pmatrix} \frac{\sigma}{4} & \frac{\sigma}{4}s \\ -\frac{\sigma}{4s} & -\frac{\sigma}{4} \end{pmatrix}$$

$$\tilde{B}_1 = \hat{G}^{-1} A_x^0 \hat{G}_0 = \begin{pmatrix} \frac{\sigma}{4} & -\frac{\sigma}{4}s \\ \frac{\sigma}{4s} & -\frac{\sigma}{4} \end{pmatrix}$$

This time then

$$\begin{cases} \alpha_0 = -\frac{\sigma}{2} \\ \beta_0 = \frac{\sigma}{2} + 1 \\ \gamma_0 = 1 \end{cases}$$

$$\begin{cases} \alpha_1 = -\frac{\sigma}{2} \\ \beta_1 = \frac{\sigma}{2} + 1 \\ \gamma_1 = 1 \end{cases}$$

$$\begin{cases} \alpha_\infty = -\frac{\sigma}{2} \\ \beta_\infty = \frac{\sigma}{2} \\ \gamma_\infty = \sigma \end{cases}$$

If follows that both at $z = 0$ and $z = 1$ there are logaritmic solutions. We skip all the derivation of the connection formulae, which is done as in the previous cases, with some more technical complications. Before giving the results we observe that

$$Y^{(2)}(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{\mathrm{diag}(\frac{\sigma}{2}, -\frac{\sigma}{2})}, \qquad z \to \infty$$

$$= \hat{G}_0^{-1}\tilde{G}_0 \ (1 + O(z)) z^J \ C_0', \qquad z \to 0$$

$$= \hat{G}_0^{-1}\tilde{G}_1 \ (1 + O(z-1))(z-1)^J \ C_1', \qquad z \to 1$$

where

$$C_i' := \tilde{C}_1 \hat{G}_0, \qquad i = 0, 1$$

Then

$$\tilde{M}_0 = \hat{G}_0 \ (C_0')^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_0' \ \hat{G}_0^{-1}$$

$$\tilde{M}_1 = \hat{G}_0 \ (C_1')^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_1' \ \hat{G}_0^{-1}$$

Then, the connection problem may be solved computing C_i'. The result is

$$C_0' = \begin{pmatrix} (C_{0\infty}')_{11} & \frac{\sigma}{\sigma+1}\frac{s}{4}(C_{0\infty}')_{12} \\ (C_{0\infty}')_{21} & \frac{\sigma}{\sigma+1}\frac{s}{4}(C_{0\infty}')_{22} \end{pmatrix}$$

$$C_1' = C_{01}' C_0'$$

where

$$(C_{0\infty}')^{-1} = \begin{pmatrix} \frac{\Gamma(\beta_0-\alpha_0)}{\Gamma(\beta_0)\Gamma(1-\alpha_0)} e^{i\pi\alpha_0} & 0 \\ \frac{\Gamma(\alpha_0-\beta_0)}{\Gamma(\alpha_0)\Gamma(1-\beta_0)} e^{i\pi\beta_0} & -\frac{\Gamma(1-\alpha_0)\Gamma(\beta_0)}{\Gamma(\beta_0-\alpha_0+1)} \ e^{i\pi\beta_0} \end{pmatrix}$$

$$C_{01}' = \begin{pmatrix} 0 & -\frac{\pi}{\sin(\pi\alpha_0)} \\ -\frac{\sin(\pi\alpha_0)}{\pi} & -e^{-i\pi\alpha_0} \end{pmatrix}$$

To prove theorem 2 it is now enough to compute

$$2 - x_0^2 = \operatorname{tr}(M_0 M_x) \equiv \operatorname{tr}(e^{2\pi i J}(C'_{01})^{-1} e^{2\pi i J} C'_{01})$$

$$2 - x_1^2 = \operatorname{tr}(M_x M_1) \equiv \operatorname{tr}((C'_1)^{-1} e^{2\pi i J} C'_1 C_{01}^{-1} e^{2\pi i J} C_{01})$$

$$2 - x_\infty^2 = \operatorname{tr}(M_0 M_1) \equiv \operatorname{tr}((C'_0)^{-1} e^{2\pi i J} C'_0 C_{01}^{-1} e^{2\pi i J} C_{01})$$

Note the remarkable simplifications obtained from the cyclic property of the trace (for example, $\hat{C}_0$ disappears). We obtain formulae I), II) and III) of theorem 2. All the other formulas are deduced from them.

The case $\sigma \in \mathbf{Z}$ interests us only if $\sigma = 0$ (otherwise $\sigma \notin \Omega$). We observe that the system (19) is precisely the system for $Y^{(2)}(z)$ with the substitution $\sigma \mapsto -2\mu$. In the formulae for x_i^2, $i = 0, 1, \infty$ we only need C_{01}, which is obtained from C'_{01} with $\alpha_0 = \mu$.

As for the system (20), the gauge $Y^{(2)} = \hat{G}_0^{-1} \tilde{Y} \hat{G}_0$ yields $\tilde{B}_0 = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$, $\tilde{B}_1 = \begin{pmatrix} 0 & 1-s \\ 0 & 0 \end{pmatrix}$. Here $\hat{G}_0$ is the matrix such that $\hat{G}_0^{-1} \Lambda \hat{G}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The behaviour of $Y^{(2)}(z)$ is now:

$$Y^{(2)}(z) = (I + O(\frac{1}{z}))\ z^J \qquad z \to \infty$$

$$= \tilde{\tilde{G}}_0^{-1}\ (1 + O(z)) z^J\ C'_0, \quad z \to 0$$

$$= \tilde{\tilde{G}}_1\ (1 + O(z-1))(z-1)^J\ C'_1, \quad z \to 1$$

Here $\tilde{\tilde{G}}_i$ is the matrix that puts $\tilde{B}_i$ in Jordan form, for $i = 0, 1$. $Y^{(2)}$ can be computed explicitely:

$$Y^{(2)}(z) = \begin{pmatrix} 1 & s\log(z) + (1-s)\log(z-1) \\ 0 & 1 \end{pmatrix}$$

If we choose $\tilde{\tilde{G}}_0 = \operatorname{diag}(1, 1/s)$, then

$$C'_0 = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$$

In the same way we find

$$C'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-s \end{pmatrix}$$

To complete the proof of theorem 2 (points i), ii), iii)), we just have to compute the square roots of the x_i^2 $(i = 0, 1, \infty)$ in such a way that (2) is satisfied.

4 Appendix: Monodromy Data

The equation PVI_μ is equivalent to the equations of isomonodromy deformation of the fuchsian system discussed in section 3. In particular, the solution $y(x)$ is obtained from the coefficients $A_i(x)$, $i = 0, x, 1$. More generally, the system

$$\frac{dY}{dz} = \left[\frac{A_1(u)}{z-u_1} + \frac{A_2(u)}{z-u_2} + \frac{A_3(u)}{z-u_3}\right] Y := A(z; u_1, u_2, u_3)\ Y$$

$$\operatorname{tr}(A_i) = \det A_i = 0, \qquad \sum_{i=1}^{3} A_i = -\operatorname{diag}(\mu, -\mu)$$

with isomonodromic dependence of the matrix coefficints A_i on $u = (u_1, u_2, u_3)$ is associated with PVI_μ by

$$x = \frac{u_2 - u_1}{u_3 - u_1}, \quad y(x) = \frac{q(u) - u_1}{u_3 - u_1}$$

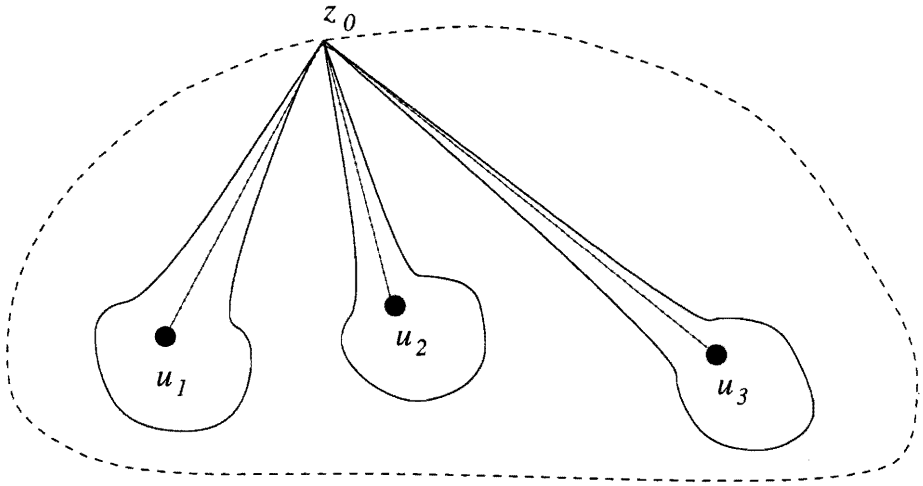

Figure 8: $M_3 M_2 M_1 = M_\infty$

where $q(u_1, u_2, u_3)$ is the root of

$$[A(q; u_1, u_2, u_3)]_{12} = 0 \qquad \text{if } \mu \neq 0$$

The case $\mu = 0$ is disregarded, because $PVI_{\mu=0} \equiv PVI_{\mu=1}$. $A_i(u)$ can be obtained as solutions of Schlesinger equations. In section 3 we have chosen $u_1 = 0$, $u_2 = x$, $u_3 = 1$. For references, see [5], [8], [11].

The problem is equivalent to a Riemann-Hilbert problem (R.H.): find the coefficients $A_i(u)$, $i = 1, 2, 3$ from the following monodromy data:

a) the matrices

$$\hat{\mu} = \mathrm{diag}(\mu, -\mu), \qquad \mu \in \mathbf{C}\backslash\{0\}$$

$$R = \begin{cases} 0 \quad , & \text{if } 2\mu \notin \mathbf{Z} \\ \left.\begin{array}{ll} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, & \mu > 0 \\ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, & \mu < 0 \end{array}\right\} & \text{if } 2\mu \in \mathbf{Z} \end{cases}$$

$b \in \mathbf{C}$.

b) three poles u_1, u_2, u_3, a basepoint and a base of loops in $\pi(\mathbf{C}\backslash\{u_1, u_2, u_3\}; z_0)$. See figure 8

c) three monodromy matrices M_1, M_2, M_3 relative to the loops and a matrix M_∞ similar to $e^{-2\pi i(\hat{\mu}+R)}$, and satisfying

$$\mathrm{tr}(M_i) = 2, \quad \det(M_i) = 1, \quad i = 1, 2, 3$$
$$M_3\ M_2\ M_1 = M_\infty$$
$$M_\infty = C_\infty^{-1}\ e^{-2\pi i(\hat{\mu}+R)}\ C_\infty \tag{33}$$

where C_∞ realizes the similitude. We also choose the indices of the problem, namely we fix the $\frac{1}{2\pi i}\log M_i$ as follows: let

$$J := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We require there exist three *connection matrices* C_1, C_2, C_3 such that

$$C_i^{-1} e^{2\pi i J} C_i = M_i, \qquad i = 1, 2, 3 \tag{34}$$

and we look for a matrix valued meromorphic function $Y(z; u)$ (branch cuts are understood as in figure 5) such that

$$Y(z; u) = \begin{cases} G_\infty(I + O(\frac{1}{z}))\ z^{-\hat{\mu}} z^{-R}\ C_\infty, & z \to \infty \\ G_i(I + O(z - u_i))\ (z - u_i)^J\ C_i, & z \to u_i, \quad i = 1, 2, 3 \end{cases} \tag{35}$$

G_∞ and G_i are constant inverible matrices. The coefficient of the fuchsian system are then given by

$$A(z;u_1,u_2,u_3) := \frac{dY(z;u)}{dz} Y(z;u)^{-1}$$

Recall that a 2×2 R.H. is always solvable [1]. The monodromy matrices are considered up to the conjugation

$$M_i \mapsto M_i' = B^{-1}M_iB, \qquad \det B \neq 0, \qquad i=1,2,3,\infty \tag{36}$$

and the coefficients of the fuchsian system itself are considered up to conjugation $A_i \mapsto F^{-1}A_iF$ ($i = 1,2,3$), by an invertible matrix F . Actually, two conjugated fuchsian systems admit fundamental matrix solutions with the same monodromy, and a given fuchsian system defines the monodromy up to conjugation (depending on the choice of the fundamental solution).

On the other hand, a triple of monodromy matrices M_1, M_2, M_3 may be realized by two fuchsian systems which are not conjugated. This corresponds to the fact that the solutions C_∞, C_i of (33), (34) are not unique, and the choice of different paricular solutions may give rise to fuchsian systems which are not conjugated. If this is the case, there is no one to one correspondence between monodromy matrices (up to conjugation) and solutions of PVI_μ.

We show now that the only case in which this happens is for $2\mu \in \mathbf{Z}$ and $R=0$. This is easily done in the following way: consider two solutions C and $\tilde{C}$ of the equations (33), (34). Then

$$(C_i\tilde{C}_i^{-1})^{-1}e^{2\pi iJ}(C_i\tilde{C}_i^{-1}) = e^{2\pi iJ}$$

$$(C_\infty\tilde{C}_\infty^{-1})^{-1}e^{-2\pi i(\hat{\mu}+R)}(C_\infty\tilde{C}_\infty^{-1}) = e^{-2\pi i(\hat{\mu}+R)}$$

We find

$$C_i\tilde{C}_i^{-1} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \qquad a,b\in\mathbf{C}, \quad a\neq 0$$

Note that this matrix commutes with J, then

$$(z-u_i)^J C_i = (z-u_i)^J \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \tilde{C}_i = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} (z-u_i)^J \tilde{C}_i$$

We anso find

$$C_\infty\tilde{C}_\infty^{-1} = \begin{cases} i) \quad \mathrm{diag}(\alpha,\beta), \quad \alpha\beta\neq 0; & \text{if } 2\mu\notin\mathbf{Z} \\ ii) \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \ (\mu>0), \quad \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} \ (\mu<0), \quad \alpha\neq 0, & \text{if } 2\mu\in\mathbf{Z}, R\neq 0 \\ iii) \ \text{Any invertible matrix} & \text{if } 2\mu\in\mathbf{Z}, R=0 \end{cases}$$

Then

$$i)\ z^{-\hat{\mu}}C_\infty = z^{-\hat{\mu}}\mathrm{diag}(\alpha,\beta)\tilde{C}_\infty = \mathrm{diag}(\alpha,\beta)z^{\hat{\mu}}\tilde{C}_\infty$$

$$ii)\ z^{-\hat{\mu}}z^{-R}C_\infty = \ \dots\ = \left[\alpha I + \frac{1}{z^{|2\mu|}}Q\right] z^{-\hat{\mu}}z^{-R}\tilde{C}_\infty$$

where $Q = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$, or $Q = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$.

$$iii)\ i)\ z^{-\hat{\mu}}C_\infty = \ \dots\ = \left[\frac{Q_1}{z^{|2\mu|}} + Q_0 + Q_{-1}z^{|2\mu|}\right] z^{-\hat{\mu}}\tilde{C}_\infty$$

where $Q_0 = \mathrm{diag}(\alpha,\beta)$, $Q_{\pm 1}$ are respectively upper and lower triangular (or lower and upper triangular, depending on the sign of μ), and $C_\infty\tilde{C}_\infty^{-1} = Q_1 + Q_0 + Q_{-1}$

This implies that the two solutions $Y(z;u)$, $\tilde{Y}(z;u)$ of the form (35) with C and $\tilde{C}$ respectively, are such that $Y(z;u)\ \tilde{Y}(z;u)^{-1}$ is holomorphic at each u_i, while at $z=\infty$ it is

$$Y(z;u)\ \tilde{Y}(z;u)^{-1} = \begin{cases} i)\ G_\infty\mathrm{diag}(\alpha,\beta)G_\infty^{-1} \\ ii)\ \alpha I \\ iii)\ \text{divergent} \end{cases}$$

Thus the two fuchsian systems are conjugated only in the cases *i*) and *ii*), as we wanted to prove. In other words *the R.H. has a unique solution, up to conjugation, for* $2\mu \notin \mathbf{Z}$ *or for* $2\mu \in \mathbf{Z}$ *and* $R \neq 0$.

Once the R.H. is solved, the sum of the matrix coefficients $A_i(u)$ of the solution $A(z; u_1, u_2, u_3) = \sum_{i=1}^{3} \frac{A_i(u)}{z-u_i}$ must be diagonalized (to give $-\operatorname{diag}(\mu, -\mu)$). After that, the solution of PVI_μ can be "computed".

Note however that if $G_\infty = C_\infty = I$, then $\sum_{i=1}^{3} A_i$ is already diagonal. Moreover, for $2\mu \notin \mathbf{Z}$, M_1, M_2, M_3 and the choice of normalization $Y(z;u)z^{\hat{\mu}} \to I$ if $z \to \infty$ determine uniquely A_1, A_2, A_3. Actually, for any diagonal invertible matrix D, the matrices $M_1' = D^{-1}M_1D$, $M_2' = D^{-1}M_2D$, $M_3' = D^{-1}M_3D$ determine the coefficients $D^{-1}A_iD$, whose sum is still diagonal (the normalization of Y is the same).

Finally, let us recall that the parameters in the space of the monodromy representation independent of conjugation of the M_i, are

$$2 - x_1^2 := \operatorname{tr}(M_1M_2), \quad 2 - x_2^2 := \operatorname{tr}(M_2M_3), \quad 2 - x_3^2 := \operatorname{tr}(M_1M_3)$$

The triple (x_0, x_1, x_∞) of sections 2 and 3 corresponds to (x_1, x_2, x_3).

We review some known results [4] [14].

1) One $M_i = I$ (and then two of the x_j's are equal to zero) if and only if the Schlesinger equations yield $q(u) \equiv u_i$. This does not correspond to a solution of PVI_μ.

2) If the M_i's, $i = 1, 2, 3$, commute, then μ is integer (as it follows from the fact that the 2×2 matrices commute if and only if they can be simultaneously put in upper or lower triangular form, with 1 on diagonals in our case); $(x_1, x_2, x_3) = (0, 0, 0)$. There are solutions of PVI_μ only for

$$M_1 = \begin{pmatrix} 1 & i\pi a \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & i\pi \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & i\pi(1-a) \\ 0 & 1 \end{pmatrix}, \qquad a \neq 0, 1$$

For $\mu = 1$ the solution is

$$y(x) = \frac{ax}{1-(1-a)x}$$

and for other integers μ the solution is obtained from $\mu = 1$ by a birational transformation [4] [14]. In particular, these solutions are rational.

3) Non commuting M_i's.

3.1) If (at least) two of the x_j's are zero, then one of the M_i's is I. We return to the case 1.

3.2) At most one of the x_j's is zero. Namely, the triple (x_1, x_2, x_3) is *admissible*. In this case there exists a basis such that, if for example $x_1 \neq 0$:

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 + \frac{x_2x_3}{x_1} & -\frac{x_2^2}{x_1} \\ \frac{x_3^2}{x_1} & 1 - \frac{x_2x_3}{x_1} \end{pmatrix}$$

The relation

$$M_3M_2M_1 \text{ similar to } e^{-2\pi i(\hat{\mu}+R)}$$

implies

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4\sin^2(\pi\mu)$$

The signs of the x_i's must be chosen in such a way that the above relation is satisfied. The conjugation (36) changes the triple by two signs. Thus the true parameters for the monodromy data are classes of equivalence of triples (x_1, x_2, x_3) defined by the change of two signs.

We distingush three cases:

i) $2\mu \notin \mathbf{Z}$. Then there is a one to one correspondence between monodromy data and the solutions of PVI_μ. For $x_j^2 \neq 4$ the solutions are those of section 1. In particular, the algebraic solutions of [4] are included in this case.

ii) For μ half integer there is an infinite set of *Picard type solutions* (see [14]), in one to one correspondence to triples of monodromy data ($R \neq 0$) not in the equivalence class of $(2, 2, 2)$. These solutions form a two parameter family, behave asymptotically as the solutions of the case *i*), and comprise

a denumerable subclass of algebric solutions. For any half integer $\mu \neq \frac{1}{2}$ there is also a one parameter family of *Chazy solutions* (non algebric). For them the one to one correspondence with monodromy data is lost. In fact, they form an infinite family but any element of the family corresponds to a triple (x_1, x_2, x_3) in the orbit (of the braid group) of the triple $(2, 2, 2)$ (this orbit is simply obtained by changing two signs in all possible ways). Actually, they appear in the case $R = 0$ (no other solutions of PVI correspond to $R = 0$ and μ half integer).

iii) μ integer. Non algebraic solutions. To our knowledge, they have not yet been studied. There are relevant examples of Frobenius manifolds where these solutions must appear, like the case of Quantum Cohomology of CP^2 [6], [12], [2], [7]. In this case $\mu = 1$, the triple $(x_0, x_1, x_\infty) = (3, 3, 3)$ (the monodromy data coincide with the elements of the Stokes'matrix of the corresponding Frobenius manifold [5] [7]) and the real part of $\sigma = 1$. Although the case $\mu = 1$ is not formally developed here, lemma 1 still holds. As a consequence, we realize that the method developed in this paper and applied to Quantum Cohomology cannot give information about the behaviour of $y(x)$ as x converges to a critical point along a radial path. It would be desirable to obtain such an information to study the analytic properties of the solution of the WDVV.

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